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INTELLECTUAL IMPLICATIONS OF THE COMPUTER REVOLUTION*

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Abstract. Most descriptions of the current computer revolution concentrate on the material aspects. This paper concentrates on the intellectual aspects.

Computers have improved in speed by a factor of more than a million, while the cost has decreased by more than a thousand. As a result completely new effects have appeared.

Some of the new effects on engineering, science, language, music, and the process of thinking are discussed, and some of the consequences to education are mentioned. The main thesis of the paper is that computers are producing a revolution in the world of ideas.

You have probably heard and read about the computer revolution, the control revolution if you prefer, that is presently occurring. But you have heard and read mainly about the material aspects of the revolution; I propose to show you that the intellectual aspects of this revolution are at least as interesting and important. Perhaps it should be mentioned here that computing machines are frequently called "information processing machines," and information is the soul of modern life.

The computer revolution is often compared with the famous industrial revolution in importance and scope. The industrial revolution effectively released man from being a beast of burden; the computer revolution will similarly release him from slavery to dull, repetitive routine. The computer revolution is, however, perhaps better compared with the Copernican revolution, or the Darwinian revolution, both of which greatly changed man's idea of himself and the world in which he lives.

Before discussing the main points of the paper, it is necessary to discuss briefly the idea of a change in a technology. Change is often measured in units of "an order of magnitude" meaning roughly a factor of ten—ten times as much. It is a common observation that a change of an order of magnitude in a technology produces *fundamentally new effects.* As illustrations consider the following examples. Modern jet planes are about one order of magnitude faster than the Wright Brother's first plane. The fastest missiles are somewhat more than two orders faster—meaning about three hundred times faster—than the first plane. Automobiles are used at speeds about one order of magnitude faster than a horse and wagon. Each of these has produced whole new effects; indeed the automobile is said to have caused even a change in our morals!

Computers have improved in speed by at least six orders of magnitude over hand calculations—a million fold. In order to understand the factor of a million consider the following two situations: first, you have only one dollar, and second, you have one million dollars. You can readily see that in the two different situa-

* Presented to Section A of the AAAS, Denver, December 1961.

tions there are *fundamental* differences in the view you adopt of yourself and of the possibilities that are open to you.

Along with the change in speed there has been a great increase in reliability of operation so that we now do much longer computations than were practical by hand.

Finally, with the increase in speed and reliability there has been a corresponding decrease in the cost per operation—something more than one thousand times cheaper. It is as if suddenly automobiles now cost two to four dollars, houses twenty to sixty dollars. And the changes in the computer technology are still going on!

These are the bases of the computer revolution; at least six orders of magnitude increase in speed, at least three orders of magnitude decrease in cost, and an increase in reliability which makes practical computations involving billions of arithmetical operations.

As I said, it is customary to recognize that a change of a single order of magnitude in a technology often produces fundamentally new effects. Think, then, of the six orders of magnitude in speed and the three in cost, and the new effects they will produce. The change has been so rapid that we in the computer field are only just starting the exploration and exploitation of these new effects. Our first approach was to carry on in the same old ways, only bigger, faster, and cheaper, and we ignored the order of magnitude change and the new effects. Now we are beginning to explore them.

The impact of the computer revolution can already be seen in a number of different fields. The most obvious application of computers is to the field of engineering where extensive hand calculations have long been used. Here, although computers have long been used, they have, in fact, had comparatively little effect when judged by the criterion of *new ideas*. Old ideas have been expanded and developed, jobs which were never computed by hand because they were uneconomical or too slow, are now routinely done, but fundamentally new ideas and approaches are comparatively rare. Perhaps the biggest change in the engineering field is in the area of education where the availability of a large scale computer on a campus has changed some of the curriculum (though, of course, the computer was not the only force producing change).

Next let us look at the field of science. Turning first to the laboratory side, the work in acoustics research at the Bell Telephone Laboratories provides a good example of the effect of having a large scale digital computer available. Before we had computers the scientists frequently had to build laboratory equipment to study various proposed transmission systems. Now about all they use is an analog to digital converter which takes in a sound track on a magnetic tape and from it produces a sampled digital tape. The digital tape is then run through a large digital computer which has been provided with a program to simulate the proposed transmission system. The output tape represents what would have been produced by the (idealized) system had it been built. This tape can then be run through the converter and the output heard directly.

Some of the important points to note are: first, the results are typically obtained in days rather than the months which would have been required had the equipment been built and debugged (that is, rendered error free). Second, expensive as large scale digital computers are, it is usually much cheaper to simulate on a computer than to build equipment. Third, because the results are obtained cheaply and quickly, more speculative systems can be examined and the enthusiasm does not get killed by the long waiting time between idea and result. Last, we can examine systems which at the moment we do not necessarily know how to build in the form of hardware—thus the range of possible exploration is greatly increased.

At present I would guess that perhaps 10% of the experiments in the Bell Telephone Laboratories are done on the computer rather than in the laboratory; I expect that in time the reverse will be true, only about 10% will be done in the laboratory. The advantages in speed, cost, and effort, favor the computer over the traditional laboratory approach.

On the theoretical side computers are influencing research even more profoundly. Let me illustrate this point from my personal experience by discussing a book I wrote last year. The book is concerned with the use of computers and would naturally contain many formulas. My first thoughts were to arrange all the formulas so that they could be put on a magnetic tape and thus supplied to the machine. The prospective user would call for the particular formula he wished to use by the label I gave it, and the machine would then find it on the tape and bring it into use. More mature thought suggested it would be better for me to supply the machine with the methods of derivation of the formulas and let the machine derive the particular formula when needed rather than try to find it on a tape. Actually it would be cheaper for the machine to do so. It would also mean that the prospective user would have available not only those formulas I had derived, but *any* that were derivable by the techniques I gave the machine. This point can be summarized by the words "information regeneration, rather than information retrieval," and illustrates how the order of magnitude change in computing capacity changes one's basic strategy. Indeed, when there are potentially an infinite number of formulas regeneration is the only practical approach.

As a result, my research for writing the book had a profound change in emphasis; I did not try to find all the formulas that might be useful, rather I sought uniform methods for finding formulas so that the machine could actually apply the methods to produce the specific ones needed at any particular time. And this is a sensible division of labor between the human and the machine. The machine is best at working out details—far faster and more accurate than I am —while I am, at least for the moment, better able to suggest the broad lines of attack, the uniform methods for approaching problems.

This suggests the obvious future. Man will concentrate more on the conceptions in a field, leaving the computers to work out the details and the checks

with experiments. As we learn more about how to use machines, we will push the line marking the division of labor so that machines take over increasing amounts of routine work, leaving man free to speculate, to imagine, and to conceive of new ideas, where he seems to be better than the machine.

For the next example we turn to the field of language. It is said that man is a tool-making animal. A hammer increases his muscle power, a microscope increases his range of vision, an oscilloscope gives him a new sense—all these are tools to aid, improve, and extend the body. Language is also a tool, but it is a tool of the mind rather than of the body. Computers, like languages, are mainly tools to aid the mind rather than the body.

This being the case, it is not surprising that computers have raised many new questions in the general area of language, and because the questions are new the answers will probably be new too. To be more specific, in order to control a digital computer it is necessary to tell the machine what to do in some mutually understood language. The instructions which the engineers build into the machine really constitute a language of communication between the human and the computer. The language is sometimes said to be in microsyllables. We soon got tired of speaking to the machine in microsyllables and started to invent languages more suitable for humans to use. We then program the machine to translate from these languages into its own language. We have now been inventing and using artificial languages of increasing sophistication for almost ten years. The languages are still far from satisfactory but the process of inventing and using them has raised many new and interesting questions.

The natural languages which humans speak, such as English, Russian, German, are all the result of long evolution. I take it as axiomatic that being the result of long evolution they have many properties which make them well adapted for the use they are put to. It must be remembered that much of the selection was made, as with humans, under circumstances far different from those in which they are now used. This being granted, I have asked *why* many of the features have persisted—"why" in a more engineering sense than it has been traditional to ask. For example, what are the survival values of regular vs. irregular verbs? Regular verbs make easy learning and recall; irregular verbs make for sounds which are quite different and hence easily separated. One would expect to find that the common verbs tend to be irregular, and the less common ones to be regular, and to a great extent this is what we find. Linguists have an alternative explanation, and the two do not necessarily contradict each other.

The common languages have a redundancy of 60% or so—that is, about 60% of a written message can be suitably deleted and still the original message can be reconstructed fairly accurately. The spoken language tends to have an even higher redundancy. The value of the redundancy is that when I talk, and a single word or phrase is poorly selected or spoken, the audience does not lose the whole talk, but only, at most, a small local part. Thus redundancy gives pro-

tection against the unreliability of the human nervous system that generates the words and sentences as well as the one that receives it, plus protection against noise in the acoustic channel over which it is sent—the value is obvious. But how it is accomplished is another matter. How is the redundancy built into the language so that it protects the human from his unreliable nervous system? All the synthetic languages we have devised so far for machines have had only a small amount of redundancy for protection, and that little not deliberately designed but rather a matter of chance.

Another aspect of language I would like to understand, and hence be able to incorporate into a synthetic language, is the well-known observation that trying to put an idea into words is often a great help to clear thinking. I would like to design a language that was particularly adapted to help humans to think clearly when they describe their problem to the machine. Alas, I do not know how.

Besides designing and using artificial languages for machine-human use, we have also attacked the problem of translating from one natural language to another, such as from Russian to English. Both the great success and the great difficulties left shed a good deal of light on various aspects of language. To me, a comparative outsider, it seems that the major stumbling block centers around the vexing question, "What did the author mean?" Perhaps this is a real question, and perhaps it is not, but it seems to me to be one of the points where progress in machine translation of languages is presently in difficulties.

We have also begun to make studies of authors' styles, to build concordances, etc. These are necessary chores for many fields of research in the language area, but by hand they are very time consuming. With machines to do the dog work, more sophisticated and ambitious plans can be carried through, and more attention paid to the interpreting, rather than the gathering, of data.

Among the creative arts, music has received a great deal of attention in computer circles. The principles of musical composition have been described to a machine and music composed. I am not an expert, but others who are tell me, just as my own ears do, that the music so composed is quite interesting in the small, but in the large a great deal is lacking. The machine seems to have a novelty that humans lack, but so far the machine programs describing the methods of composition have not been able to create any long range interest for humans. Will the future prove that such programs are impossible, or that they are possible? I do not know.

But one point is already clear. Suppose you were very much interested in musical composition theory and you had a conjecture that one particular theory was better than another. You would be hard put to test the theories objectively. If you had a program that composed reasonably good music, you could have the machine compose several pieces in the two styles and let various people hear them. The result would be highly suggestive as to which style was the better, but, of course, not definitive. The lack of the mathematical model which faith-

fully imitated live composers would lend an element of doubt to the answer. But again I say it would be highly suggestive, and would provide one more tool in the study of composition.

Machine routines have been written which compute the sound track of any ideal musical instruments which are described by suitable programs. Thus the attack, the vibrato, the overtone pattern, etc., may be defined, and music produced which will sound as it should. Again, this will probably not be exactly like currently used instruments, but such an approach can be used to suggest many new things to try.

In any case, even with only a few years of using computers in the general area of music, many new ideas have been raised, and the answers are highly suggestive for further research.

One of the most common questions asked is, "Can computers think?" The answer, of course, depends on the precise definition of "thinking," which practically no one is prepared to define sharply.

Let us examine some of the available data. Machines have been "programmed" to play such thinking games as chess and checkers, and all things considered, they play fairly well—in checkers well enough to beat many human players. They have also been programmed to solve high school geometry problems. By "solve" and "play a game" we do not mean the machine consults some large table and selects the correct move. Rather, we mean some human or humans have described a method by which such problems can be approached—usually with no guarantee of success. The methods are not necessarily those which humans themselves use, but they tend to resemble our rationalizations of how we solve problems and play games.

Inevitably the question comes up, "Can a computer produce a new result?" Certainly machines have produced results which surprised the humans who planned the program. To take one classic example, the geometry proving routine was asked to prove that in a triangle ABC if AB equals AC then angle ABC equals angle ACB—the well-known theorem that if two sides of a triangle are equal then the base angles are equal. Most humans faced by this problem either bisect the angle A and produce two congruent triangles, or else draw a line from the vertex A to the midpoint of the base BC, again producing two congruent triangles. The machine routine merely observed that triangle ABC was congruent to triangle ACB and hence corresponding angles were equal. The proof is both short and elegant. It was known to Pappus. (See Sir T. L. Heath, Euclid's Elements, Dover Publications, 1956, p. 254.) But if you were to examine the routine the machine used you would probably find that the machine was programmed first to see if it could prove the theorem, and if not then try to add a suitable line and try again. The result is, then, easily explained—the machine did what it was told to do. But then are we so different? Were we not programmed, haphazardly to be sure, to solve problems? What was our high school course in geometry all about except to program us to solve problems?

Thinking is closely associated with learning, and perhaps you feel that the crux of the matter is self-improvement. Consider, then, the fact that I could take two copies of a chess playing program, put them both in the same computer, but with one coefficient in one program changed. I could then let the machine play one formula against the other. Due to the fact that we almost always include in a game playing program a random choice to be used when two or more moves are rated as about equal, the machine will play, say, ten different games. Suppose one formula wins seven out of ten games. I could then have the machine continue to change the coefficient in the favorable direction until no further improvement was observed. In this fashion I could have the machine go through the coefficients one at a time until all of them had been improved. I would also probably try to change combinations of two at a time. Thus the machine from experience would improve the quality of its game. In evolutionary terms, it would be the survival of the fittest program.

We can imagine going further. Suppose I had a collection of small strategies. I could have the machine substitute whole pieces here and there into the program to see if the program were thereby improved. In the biological analogy, these are mutations to be selected for their survival by success or failure in competition with other programs.

Now that you can see the survival of the fittest using both small variations as well as occasional large mutations, are you so sure that a program cannot produce "thinking"—whatever the word means?

Very frequently "thinking" is defined to be what Newton did when he discovered gravitation. By this definition most of us cannot think! As an exercise I suggest you try framing a test that is the least, or close to the least, which you will accept as demonstrating that a machine can think. I have been unable to devise one that would suit myself, let alone others, and have tentatively considered the hypothesis that "thinking" is not measured by what is produced, but rather is a property of the way something is done.

Before leaving the general area of thinking I wish to point out the fact that machines provide a fruitful approach to many questions. Back in 1939 Turing, a British logician, imagined a computing machine, now called a Turing machine, to prove some theorems in abstract logic. The machine was a paper machine in the sense that no actual machine was ever contemplated, rather the conception of a machine was used to aid in the analysis and proof of the results.

Such an approach has been used many times, but the presence of actual machines has greatly stimulated the general field. Thought experiments are now fairly common in some fields. The discussion of chess playing I gave is an example of a thought experiment. To carry out the idea on an actual computer would be very expensive in money and time. I have also on several occasions examined a problem as if I were going to put it on a computer, though I had no intention of actually doing so, and in the examination found the answer I was looking for.

Such an approach requires you to give an absolutely complete description without skipping lightly over some detail that you think is obvious. As an example of oversight, years ago in a calculus class I taught a certain process called "integration by parts," yet when I now try to give a description to a machine I find that there are many details I do not understand well enough to write out a program for the machine. The students had the impression, along with me, that they understood the process, but they too probably cannot give a detailed description to a machine.

At times the machine approach can be very fruitful—and it can certainly pinpoint obscurities very rapidly as well as expose ignorance. I note that increasingly in abstract books authors are appealing to a machine model for clarity of expression. I suggest, therefore, that the habit of asking for a machine description of something will become widespread wherever it is desired to know clearly what one is talking about. Without a detailed description in some language a machine can use there is no conviction that you know what you are talking about; with it there is at least the illusion you do.

The general areas I have covered, engineering, science, language, music and thinking, are only some of the examples I could have used. In each case it seems to me that the computer enables us to pose new types of questions and to find new answers, though, of course, many of the old questions will remain unanswered. It is the new view that I wish to emphasize.

Finally, let me discuss the problems of education which are close to many of us in these days of rapidly expanding knowledge. I hope that by the above examples I have convinced you that at least part of my vision has some reality—that computers are an intellectual phenomenon of our society as well as a material phenomenon, and that they are having, and will continue to have, large effects on various fields of human thought. If you grant this then it is time we began to adjust our educational system to these new ideas. Students now in college can be expected to be still working in the year 2000, yet how is their present education preparing them to live in a world full of machines? Some thought is going into the preparation for the physical consequences of the revolution, but who is trying to teach the new ideas?

I believe, though I have not discussed it here, that the techniques of using computers, like mathematics and language, will be common to many fields and hence provide a unifying thread to our rapidly fragmenting education.

In closing let me repeat my main thesis; I believe that the points I have raised, and many more, will require a gradual reorganization of man's conception of himself and his relation to the rest of the universe. A philosophy for the future man-machine combination is yet to be created, but it is time to start searching for one.

THE SEARCH FOR HADAMARD MATRICES

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Consider the square array in Figure 1.

$$\begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix}.$$

FIG. 1

Any two rows of this array "agree" in half their places, and "disagree" in the other half. The same phenomenon occurs for the columns of the array as well. Any such array of + and − signs, with an equal number of agreements and disagreements between any pair of rows, is called an "Hadamard matrix," after the French mathematician Jacques Hadamard [1]. In matrix notation, these arrays satisfy the relation $AA^T = kI$, where k is the order of the matrices. The problem is to discover all the possible Hadamard matrices, or at least to describe the *sizes* for which such arrays exist.

In 1933, R. E. A. C. Paley [2] described a variety of methods for constructing Hadamard matrices. Beyond the 1×1 case, "+", and the 2×2 case, "$\substack{++\\+-}$", Paley shows that all such arrays must have a number of rows (and of columns) which is a multiple of 4. Up to 200×200, he was able to describe examples of $4t\times 4t$ Hadamard matrices in all but six cases, the exceptional values of $4t$ being 92, 116, 156, 172, 184, 188. Using black and white squares, instead of plus and minus signs, Hadamard matrices of the first eight sizes (1×1, 2×2, 4×4, 8×8, 12×12, 16×16, 20×20, 24×24) are shown in Figure 2.

The six exceptional cases of Paley have never been proved *not* to exist. It was merely that Paley was unable to construct them. In fact, in 1944, John Williamson [3] succeeded in constructing an Hadamard matrix of size 172×172. Moreover, no one has shown that Hadamard matrices *ever* fail to exist for any size $4t\times 4t$, and since 1933 several attempts have been made to prove or disprove the existence of the 92×92 array, as an indication of whether or not to believe the conjecture that an Hadamard matrix of size $4t\times 4t$ exists for all values of $t = 1, 2, 3, 4, 5, \cdots$.

There is a simple method of getting bigger Hadamard matrices from smaller ones. For example, to get an 8×8 array from a 2×2 array and a 4×4 array, one substitutes the 2×2 array instead of each + sign in the 4×4 array, and one substitutes the *negative* of the 2×2 array instead of each − sign in the 4×4 array, as shown in Figure 3.

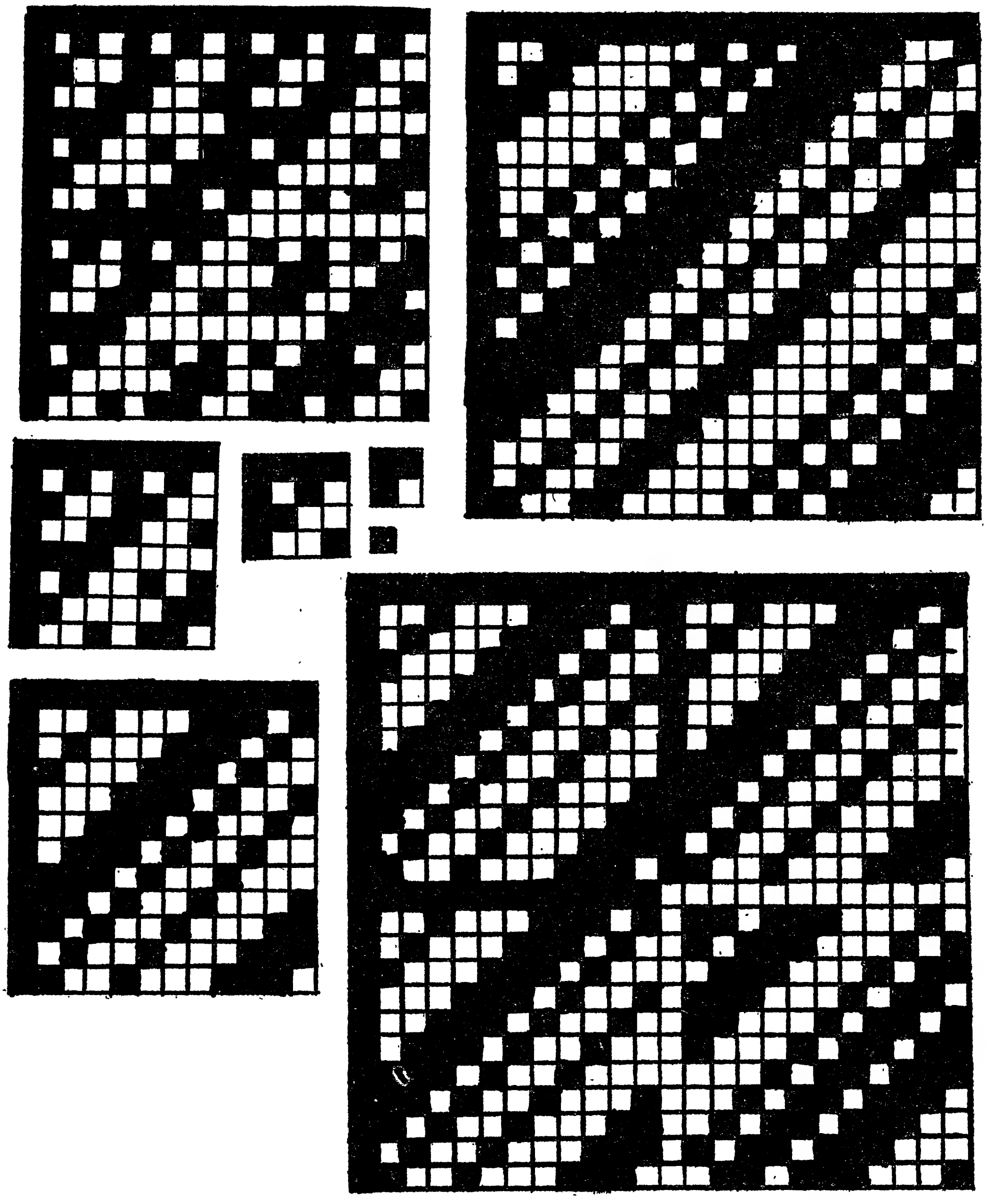

FIG. 2. Hadamard Matrices of orders 1, 2, 4, 8, 12, 16, 20, and 24.

$$
\begin{bmatrix} + & + \\ + & - \end{bmatrix} * \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix} = \begin{bmatrix}
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} \\
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} \\
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} \\
\begin{bmatrix} + & + \\ + & - \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} - & - \\ - & + \end{bmatrix} & \begin{bmatrix} + & + \\ + & - \end{bmatrix}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
+ & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - \\
+ & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + \\
+ & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + \\
+ & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & -
\end{bmatrix}.
$$

FIG. 3

This method of combining two smaller matrices to get a bigger matrix is called the "tensor product" of the two matrices. Letting H_n stand for the Hadamard matrix of order n, the illustrations in Figure 2 satisfy $H_4 = H_2 * H_2$, $H_8 = H_4 * H_2$, $H_{16} = H_8 * H_2$, and $H_{24} = H_{12} * H_2$.

Starting with the 2×2 Hadamard matrix, repeated tensor products give the 4×4 matrix, the 8×8 matrix, the 16×16 matrix, the 32×32 matrix, the 64×64 matrix, the 128×128 matrix, etc. Starting with the 12×12 matrix (which comes from a special construction that works because $4t-1=11$ is a *prime*), we also get a 24×24 matrix, a 48×48 matrix, a 96×96 matrix, a 192×192 matrix, etc., using repeated tensor products with the 2×2 matrix; and also the 144×144 matrix, the 1728×1728 matrix, etc., taking the tensor product of the 12×12 matrix with itself.

A variety of special circumstances were used for the other examples in Paley's catalog of Hadamard matrices. Thus, the 20×20 example arises from $4t-1=19$ being a prime, while the 28×28 example comes from $4t-1=3^3$ being a perfect *power* of a prime.

Several years ago, at the Jet Propulsion Laboratory of Caltech, we became interested in the problem of the optimum codes for communicating through space. The rows of an Hadamard matrix form an ideal set of "code words" for

this purpose, because of the high degree of mutual distinguishability (as many disagreements as agreements) between any two such rows. A row of $+$'s and $-$'s can be regarded as a "square wave" of pulses and spaces, or of $+90°$ and $-90°$ phase shifts, for purposes of radio communication. We resolved to determine once and for all, whether or not the 92×92 Hadamard matrix exists.

There are 8464 entries in a 92×92 matrix, and hence 2^{8464} ways to form a 92×92 matrix of $+$ and $-$ signs. Even if a computer could form and investigate these matrices at the rate of a million a second, the problem would run for too many eons. It was necessary to restrict the problem in some drastic fashion in order to succeed by computer search.

Professor Marshall Hall suggested that we run a program using the same method which enabled Williamson to find a 172×172 matrix. Williamson's idea for finding an Hadamard matrix of size $4t\times 4t$ was to look for a matrix of the form shown in Figure 4,

$$\begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix}$$

FIG. 4

where each of A, B, C, and D is a $t\times t$ matrix. An example of such a structure is shown in Figure 5.

$$\begin{bmatrix}
+ & + & + & + & - & - & + & - & - & + & - & - \\
+ & + & + & - & + & - & - & + & - & - & + & - \\
+ & + & + & - & - & + & - & - & + & - & - & + \\
- & + & + & + & + & + & - & + & + & + & - & - \\
+ & - & + & + & + & + & + & - & + & - & + & - \\
+ & + & - & + & + & + & + & + & - & - & - & + \\
- & + & + & + & - & - & + & + & + & - & + & + \\
+ & - & + & - & + & - & + & + & + & + & - & + \\
+ & + & - & - & - & + & + & + & + & + & + & - \\
- & + & + & - & + & + & + & - & - & + & + & + \\
+ & - & + & + & - & + & - & + & - & + & + & + \\
+ & + & - & + & + & - & - & - & + & + & + & +
\end{bmatrix}$$

FIG. 5

The other conditions which Williamson imposed were that each of A, B, C, and D be a cyclic symmetric matrix (each row being merely a cyclic permutation of the previous row); and that $A^2+B^2+C^2+D^2=4tI$, where A^2, B^2, C^2, D^2 refer to the squares of the respective matrices as usually defined for matrices (*not* in the sense of tensor products), and I is the unit matrix. For purposes of these operations, $+$ is considered to be $+1$, and $-$ is considered to -1. For the case of the 12×12 matrix in Figure 5, Williamson's "four squares" relation takes the form:

$$A^2 + B^2 + C^2 + D^2$$

$$= \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}^2 + \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}^2 + \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}^2 + \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}^2$$

$$= \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 3 & - & - \\ - & 3 & - \\ - & - & 3 \end{bmatrix} + \begin{bmatrix} 3 & - & - \\ - & 3 & - \\ - & - & 3 \end{bmatrix} + \begin{bmatrix} 3 & - & - \\ - & 3 & - \\ - & - & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = 12 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 12I.$$

Williamson's construction [3] is intimately related to Lagrange's theorem that every positive integer is the sum of four squares.

The process of searching through all possible matrices of this special form was programmed by one of us (L. D. Baumert) for the I.B.M. 7090 computer at the Jet Propulsion Laboratory, and on the night of September 27, 1961, an example of the long-awaited 92×92 Hadamard matrix was discovered, in less than an hour of machine time. In fact, there turned out to be *one and only one* example of the Williamson type for size 92. This is shown in Figure 6. (There are many "trivial" rearrangements of A, B, C, D, and of row elements and column elements within A, B, C, D, which lead to superficially different examples. It is by excluding such manipulations that uniqueness is achieved.)

Taking the tensor product of the 92×92 Hadamard matrix with the 2×2 Hadamard matrix yields a 184×184 Hadamard matrix, thereby removing yet another entry from Paley's list of six doubtful cases, and leaving only the sizes 116, 156, and 188.

Elimination of all six cases from Paley's list will justify the "engineering conclusion" that Hadamard matrices "always exist" (that is, for size $4t\times 4t$). However, the mathematical goal of *proving* this conclusion will still be far from attained. It does, however, seem quite likely that not merely Hadamard ma-

trices, but Hadamard matrices of the Williamson type, "always exist," and this assertion, stating as it does some specific structural properties for Hadamard matrices, may be easier to prove than the *unrestricted* existence theorem. In any case, there is still much work to be done.

FIG. 6

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A SPECIAL HILL'S EQUATION WITH DISCONTINUOUS COEFFICIENTS*

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1. Introduction. A differential equation of the form

$$y'' + p(t)y = 0, \tag{1}$$

where $p(t)$ is a periodic function of t is known as an equation of the Hill type. In many physical problems the function $p(t)$ involves an eigenvalue parameter in the form

$$p(t) = \lambda Q(t)$$

or

$$p(t) = \lambda + Q(t).$$

The permissible values are then determined by the requirement that a solution of (1) also be periodic. If $2L$ denotes the period of $Q(t)$ then the λ are determined by the conditions

$$y(t + 2L) = y(t)$$

in order for y to have period $2L$ or by

$$y(t + 2L) = - y(t)$$

in order for y to have a minimal period of $4L$. Evidently in the latter case

$$y(t + 4L) = - y(t + 2L) = y(t).$$

The theory of Hill's equation is far from trivial and somewhat inaccessible on a purely elementary basis. The only equation, which can be investigated by the use of elementary functions is the Meissner equation [1]. In this equation the function $Q(t)$ is taken to be a piece-wise constant function. This type of equation was also used by Kronig and Penney [2] and Sommerfeld [3] as a model for a one-dimensional crystal.

The purpose of this article is not merely to review the results obtainable from Meissner's equation but also to demonstrate some hitherto unobserved properties of this equation. It will be shown that the characteristic values can be arranged in a sequence which holds for the characteristic values of a general class of Hill's equations. This was first proved by Lyapunov [4] and later independently by Haupt [5]. A general question regarding the coexistence of periodic solutions can also be answered for this equation. In general, when λ is an eigenvalue only one of the two independent solutions of the differential equation is periodic, but if λ is a double root of the characteristic equation two and therefore all solutions will be periodic. In this case two periodic solutions are

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said to coexist. A necessary and sufficient condition for such coexistence for Meissner's equation will be derived. Furthermore some approximations for characteristic values of large index will be derived.

When λ is not a characteristic value either both solutions of the differential equation are bounded or one increases beyond all bounds as t tends to infinity. In the former case the solutions are said to be stable and in the latter unstable. It is of interest to know which condition prevails for any one value of λ. A stability chart will be constructed, for Meissner's equation, which clearly indicates which are regions of stability and which are regions of instability.

2. The characteristic equation. We now turn our attention to the equation

$$y'' + \lambda^2 Q(t)y = 0, \tag{2}$$

where

$$\begin{aligned} Q(t) &= 1, & |t| &\leqq 1 \\ &= a^2, & 1 < |t| &\leqq L \end{aligned}$$

and outside the interval $(-L, L)$ $Q(t)$ is to be continued as a periodic function of period $2L$. In any interval in which $Q(t)$ is constant the equation can be solved in terms of trigonometric functions and continued outside the interval by making y and y' continuous at the endpoint. Let y_1 and y_2 denote the solutions satisfying the initial conditions

$$\begin{aligned} y_1(0) &= 1 & y_2(0) &= 0 \\ y_1'(0) &= 0 & y_2'(0) &= 1; \end{aligned}$$

then any other solution can be expressed in the form

$$y = c_1y_1 + c_2y_2,$$

where c_1 and c_2 are suitably chosen constants.

Next we pose the question whether we can select c_1 and c_2 so that

$$\begin{aligned} y(t + 2L) &= \rho y(t) \\ y'(t + 2L) &= \rho y'(t) \end{aligned}$$

for some constant ρ. We thus find that

$$\begin{aligned} c_1y_1(t + 2L) + c_2y_2(t + 2L) &= \rho c_1y_1(t) + \rho c_2y_2(t), \\ c_1y_1'(t + 2L) + c_2y_2'(t + 2L) &= \rho c_1y_1'(t) + \rho c_2y_2'(t). \end{aligned}$$

This is a system of homogeneous equations for c_1 and c_2 and can have a solution only if its determinant vanishes. In particular for $t=0$, the determinantal equation becomes

$$\begin{vmatrix} y_1(2L) - \rho & y_2(2L) \\ y_1'(2L) & y_2'(2L) - \rho \end{vmatrix} = 0.$$

Since the expression

$$y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

is the Wronskian of the differential equation we find that it is a constant and from the initial conditions we find that its value is unity. Thus the determinantal equation for ρ becomes

$$\rho^2 - (y_1(2L) + y_2'(2L))\rho + 1 = 0.$$

This equation is the characteristic equation. If this equation has two distinct roots we can find two distinct solutions u and v with the properties

$$u(t + 2L) = \rho_1 u(t),$$
$$v(t + 2L) = \rho_2 v(t).$$

If $|y_1(2L)+y_2'(2L)| < 2$, we see that ρ_1 and ρ_2 are complex conjugates of unit magnitude and then all solutions are bounded. If

$$|y_1(2L) + y_2'(2L)| > 2,$$

ρ_1 and ρ_2 are real and since $\rho_1\rho_2 = 1$ at least one, say ρ_1, is greater than unity in absolute value. Then it follows that

$$u(t + 2nL) = \rho_1^n u(t)$$

and we see that u must grow beyond all bounds as t approaches infinity.

When $y_1(2L)+y_2'(2L) = 2$, $\rho = 1$ and the corresponding solution has period $2L$ since in that case

$$u(t + 2L) = u(t).$$

Since $\rho = 1$ is a double root of the characteristic equation we may be able to determine only one set of coefficients c_1 and c_2 so that

$$y(t + 2L) = y(t).$$

The other linearly independent solution need not be periodic.

When $y_1(2L)+y_2'(2L) = -2$, $\rho = -1$ is a double root of the characteristic equation and we can find a solution so that

$$y(t + 2L) = - y(t),$$

in which case

$$y(t + 4L) = y(t).$$

Here also only one solution need be periodic.

3. The characteristic value. A straightforward calculation shows that for equation (2)

$$y_1(2L) + y_2'(2L) = 2\cos[2\lambda a(L-1)]\cos 2\lambda - \left(a + \frac{1}{a}\right)\sin 2\lambda \sin[2\lambda a(L-1)].$$

We are now interested in gathering as much information as possible about those values of λ for which

$$y_1(2L) + y_2'(2L) = \pm 2.$$

For these values of λ the differential equation will have solutions of period $2L$ or $4L$. One can easily verify that

$$\begin{aligned} y_1(2L) + y_2'(2L) - 2 &= -4\cos^2\lambda\cos^2\lambda a(L-1)[\tan\lambda + a\tan\lambda a(L-1)] \\ &\quad \cdot\left[\tan\lambda + \frac{1}{a}\tan\lambda a(L-1)\right] \\ y_1(2L) + y_2'(2L) + 2 &= 4\cos^2\lambda\sin^{-2}\lambda a(L-1)[\tan\lambda - a\cot\lambda a(L-1)] \\ &\quad \cdot\left[\tan\lambda - \frac{1}{a}\cot\lambda a(L-1)\right]. \end{aligned}$$

The above equations show that the characteristic values are furnished by the following four equations:

1) $$\lambda a(L-1) = -\tan^{-1}\frac{1}{a}\tan\lambda$$

2) $$\lambda a(L-1) = -\tan^{-1} a\tan\lambda$$

3) $$\lambda a(L-1) = \cot^{-1}\frac{1}{a}\tan\lambda$$

4) $$\lambda a(L-1) = \cot^{-1} a\tan\lambda.$$

Cases 1) and 2) yield characteristic values corresponding to solutions of period $2L$, and cases 3) and 4) yield solutions of period $4L$.

We can easily show that when periodic solutions exist, one can find periodic solutions which are either even or odd. Since $Q(t)$ is an even function the equation is invariant if t is replaced by $-t$. Therefore if $y(t)$ is a solution so is $y(-t)$ and, unless $y(t)$ is even or odd, these are linearly independent solutions. Thus if $y(t)$ is a periodic solution it is either even or odd, or else the solutions

$$y(t) + y(-t), \qquad y(t) - y(-t)$$

are also periodic and even and odd respectively. One can easily check that cases 1) to 4) correspond to solutions of the following character.

1) even of period $2L$ 3) even of period $4L$
2) odd of period $2L$ 4) odd of period $4L$.

The following method will be used to obtain the characteristic values. We treat case 1), where

$$\lambda a(L-1) = -\tan^{-1}\frac{1}{a}\tan\lambda,$$

as typical.

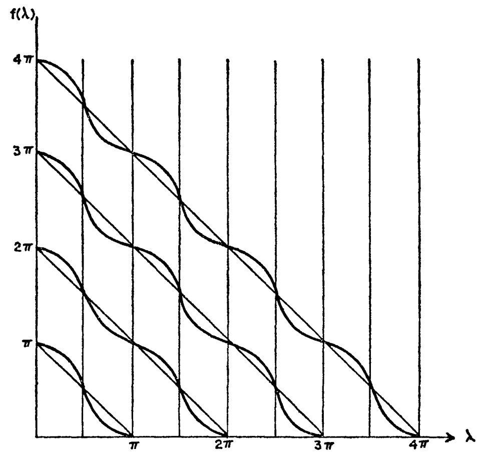

FIG. 1

In the evaluation of the inverse tangent we must take the multivalued character of the function into account. We will graph both the right side and the left side of the equation and wherever the curves intersect, we evidently have a characteristic value. Figure 1 shows the function on the right as a function of λ. Figure 1 was drawn for $a>1$. For $a=1$ the wavy lines become straight and have slope -1. If we superimpose on this figure the other three cases we obtain Figure 2.

Figure 2 was drawn for $a>1$. For $a=1$ the curves of the type 1) and 2) become identical straight lines of slope -1, and similarly for the curves of type 3) and 4). Wherever the line $\lambda a(L-1)=f(\lambda)$ intersects these curves, we obtain characteristic values. If $\{\lambda_i\}$ denote the characteristic λ's corresponding to solutions of period $2L$ and $\{\lambda_i'\}$ the ones corresponding to solutions of period $4L$, we find that

$$0=\lambda_0<\lambda_1'\leqq\lambda_2'<\lambda_1\leqq\lambda_2<\lambda_3'\leqq\lambda_4'<\lambda_3\leqq\lambda_4\cdots.$$

That the characteristic values of a wide class of Hill's equations are arranged in such a sequence is well known ([4], [5]), but is in this case selfevident from Fig. 2. This is the only known case in which this fact can be demonstrated explicitly.

One can derive more explicit information regarding the characteristic values. For the case $a=1$ equation (2) takes the form

$$y'' + \lambda^2 y = 0.$$

In this case we find that

$$y_1(2L) + y_2'(2L) = 2 \cos 2\lambda L.$$

Then the $\{\lambda_i\}$ are roots of the equation

$$2 \cos 2\lambda L = 2$$

so that $\lambda L = n\pi$. These are double roots of the equation, since the derivative of $\cos 2\lambda L$ vanishes for these values. Thus we find $\lambda_0 = 0$, $\lambda_{2n-1} = \lambda_{2n}$, for $n = 1, 2, \cdots$.

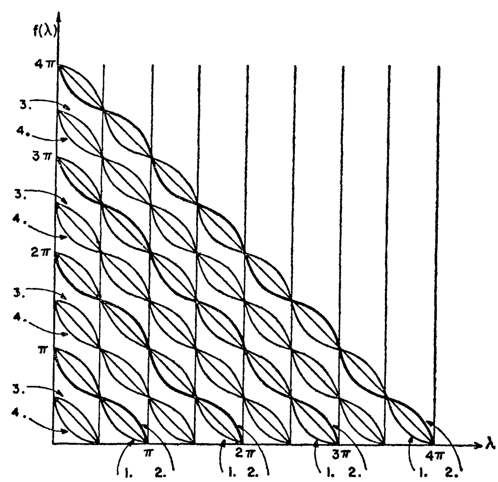

FIG. 2

Similarly we have $2 \cos 2\lambda L = -2$ so that $\lambda L = (2n+1)\pi/2$ and again these are double roots so that

$$\lambda'_{2n-1} = \lambda'_{2n}, \qquad n = 1, 2, \cdots.$$

In general, when $a \neq 1$ we can have multiple roots only when $a(L-1)$ is rational. In this case the line of slope $a(L-1)$ will pass through lattice points whose coordinates are integral multiples of $\pi/2$ where the families of curves 1) and 2) or 3) and 4) intersect. Suppose $a(L-1) = p/q$, where p and q are relatively prime. Then we will have multiple roots at the lattice points $(kp\pi/2, kq\pi/2)$ such that

$$\lambda_{2n-1} = \lambda_{2n} = kq \frac{\pi}{2} \qquad \text{for } k(p + q) = 2n$$

and

$$\lambda'_{2n+1} = \lambda'_{2n+2} = kq\frac{\pi}{2} \qquad \text{for } k(p+q) = 2n+1.$$

Evidently in these cases one needs to compute only the first $2(p+q)$ characteristic values of type λ or λ' and all others of that type must be congruent to these modulo $q\pi$. An inspection of Figure 2 shows this.

But if one knows only the first $p+q$ characteristic values the next $p+q$ can be determined from

$$\lambda'_n + \lambda'_{2(p+q)-n+1} = q\pi, \qquad n = 1, 2, \cdots, p+q$$

$$\lambda_n + \lambda_{2(p+q)-n-1} = q\pi, \qquad n = 1, 2, \cdots, p+q-2$$

which are also evident from an inspection of Figure 2.

By approximating all these curves by straight lines of slope -1, one can easily obtain the following estimates for the characteristic values.

$$\lambda_{2n-1} = \left[\frac{2n}{1+a(L-1)}\right]\frac{\pi}{2} + \theta\frac{\pi}{2}, \qquad 0 \leqq \theta < 1,$$

$$\lambda_{2n} = \left[\frac{2n}{1+a(L-1)}\right]\frac{\pi}{2} + \theta\frac{\pi}{2},$$

$$\lambda'_{2n-1} = \left[\frac{2n-1}{1+a(L+1)}\right]\frac{\pi}{2} + \theta\frac{\pi}{2},$$

$$\lambda'_{2n} = \left[\frac{2n-1}{1+a(L-1)}\right]\frac{\pi}{2} + \theta\frac{\pi}{2}.$$

(Here $[x]$ represents the greatest integer less than or equal to x.)

To obtain better numerical estimates in particular cases we can use Newton's method.

4. The stability chart. We can make a rough stability chart from Fig. 2. That is, we will make a graph showing the relationship between the characteristic λ's and a. These curves subdivide the first quadrant into regions of stability and instability. That is, for any λ and a from a stable region, all solutions of the differential equation are stable. Outside the stability regions there is only one bounded solution for $t \geqq 0$.

To obtain Figure 3 the value of L is held fixed and the value of a is allowed to vary. One proceeds as follows. For any fixed value of a the line of slope $a(L-1)$ will intersect each of the curves of Figure 2 exactly once. In this fashion an infinity of characteristic values is obtained. One then observes how each of these values of λ varies as a is varied and these functions are plotted on

Figure 3. For any values of λ and a in the shaded region

$$|y_1(2L) + y_2'(2L)| < 2,$$

so that all solutions are bounded. Therefore these regions represent regions of stability. Outside these regions

$$|y_1(2L) + y_2'(2L)| > 2,$$

so that the solutions are not bounded and these are regions of instability.The borderlines are defined by

$$|y_1(2L) + y_2'(2L)| = 2,$$

so that for any such pair of values (λ, a) at least one solution of the differential equation has a period $2L$ or $4L$. At any point of intersection two such periodic solutions will coexist. From Figure 2 it is evident that the first pair of λ curves will have exactly one point of intersection, the second pair two such points and so on. These points of intersection correspond to lattice points in Figure 2 and are therefore known exactly.

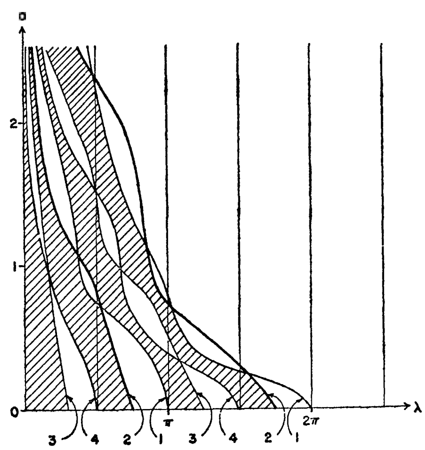

FIG. 3

Another Hill's equation, which arises very frequently in mathematical physics is the Mathieu equation

$$y'' + [\lambda - 2h^2 \cos 2t]y = 0.$$

This equation cannot be solved in terms of elementary functions, but many of the properties displayed by the particular example treated here also hold for the Mathieu equation. But there are some differences. One of the most important is that coexistence of periodic solutions is impossible for the Mathieu equation. That is the characteristic equation has no double roots and the different curves in the stability chart never intersect.

At present very little is known regarding the question of coexistence of periodic solutions. Some recent results may be found in [6]. The example treated here seems to be the only one known, where the double characteristic values are known precisely.

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THE FINITE DIFFERENCE EXTENSION OF ROLLE'S THEOREM

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By a lemma of S. Bernstein [1] extending Rolle's theorem to the calculus of finite differences, if f is a continuous real function on the closed interval $[a, b]$, and

$$f(a) = f(b) = 0, \tag{1}$$

then the difference $\Delta_h f$, defined by

$$\Delta_h f(x) = f(x + h) - f(x),$$

has a zero in the interval provided h is sufficiently small. How small must h be to justify this conclusion? To phrase the question geometrically, a function f defined on $[a, b]$ is said to have a *horizontal chord of length h* if $f(x+h) = f(x)$ for some x such that $a \leqq x < x+h \leqq b$. We are then asking for a bound $H > 0$ such that any function f continuous on $[a, b]$ and satisfying (1) has horizontal chords of every positive length $h \leqq H$. Such a bound cannot apply uniformly to every such function. This follows from the universal chord theorem of P. Lévy [2] (see discussion in [3]), which states that the necessary and sufficient condition that every continuous function f on $[a, b]$ satisfying (1) have a horizontal chord of a given length h is that $h = (b-a)/m$ for some positive integer m. The bound H must, therefore, depend upon some property of the function f. The purpose of this note is to derive the best possible bound H depending only upon the number of changes in sign or the number of zeros of f in the open interval (a, b).

THEOREM. *If a real function f is continuous on a closed interval $[a, b]$ and changes sign exactly n times in (a, b), and if*

$$f(a) = f(b) = 0, \tag{2}$$

then for any positive h such that

$$h \leqq H_n = \frac{b-a}{[(n+3)/2]} \tag{3}$$

there is a number r such that

$$a \leqq r \leqq b - h \quad \textit{and} \quad \Delta_h f(r) = 0. \tag{4}$$

First we establish a lemma.

LEMMA. *Let f satisfy the conditions of the theorem, and let $0<h\leqq b-a$. If*

$$n \leqq 2\left[\frac{b-a}{h}\right] - 2, \tag{5}$$

then there is a number r satisfying (4).

Proof. Let

$$m = \left[\frac{b-a}{h}\right]. \tag{6}$$

If $(b-a)/h=m$, (4) follows at once by the universal chord theorem; so we may suppose that $m<(b-a)/h<m+1$. Then, $a<b-mh<a+h$; and, if we define

$$a_i = a + ih, \qquad b_i = b - ih \qquad (i = 0, 1, \cdots, m), \tag{7}$$

we have $a_i<b_{m-i}<a_{i+1}$, so that these points are interspersed as follows:

$$a = a_0 < b_m < a_1 < b_{m-1} < \cdots < a_i < b_{m-i} < a_{i+1} < \cdots < b_1 < a_m < b_0 = b. \tag{8}$$

Now suppose that

$$\Delta_h f(x) > 0 \qquad\qquad a \leqq x \leqq b - h. \tag{9}$$

Then, $f(a_i)-f(a_{i-1})=\Delta_h f(a_{i-1})>0$, and $f(b_{i-1})-f(b_i)=\Delta_h f(b_i)>0$; so that $f(a_i)>f(a_{i-1})>f(a_{i-2})> \cdots >f(a_0)=f(a)=0$ by (2), and $f(b_i)<f(b_{i-1})< \cdots <f(b_0)=f(b)=0$ for $i=1, 2, \cdots, m$. Accordingly, as we traverse the sequence of points (8) in the order indicated, f changes sign between every pair of consecutive points beginning with b_m and ending with a_m, a total of $2m-1$ changes in sign. Thus, $n\geqq 2m-1=2[(b-a)/h]-1$, contrary to (5). The assumption (9) is thereby disproved, and we conclude that there is a number x_1, $a\leqq x_1\leqq b-h$, such that $\Delta_h f(x_1)\leqq 0$. Similarly, there is a number x_2, $a\leqq x_2\leqq b-h$, such that $\Delta_h f(x_2)\geqq 0$. Since f is continuous in $[a, b]$, $\Delta_h f$ is continuous in $[a, b-h]$. Consequently, there is a number r in the closed interval bounded by x_1 and x_2 and hence in $[a, b-h]$ such that $\Delta_h f(r)=0$.

Proof of Theorem. For n odd, by (3), $h \leqq 2(b-a)/(n+3)$, or $n \leqq \{2(b-a)/h\} - 3$. Then, since n is integral, $n \leqq [2(b-a)/h] - 3 \leqq 2[(b-a)/h] - 2$, so that (5) holds for n odd. For n even, (3) becomes $h \leqq 2(b-a)/(n+2)$. Hence, $n \leqq [2(b-a)/h] - 2 \leqq 2[(b-a)/h] - 1$. But n is an even integer; so (5) holds in this case also. The conclusion of the theorem now follows directly from the lemma.

The bound H_n in (3) cannot be improved. For any integer $n \geqq 0$ and any $H > H_n$ there is a positive number $h < H$ and a function f satisfying the conditions of the theorem but such that

$$\Delta_h f(x) \neq 0 \qquad \text{for } a \leqq x \leqq b - h. \tag{10}$$

In fact we may choose h as any number in the interval,

$$\frac{b-a}{[(n+3)/2]} < h < \min\left(\frac{b-a}{[(n+1)/2]},\ H\right), \tag{11}$$

and then proceed to construct a function f with the required properties in the following manner. Define m by (6) and a_i, b_i by (7); also define

$$w = \frac{b-a}{h} - m, \quad b_0' = b - \frac{wh}{m+1}, \quad b_i' = b_0' - ih \quad (i = 1, 2, \cdots, m). \tag{12}$$

We note that from the definitions (12) and (7) and with the aid of (8) it follows that

$$b_0' - a_m = b - \frac{b-a-mh}{m+1} - a_m = (b - a_m)\left(1 - \frac{1}{m+1}\right) > 0$$

and $b - b_0' = wh/(m+1) > 0$ so that $a_m < b_0' < b$ and hence $a_i = a_m - (m-i)h < b_0' - (m-i)h = b'_{m-i} < b - (m-i)h = b_{m-i} < a_{i+1}$. Thus, the abcissas b_i' are interspersed with the a_i in the order,

$$a < b_m' < a_1 < b'_{m-1} < \cdots < a_i < b'_{m-i} < a_{i+1} < \cdots < a_m < b_0' < b. \tag{13}$$

Let P_i, Q_i, Q_i' be points with rectangular coordinates

$$P_i(a_i, i), \quad Q_i(b_{m-i}, i-m), \quad Q_i'\left(b'_{m-i}, i - m - \frac{w}{1-w}\right) \quad (i = 0, 1, \cdots, m). \tag{14}$$

For each integer $n \geqq 0$ and each h in the interval (11) we define $F_{n,h}$ as the function having as its Cartesian graph the polygonal arc,

$$P_0Q_0P_1Q_1 \cdots P_iQ_iP_{i+1} \cdots P_mQ_m \qquad \text{when } n \text{ is odd,}$$

$$P_0Q_0'P_1Q_1' \cdots P_iQ_i'P_{i+1}Q_{i+1}' \cdots P_mQ_m'Q_m \qquad \text{when } n \text{ is even.}$$

By (8) and (13) these graphs determine a unique value $F_{n,h}(x)$ for each x in $[a, b]$. We now show that $F_{n,h}$ meets the requirements for the function f. Clearly, $F_{n,h}$ is continuous on $[a, b]$ and $F_{n,h}(a) = F_{n,h}(b) = 0$. To determine the number

of times $F_{n,h}$ changes sign note that each of the points $P_1, P_2, \cdots, P_m$ lies above the X-axis while each of the points $Q_0, Q_1, \cdots, Q_{m-1}$ and $Q_0', Q_1', \cdots, Q_m'$ lies below the X-axis. Thus, each of the segments, P_iQ_i $(i=1, 2, \cdots, m-1)$, P_iQ_i' $(i=1, 2, \cdots, m)$, Q_iP_{i+1} and $Q_i'P_{i+1}$ $(i=0, 1, \cdots, m-1)$ crosses the X-axis. Consequently, $F_{n,h}$ changes sign exactly $(m-1)+m=2m-1$ times for n odd and $m+m=2m$ times for n even. But by (11) $[(n+1)/2]<(b-a)/h<[(n+3)/2]$, and hence by (6) $m=[(n+1)/2]$. Therefore, whether n is odd or even, $F_{n,h}$ changes sign exactly n times in (a, b). It remains to show that (10) holds for $f=F_{n,h}$. Direct computation from (14) shows that all the segments P_iQ_i are parallel with slope $k=-m/(a_i-b_{m-i})=-m/wh$ by (7) and (12). Let $a_i\leqq x\leqq b_{m-i}$. Then for n odd, $F_{n,h}(x)=F_{n,h}(a_i)+k(x-a_i)$; while $a_{i+1}\leqq x+h \leqq b_{m-i-1}$ and $F_{n,h}(x+h) = F_{n,h}(a_{i+1}) + k(x+h-a_{i+1}) = 1+i+k(x-a_i) = 1+F_{n,h}(x)$ so that

$$\Delta_h F_{n,h}(x) = 1 \tag{15}$$

for n odd and x in any interval $[a_i, b_{m-i}]$ $(i=0, 1, \cdots, m-1)$. All segments Q_iP_{i+1} are also parallel as are the segments P_iQ_i'. The segments $Q_i'P_{i+1}$ are likewise all parallel to each other and to $Q_m'Q_m$. We can therefore show in the same way that (15) holds for all x in $[a, b-h]$ and for n even as well as odd. Thus, $F_{n,h}$ is a function with the required properties.

To guarantee the existence of a zero of $\Delta_h f$ in the open interval (a, b) a slightly more restrictive bound on h is required.

COROLLARY 1. *Let f satisfy the conditions of the theorem. Then for any positive h such that*

$$h \leqq H_n^* = \frac{b-a}{[(n+5)/2]} \tag{16}$$

there is a number r such that

$$a < r \leqq b - h \quad \text{and} \quad \Delta_h f(r) = 0. \tag{17}$$

Proof. If $\Delta_h f(a) \neq 0$, the conclusion is an immediate consequence of the theorem. If $\Delta_h f(a)=0$, we have $f(a+h)=f(a)=0$. Let $g=[(n+5)/2]$, and let n^* be the number of times f changes sign in the interval $(a+h, b)$. Then,

$$h + \frac{h}{g-1} = \frac{gh}{g-1} \leqq \frac{b-a}{g-1}$$

by (16), and hence

$$h \leqq \frac{b-a-h}{g-1} = \frac{b-(a+h)}{[(n+3)/2]} \leqq \frac{b-(a+h)}{[(n^*+3)/2]}\,.$$

We can now apply the theorem to f on the interval $[a+h, b]$ to conclude that there is a number r such that $a<a+h\leqq r\leqq b-h$ and $\Delta_h f(r)=0$.

The bound H_n^* in (16) is also the best possible. This is proved by modifying the construction of $F_{n,h}$ by replacing the points P_i in (14) with points P_i^* having Cartesian coordinates $P_0^*(a, 0)$; $P_i^*(a_i, i-1)$ $(i=1, 2, \cdots, m)$. The resulting function $F_{n,h}^*$ is continuous on $[a, b]$, vanishes at the endpoints, and, if $[(n+5)/2] < h < [(n+3)/2]$, changes sign exactly n times in (a, b). But $\Delta_h F_{n,h}^*(x)=1$, $b_m \leqq x \leqq b-h$, or $b'_m \leqq x \leqq b-h$, accordingly as n is odd or even respectively; while $\Delta_h F_{n,h}^*(x)$ increases strictly monotonically from zero to one in the interval $a \leqq x \leqq b_m$ (n odd) or $a \leqq x \leqq b'_m$ (n even). In any case, $\Delta_h F_{n,h}^*(x) \neq 0$ for all x in the interval $a < x \leqq b-h$.

COROLLARY 2. *If f is continuous on $[a, b]$ and $f(a)=f(b)=c$, and if the Cartesian graph of f crosses the line $y=c$ exactly n times in (a, b), then for any positive h satisfying* (3) *there is a number r satisfying* (4), *and for any positive h satisfying* (16) *there is a number r satisfying* (17).

The proof follows at once by considering $\phi(x)=f(x)-c$.

We close with two remarks. If we drop the hypothesis that f is continuous throughout the foregoing and instead assume merely that it is defined at every point of $[a, b]$, we can at least conclude that $\Delta_h f$ either vanishes or changes sign in $[a, b-h]$ or $(a, b-h]$, as the case may be. On the other hand, if we retain the assumption of continuity but alter the hypotheses of the lemma, theorem, and corollaries to specify that f has exactly n zeros in (a, b) in place of n changes in sign, the conclusions continue to hold a fortiori. Moreover, the same function $F_{n,h}$ can be used to show that the bound H_n in (3) is again the best possible. (Of course, the original criteria in terms of sign changes are preferable, since the resulting bound on h is at least as large.)

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ON THE POLYNOMIAL OF A GRAPH

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1. Introduction. Several recent investigations in graph theory and studies of "association schemes" arising in the design of experiments (all references other than [1], [5], and [10]) have been variations on the following theme. For each pair of (not necessarily distinct) vertices i, j of a graph G, let $p_t(i, j)$ be the number of different paths in G from i to j of length t (we allow paths to be re-

entrant and cross themselves without restriction, and we also stipulate that $p_0(i, j) = \delta_{ij}$). The question pursued in these investigations is: given a positive integer n, and rational coefficients $a_0, a_1, \cdots, a_k$, to find all graphs G with n vertices such that

$$a_0p_0(i, j) + a_1p_1(i, j) + \cdots + a_kp_k(i, j) = 1 \tag{1.1}$$

for all pairs of vertices i, j of G.

This paper:

(i) points out that a graph satisfies (1.1) for some set of coefficients if and only if it is regular (the number of edges meeting each vertex is a constant) and connected, and suggests some appropriate terminology for considering (1.1);

(ii) characterizes bicolored (every cycle is of even length) regular and connected graphs by properties of the coefficients;

(iii) applies this characterization to the study of the graphs formed by the vertices and edges of the m-dimensional cube, for $m \leqq 4$.

The intent of (ii) and (iii) is to exhibit simple instances of the methods used to investigate particular instances of (1.1), namely an interplay among properties of matrices, polynomials, and graphs.

2. Notation and Main Theorem. Let G be an unoriented* graph with vertices $1, \cdots, n$ with no edges from i to i, and at most one edge joining i and $j (i \neq j)$. Let A be the square matrix of order n, called the adjacency matrix of G, given by

$$A = (a_{ij}) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are joined by an edge} \\ 0 & \text{otherwise.} \end{cases}$$

Let u be the vector of order n, every entry of which is unity, and let J be the square matrix of order n, every entry of which is unity. Let d_i (called the degree of i) be the number of edges meeting vertex i. G is said to be regular (of degree d) if $d = d_i$ for each i.

THEOREM 1. *There exists a polynomial $P(x)$ such that*

$$J = P(A) \tag{2.1}$$

if and only if G is regular and connected.

Note the equivalence of (2.1) with (1.1). E. C. Dade and K. Goldberg (and perhaps others) have known Theorem 1 for some time, but it does not seem to be in the literature. It would be worthwhile to find a proof which does not use the concept of characteristic root.

Proof. Assume (2.1). Then A commutes with J, hence $d_i = (i, j)$th entry of $AJ = (i, j)$th entry of $JA = d_j$, so G is regular. Further, if i and j are any vertices of G, there is, for some t, a nonzero number as the (i, j)th entry of A^t; otherwise, no linear combination of the powers of A could have 1 as the (i, j)th entry, and

* This hypothesis is not needed, but we use it to keep the discussion simple.

(2.1) would be false. Thus, for some t, there is at least one path of length t from i to j. But this means G is connected.

Conversely, assume G regular (of degree d) and connected. As we saw in the proof of necessity, because G is regular, A commutes with J. Thus, since A and J are symmetric commuting matrices, there exists an orthogonal matrix U such that

$$J = UJ_0U^T, \qquad A = UA_0U^T, \tag{2.2}$$

where J_0 is a diagonal matrix whose diagonal entries are the eigenvalues of J, namely $(n, 0, 0, \cdots, 0)$, and A_0 is a diagonal matrix whose diagonal entries are the eigenvalues of A, namely $(\alpha_1, \cdots, \alpha_n)$. Now u is an eigenvector of both A and J, with d and n the corresponding eigenvalues, a consequence of the fact that G is regular of degree d. It is a classic result from the theory of matrices with nonnegative entries that, because G is connected, d is an eigenvalue of A of multiplicity 1 (also, an eigenvalue of largest absolute value; see [10]). Let $d, \beta_1, \cdots, \beta_k$ be the distinct eigenvalues of A, and let

$$P(x) = \frac{n \prod_{i=1}^{k} (x - \beta_i)}{\prod_{i=1}^{k} (d - \beta_i)}. \tag{2.3}$$

Then $P(A_0) = J_0$, so (2.2) implies (2.1).

Let us call (2.3) the polynomial of the graph G, and say that the polynomial and graph belong to each other. It is clear that (2.3) is the polynomial of smallest degree for which (2.1) holds. Further, the distinct eigenvalues of A, other than d, are the roots of $P(x)$.

3. A lemma on bicolored graphs. A graph G is bicolored [5] if and only if it is possible so to number its nodes that the adjacency matrix is

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}, \tag{3.1}$$

where the 0's are square blocks along the diagonal. Of course, if G is regular, then the squares are of the same order, and B is a square matrix also of that order. To see this, it is sufficient to show that B is square. Suppose G regular, of degree d, and that B has p rows and q columns, then the number of 1's in B is pd (adding by rows) or qd (adding by columns). Hence, $p = q$.

LEMMA 1. *If G is a regular connected graph of degree d, and $P(x)$ is the polynomial of G, then G is a bicolored graph if and only if $P(-d) = 0$.*

Proof. It is clear that if A has the form (3.1), then the vector $(u; -u)$ is an eigenvector of A, corresponding to the eigenvalue $-d$. Conversely, suppose that $-d$ is an eigenvalue of A, where A is the adjacency matrix of a regular connected

graph of degree d, and that $v=(v_1, \cdots, v_n)$ is a corresponding eigenvector. There is no loss of generality in assuming that the largest absolute value of the components of v is 1, and that $v_{i_0}=1$. Since $\sum a_{i_0 j} v_j = -d$, it follows that $v_{i_1}=-1$ for all vertices i_1 joined to i_0 by an edge. Similarly, if i_2 is a vertex joined to an i_1 by an edge, then $v_{i_2}=+1$, and so on. Because G is connected, every coordinate of v is ± 1, and every edge of G joins two vertices such that the corresponding coordinates of v are different. But this is equivalent to saying that G is bicolored.

It is worth remarking, that if G is bicolored, then $(x-d)P(x)$ is an even function. This follows from Wielandt's observation [1] that the eigenvalues of A in (3.1) are the nonnegative square roots of the eigenvalues of BB' and their negatives.

4. The polynomials belonging to the graphs of low-dimensional cubes. Let Q_m be the graph of 2^m nodes and $m2^{m-1}$ arcs, $m \leqq 4$, formed by the vertices and edges of the m-dimensional cube. The adjacency matrix of Q_m is of the form (3.1), with

$$(4.1)\qquad \begin{cases} \text{(a)}\quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \text{in case } m = 2 \\[2ex] \text{(b)}\quad B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} & \text{in case } m = 3 \\[2ex] \text{(c)}\quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} & \text{in case } m = 4. \end{cases}$$

It is also easy to calculate that the polynomial $P(x)$ of smallest degree such that $J=P(A)$, i.e., the polynomial of the graph, is

$$(4.2)\qquad \begin{cases} \text{(a)}\quad P_2(x) = \dfrac{x^2}{2} + x & \text{in case } m = 2 \\[2ex] \text{(b)}\quad P_3(x) = \dfrac{x^3}{6} + \dfrac{x^2}{2} - \dfrac{x}{6} - \dfrac{1}{2} & \text{in case } m = 3 \\[2ex] \text{(c)}\quad P_4(x) = \dfrac{x^4}{24} + \dfrac{x^3}{6} - \dfrac{x^2}{6} - \dfrac{2x}{3} & \text{in case } m = 4. \end{cases}$$

The question we now raise for consideration is: for $m=2$, 3, 4 respectively, do 2^m (the number of vertices of Q_m) and $P_m(x)$, as given by (4.2), characterize Q_m? Is there possibly another graph with 2^m vertices belonging to $P_m(x)$?

5. The graphs with 2^m vertices belonging to $P_m(x)(m=2, 3, 4)$.

THEOREM 2. *For $m=2$ or 3, Q_m is the only graph with 2^m vertices belonging to $P_m(x)$. But Q_4 and exactly one other graph of* 16 *vertices (given by* (5.1) *and* (5.5) *below) belong to $P_4(x)$.*

Proof. We begin with some general considerations. Let H_m be any graph with 2^m nodes belonging to $P_m(x)$. Let d_m be the degree of H_m. Then, by (2.3), d_m is the unique positive x satisfying $P_m(x)=2^m$, namely (as can be seen from (4.2)), $d_m=m$. In particular, $d_2=2$ and it follows at once from (4.2) (a) that $H_2=G_2$.

Secondly, it also follows from (4.2) that $P_m(-m)=0$, hence (Lemma 1) H_m is a bicolored graph. It follows that if C is the adjacency matrix of H_m, we may assume

$$C = \begin{bmatrix} 0 & D \\ D^T & 0 \end{bmatrix}. \tag{5.1}$$

It is clear that the even powers of C can be nonzero only in the two blocks along the diagonal, the odd powers of C can be nonzero only in the two off-diagonal blocks.

If we let

$$F = DD^T, \tag{5.2}$$

it follows from (4.2) (b) and (c) that

$$\begin{aligned} &\text{(b)} \quad 2J = F - 1 \qquad \text{in case } m = 3 \\ &\text{(c)} \quad 24J = F^2 - 4F \qquad \text{in case } m = 4 \end{aligned} \tag{5.3}$$

when F and J are of order 4 in case $m=3$, and 8 in case $m=4$.

Now it is obvious from (5.2) and (5.3) (b) that the only possibility for D in case $m=3$ is a matrix which, under arbitrary permutations of rows and columns, is of the form (4.1) (b). Hence, $H_3=G_3$.

There remains the case $m=4$. Because $d_4=4$, we know that the diagonal entries of F are 4, and the sum of the entries in any row of F is 16. From (5.3) (c), we know that each diagonal element of F^2 is 40. Hence, since F is symmetric, the sum of the squares of the elements in any row of F is 40. It follows from these facts that the off-diagonal elements of any row of F are nonnegative integers whose sum is 12, and the sum of whose squares is 24. There are only two possibilities:

$$\begin{aligned} &\text{(a)} \quad 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 0, \quad \text{and} \\ &\text{(b)} \quad 3,\ 2,\ 2,\ 2,\ 1,\ 1,\ 1. \end{aligned} \tag{5.4}$$

We first show that (5.4) (a) and (5.4) (b) cannot occur in the same F. It is no loss of generality to assume that the first row of F is

$$4 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 0$$

and the second row contains a 3. If the 3 were in the last place, then the inner product of the two rows would be 30, whereas (5.3) (c) requires that it be 32. Hence, there is no loss of generality in having the second row begin

$$2, \ 4, \ 3.$$

Then the only way the remaining elements of the second row can be placed so that the inner product of the two rows will be 32 is so the first two rows look like

$$\begin{matrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 4 & 3 & 1 & 1 & 1 & 2 & 2. \end{matrix}$$

Then it is easy to see that the first three rows are

$$\begin{matrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 4 & 3 & 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & 1 & 1 & 1 & 2 & 2 \end{matrix} \quad \text{or} \quad \begin{matrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 4 & 3 & 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & 1 & 1 & 2 & 1 & 2. \end{matrix}$$

But then the inner product of the second and third row is 39 or 38, whereas (5.3) (c) requires that it be 36.

Hence, F is composed entirely of rows whose off-diagonal elements are all of the form (5.4) (a) or all of the form (5.4) (b). It is then possible to show by an extensive use of (5.3) (c) that, apart from renumbering of the vertices, either $D = (4.1)$ (c) or:

$$(5.5) \qquad D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

In the former case, $H_4 = Q_4$. In the latter case, we can verify that the graph H_4 does satisfy (4.2) (c), and it is obviously different from Q_4.

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SEMI-OPEN SETS AND SEMI-CONTINUITY IN TOPOLOGICAL SPACES

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1. Semi-open sets.

DEFINITION 1. *A set A in a topological space X will be termed semi-open (written* s.o.*) if and only if there exists an open set O such that $O\subset A\subset cO$ where c denotes the closure operator in X.*

THEOREM 1. *A subset A in a topological space X is* s.o. *if and only if $A\subset c\operatorname{Int} A$,* Int *denoting the interior operator.*

Proof. Sufficiency. Let $A\subset c\operatorname{Int} A$. Then for $O=\operatorname{Int} A$, we have $O\subset A\subset cO$.

Necessity. Let A be s.o. Then $O\subset A\subset cO$ for some open set O. But $O\subset\operatorname{Int} A$ and thus $cO\subset c\operatorname{Int} A$. Hence $A\subset cO\subset c\operatorname{Int} A$.

THEOREM 2. *Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of* s.o. *sets in a topological space X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is* s.o.

Proof. For each $\alpha\in\Delta$, we have an O_α such that $O_\alpha\subset A_\alpha\subset cO_\alpha$. Then $\bigcup_{\alpha\in\Delta} O_\alpha\subset\bigcup_{\alpha\in\Delta} A_\alpha\subset\bigcup_{\alpha\in\Delta} cO_\alpha\subset c\bigcup_{\alpha\in\Delta} O_\alpha$. Hence let $O=\bigcup_{\alpha\in\Delta} O_\alpha$.

THEOREM 3. *Let A be* s.o. *in the topological space X and suppose $A\subset B\subset cA$. Then B is* s.o.

Proof. There exists an open set O such that $O \subset A \subset cO$. Then $O \subset B$. But $cA \subset cO$ and thus $B \subset cO$. Hence $O \subset B \subset cO$ and B is s.o.

Remark 1. If O is open in X, then O is semi-open in X. The converse is clearly false.

DEFINITION 2. S.O.(X) *will denote the class of all semi-open sets in* X.

THEOREM 4. *Let* τ *be the class of open sets in* X. *Then* (1) $\tau \subset$ S.O.(X) *and* (2) *for* $A \in$ S.O.(X) *and* $A \subset B \subset cA$, *then* $B \in$ S.O.(X).

Proof. This follows from Theorem 3 and Remark 1.

THEOREM 5. *Let* $\mathcal{B} \equiv \{B_\alpha\}$ *be a collection of sets in* X *such that* (1) $\tau \subset \mathcal{B}$ *and* (2) *if* $B \in \mathcal{B}$ *and* $B \subset D \subset cB$, *then* $D \in \mathcal{B}$. *Then* S.O.$(X) \subset \mathcal{B}$. *Thus* S.O.(X) *is the smallest class of sets in* X *satisfying* (1) *and* (2).

Proof. Let $A \in$ S.O.(X). Then $O \subset A \subset cO$ for some $O \in \tau$. Then $O \in \mathcal{B}$ by (1) and thus $A \in \mathcal{B}$ by (2).

THEOREM 6. *Let* $A \subset Y \subset X$ *where* X *is a topological space and* Y *is a subspace. Let* $A \in$ S.O.(X). *Then* $A \in$ S.O.(Y).

Proof. $O \subset A \subset c_X O$ where O is open in X and c_X denotes the closure operator in X. Now $O \subset Y$ and thus $O = O \cap Y \subset A \cap Y \subset Y \cap c_X O$ or $O \subset A \subset c_Y O$. Since $O = O \cap Y$, O is open in Y and the theorem is proved.

Remark 2. The converse of Theorem 6 is false, as shown by

Example 1. Let X be the space of reals and $Y = \{0\}$ and $A = \{0\}$. Then A is open in Y and hence $A \in$ S.O.(Y). But $A \notin$ S.O.(X).

LEMMA 1. *Let* O *be open in* X. *Then* $cO - O$ *is nowhere dense in* X.

This is well known and a proof will not be given.

THEOREM 7. *Let* $A \in$ S.O.(X) *where* X *is a topological space. Then* $A = O \cup B$ *where* (1) $O \in \tau$, (2) $O \cap B = \emptyset$ *and* (3) B *is nowhere dense.*

Proof. $O \subset A \subset cO$ for some O open in X. But $A = O \cup (A - O)$. Let $B = A - O$. Then $B \subset cO - O$ and thus is nowhere dense by Lemma 1. Then $A = O \cup B$, and (1) and (2) immediately follow.

Remark 3. The converse of Theorem 7 is false, as shown by

Example 2. Let X be the space of reals and $A = \{x \mid 0 < x < 1\} \cup \{2\}$. Then $A \notin$ S.O.(X) although (1), (2), and (3) in Theorem 7 hold.

Remark 4. The converse of Theorem 7 is false even when connectedness is imposed upon A, as shown by

Example 3. Let X be the plane and $A = \{(x, y) \mid 0 < x < 1 \text{ and } 0 < y < 1\} \cup \{(x, 0) \mid 1 \leqq x \leqq 2\}$. It is clear that $A \notin$ S.O.(X) although again (1), (2), and (3) of Theorem 7 are satisfied.

We do however have

THEOREM 8. *Let X be a topological space and $A=O\cup B$ where* (1) *$O\neq\emptyset$ is open,* (2) *A is connected and* (3) *$B'=\emptyset$ where B' is the derived set of B. Then $A\in$S.O.(X).*

It is sufficient to show that $B\subset cO$. Deny. Then $B=B_1\cup B_2$ where $B_1\subset cO$ and $B_2\subset \mathcal{C}cO$ where $\mathcal{C}$ denotes the complement operator and $B_2\neq\emptyset$. Now $A=(O\cup B_1)\cup B_2$ and $O\cup B_1\neq\emptyset$ by (1) and $B_2\neq\emptyset$. Also, $O\cup B_1\subset cO$, a closed set, and $B_2\subset B_2$, a closed set and $B_2\cap cO=\emptyset$. Thus $O\cup B_1$ and B_2 constitute a separation for A, a contradiction. $B_2'\subset B'=\emptyset$ and thus B_2 is closed by (3).

Remark 4. It is not true that components of semi-open sets are semi-open, as shown by

Example 4. Let X be the space of reals and $A=\{0\}\cup(\frac{1}{2},1)\cup(\frac{1}{4},\frac{1}{2})\cup\cdots\cup(1/2^{n+1},1/2^n)\cup\cdots$. Then A is semi-open and $\{0\}$ is a component of A, but $\{0\}$ is not semi-open in X.

Remark 5. In general the complement of a semi-open set is not semi-open nor is the intersection of two semi-open sets semi-open. The reader can easily construct examples.

THEOREM 9. *Let $f: X\to X^*$ be continuous and interior where X and X^* are topological spaces. Let $A\in$S.O.(X). Then $f(A)\in$S.O.(X^*).*

Proof. Let $A=O\cup B$ where O is open and $B\subset cO-O$. Then $f(O)\subset f(A)=f(O)\cup f(B)\subset f(O)\cup f(cO)\subset f(O)\cup cf(O)=cf(O)$. Hence, $f(O)\subset f(A)\subset cf(O)$ and $f(O)$ is open in X^* since $f: X\to X^*$ is interior.

Remark 6. If "interior" is removed from Theorem 9, then the Theorem is in general false, as shown by

Example 5. Let X and X^* both be the space of reals and $f: X\to X^*$ as follows: $f(x)\equiv 1$ for all $x\in X$. Then X is semi-open in X but $f(X)$ is not semi-open in X^*.

DEFINITION 3. *Let X be a topological space and $\mathcal{B}\equiv\{B_\alpha\}$ a collection of subsets. Then* Int $\mathcal{B}$ *will denote* $\{\text{Int } B_\alpha\}$.

LEMMA 2. *Let τ be the class of open sets in the topological space X. Then* $\tau\equiv\text{Int S.O.}(X)$.

Proof. Let $O\in\tau$. Then $O\in$S.O.(X) and since $O=\text{Int } O$, $O\in$Int S.O.(X). Conversely let $O\in$Int S.O.(X). Then $O=\text{Int } A$ for some $A\in$S.O.(X) and thus $O\in\tau$.

THEOREM 10. *Let τ and τ^* be two topologies for X. Suppose* S.O.$(X,\tau)\subset$S.O.(X,τ^*). *Then $\tau\subset\tau^*$.*

Proof. $\tau=\text{Int S.O.}(X,\tau)\subset\text{Int S.O.}(X,\tau^*)=\tau^*$ by Lemma 2.

COROLLARY 1. *Let τ and τ^* be two topologies for X. Suppose* S.O.$(X,\tau)=$S.O.(X,τ^*). *Then $\tau=\tau^*$.*

Remark 7. It is interesting to note that the converse of Theorem 10 is false, as shown by

Example 6. Let X be the set of reals and τ the topology generated by sets of the form (x, y) where x is less than y. Let τ^* be the topology generated by sets of the form $[x, y)$ where again x is less than y. Then $\tau \subset \tau^*$, but $\text{S.O.}(X, \tau) \not\subset \text{S.O.}(X, \tau^*)$ since $(x, y] \in \text{S.O.}(X, \tau)$, but $(x, y] \notin \text{S.O.}(X, \tau^*)$.

THEOREM 11. *Let X_1 and X_2 be topological spaces and $X = X_1 \times X_2$ be the topological product. Let $A_1 \in \text{S.O.}(X_1)$ and $A_2 \in \text{S.O.}(X_2)$. Then $A_1 \times A_2 \in \text{S.O.}(X_1 \times X_2)$.*

Proof. We have $A_i = O_i \cup B_i$, where O_i is open in X_i and $B_i \subset c_i O_i - O_i$ for $i = 1, 2$. Then $A_1 \times A_2 = (O_1 \times O_2) \cup (B_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times B_2)$. But $O_1 \times O_2$ is open in $X_1 \times X_2$ and

$$(B_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times B_2) \subset c_{X_1} O_1 \times c_{X_2} O_2 = c_{X_1 \times X_2} O_1 \times O_2.$$

It follows then that $A_1 \times A_2 \in \text{S.O.}(X_1 \times X_2)$.

Remark 8. If $X = X_1 \times X_2$, X_i being topological spaces and $A \in \text{S.O.}(X)$, then it is not true in general that A is a union of sets of the form $A_1 \times A_2$ where $A_1 \in \text{S.O.}(X_1)$ and $A_2 \in \text{S.O.}(X_2)$. Consider, in fact

Example 7. Let $A = \{(x, y) \mid 0 < x < 1, 0 < y < 1\} \cup (1, 1)$. The reader will supply the remaining details.

2. Semi-continuity.

DEFINITION 4. *Let $f: X \to X^*$ be single valued (continuity not assumed) where X and X^* are topological spaces. Then $f: X \to X^*$ is termed semi-continuous (written* s.c.) *if and only if, for O^* open in X^*, then $f^{-1}(O^*) \in \text{S.O.}(X)$.*

Remark 9. Continuity implies semi-continuity, of course, but not conversely; consider, in fact,

Example 8. Let $X = X^* = [0, 1]$. Let $f: X \to X^*$ as follows: $f(x) = 1$ if $0 \leqq x \leqq \frac{1}{2}$ and $f(x) = 0$ if $\frac{1}{2} < x \leqq 1$. The easy details are left to the reader.

THEOREM 12. *Let $f: X \to X^*$ be a single valued function, X and X^* being topological spaces. Then $f: X \to X^*$ is* s.c. *if and only if for $f(p) \in O^*$, there exists an $A \in \text{S.O.}(X)$ such that $p \in A$ and $f(A) \subset O^*$.*

Proof. Sufficiency. Let O^* be open in X^* and let $p \in f^{-1}(O^*)$. Then $f(p) \in O^*$ and thus there exists an $A_p \in \text{S.O.}(X)$ such that $p \in A_p$ and $f(A_p) \subset O^*$. Then $p \in A_p \subset f^{-1}(O^*)$ and $f^{-1}(O^*) = \bigcup_{p \in f^{-1}(O^*)} A_p$. Then by Theorem 2, $f^{-1}(O^*) \in \text{S.O.}(X)$.

Necessity. Let $f(p) \in O^*$. Then $p \in f^{-1}(O^*) \in \text{S.O.}(X)$ since $f: X \to X^*$ is s.c. Let $A = f^{-1}(O^*)$. Then $p \in A$ and $f(A) \subset O^*$.

THEOREM 13. *Let $f: X \to X^*$ be* s.c. *and X^* a 2nd axiom space. Let P be the set of points of discontinuity of f. Then P is of first category.*

Proof. Let $p \in P$. Then there exists an $O^*_{i_p}$ in the countable open basis for X^* such that $p \in O$ open in X implies that $f(O) \not\subset O^*_{i_p}$. Now there exists an $A_{i_p} \in$ S.O.(X) such that $p \in A_{i_p}$ and $f(A_{i_p}) \subset O^*_{i_p}$ by Theorem 12. But $A_{i_p} = O_{i_p} \cup B_{i_p}$ where $B_{i_p} \subset cO_{i_p} - O_{i_p}$. Hence $p \notin O_{i_p}$ and thus $p \in B_{i_p}$, a nowhere dense set. It follows then that $P \subset \bigcup_{p \in P} B_{i_p}$ and since $\bigcup_{p \in P} B_{i_p}$ is of first category, it follows that P is of first category.

Remark 10. The converse of Theorem 13 is generally false, as shown by

Example 9. Let $X = (0, 1]$ and $X^* = [0, 1]$. Let $f(x) = 0$ if x is irrational and $1/q$ if $x = p/q$ and p and q are relatively prime. Then f is continuous at the irrationals and discontinuous at the rationals. Hence the set of discontinuities is of the first category. But $f: X \to X^*$ is not s.c. because $f^{-1}(\frac{1}{2}, 1]$ is a subset of the rationals and thus is not s.o.

THEOREM 14. *Let* $f_i: X_i \to X_i^*$ *be* s.c. *for* $i = 1, 2$. *Let* $f: X_1 \times X_2 \to X_1^* \times X_2^*$ *as follows*: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. *Then* $f: X_1 \times X_2 \to X_1^* \times X_2^*$ *is* s.c.

Proof. Let $O_1^* \times O_2^* \subset X_1^* \times X_2^*$ where O_i^* is open in X_i^* for $i = 1, 2$. Then $f^{-1}(O_1^* \times O_2^*) = f_1^{-1}(O_1^*) \times f_2^{-1}(O_2^*)$. But $f_1^{-1}(O_1^*)$ and $f_2^{-1}(O_2^*)$ are s.o. in X_1 and X_2 respectively and thus $f_1^{-1}(O_1^*) \times f_2^{-1}(O_2^*)$ is s.o. in $X_1 \times X_2$ by Theorem 11. Now if O^* is any open set in $X_1^* \times X_2^*$, then $f^{-1}(O^*) = f^{-1}(\bigcup O_\alpha^*)$ where O_α^* is of the form $O^*_{\alpha 1} \times O^*_{\alpha 2}$. Then $f^{-1}(O^*) = \bigcup f^{-1}(O_\alpha^*)$ which is s.o. by Theorem 2 since $f^{-1}(O_\alpha^*)$ is s.o. by the above argument.

THEOREM 15. *Let* $h: X \to X_1 \times X_2$ *be* s.c. *where* X, X_1, *and* X_2 *are topological spaces. Let* $f_i: X \to X_i$ *as follows*: *for* $x \in X$, $h(x) = (x_1, x_2)$. *Let* $f_i(x) = x_i$. *Then* $f_i: X \to X_i$ *is* s.c. *for* $i = 1, 2$.

Proof. We shall show only that $f_1: X \to X_1$ is s.c. Let O_1 be open in X_1. Then $O_1 \times X_2$ is open in $X_1 \times X_2$ and $h^{-1}(O_1 \times X_2)$ is s.o. in X. But $f_1^{-1}(O_1) = h^{-1}(O_1 \times X_2)$ and thus $f_1: X \to X_1$ is s.c.

Remark 11. The converse of Theorem 15 is generally false, as shown by

Example 10. Let $X = X_1 = X_2 = [0, 1]$. Let $f_1: X \to X_1$ as follows: $f_1(x) = 1$ if $0 \leqq x \leqq \frac{1}{2}$ and $f_1(x) = 0$ if $\frac{1}{2} < x \leqq 1$. Let $f_2: X \to X_2$ as follows: $f_2(x) = 1$ if $0 \leqq x < \frac{1}{2}$ and $f_2(x) = 0$ if $\frac{1}{2} \leqq x \leqq 1$. Then $f_i: X \to X_i$ is clearly s.c. for $i = 1, 2.$, but $h(x) = (f_1(x), f_2(x)): X \to X_1 \times X_2$ is not s.c. for $S_{\frac{1}{2}}(1, 0)$ is open in $X_1 \times X_2$, but $h^{-1}(S_{\frac{1}{2}}(1, 0)) = (\frac{1}{2})$ which is not s.o. in X. $S_{\frac{1}{2}}(1, 0)$ denotes the spherical neighborhood of $(1, 0)$, radius $\frac{1}{2}$.

Remark 12. A s.c. function of a s.c. function is not in general s.c., as shown by

Example 11. Let $X = X_1 = X_2 = [0, 1]$. Let $f_1: X \to X_1$ as follows: $f_1(x) = x$ if $0 \leqq x \leqq \frac{1}{2}$ and $f_1(x) = 0$ if $\frac{1}{2} < x \leqq 1$. Let $f_2: X_1 \to X_2$ as follows: $f_2(x) = 0$ if $0 \leqq x < \frac{1}{2}$ and $f_2(x) = 1$ if $\frac{1}{2} \leqq x \leqq 1$. The remaining easy details are left to the reader.

Remark 13. The algebraic sum and product of two s.c. functions are not in general s.c. The reader will easily construct examples.

Remark 14. The limit of a sequence of s.c. functions is not in general s.c., as shown by

Example 12. Let $X=X^*=[0, 1]$ and $f_n: X \to X^*$ be defined as follows: $f_n(x)=x^n$ for $n=1, 2, \cdots$. Then $f_0: X\to X^*$ is the limit of the sequence where $f_0(x)=0$ if $0 \leqq x < 1$ and $f_0(x)=1$ if $x=1$. But f_0 is not s.c. For $(\frac{1}{2}, 1]$ is open in X^*, but $f_0^{-1}(\frac{1}{2}, 1]=(1)$ which is not s.o. in X. We can, however, prove

THEOREM 16. *Let* $f_n: M\to M^*$, *where* M *and* M^* *are metric spaces with metrics* d *and* d^*, *be* s.c. *for* $n=1, 2, \cdots$, *and let* $f_0: M \to M^*$ *be the uniform limit of* $\{f_n\}$. *Then* $f_0: M\to M^*$ *is* s.c.

Proof. Let $f_0(x) \in O^*$. Then $f_0(x) \in S_\eta^*(f_0(x)) \subset O^*$ for some $\eta > 0$. There exists then an n^* such that $d^*(f_{n^*}(y), f_0(y)) < \eta/2$ for all $y \in M$. Then $d^*(f_{n^*}(x), f_0(x)) < \eta/2$ and thus $f_{n^*}(x) \in S_{\eta/2}^*(f_0(x)) \subset O^*$. Since f_{n^*} is s.c. there exists by Theorem 12, an A s.o. such that $x\in A$ and $f_{n^*}(A) \subset S^*_{\eta/2}(f_0(x))$. The proof will be complete when we show that $f_0(A)\subset O^*$. Let $y \in A$. Then

$$d^*(f_0(y), f_0(x)) \leqq d^*(f_0(y), f_{n^*}(y)) + d^*(f_{n^*}(y), f_0(x)) < \eta/2 + \eta/2 = \eta.$$

This proves $f_0(A) \subset S_\eta^*(f_0(x)) \subset O^*$.

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ON A THEOREM OF THRALL IN COMBINATORIAL ANALYSIS

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1. Regular tables. Let m be a positive integer and $(a)=(a_1, a_2, \cdots, a_k)$ be a partition of $m=a_1+a_2+\cdots+a_k$ into k unequal parts, $a_1 > a_2 > \cdots > a_k$. Let there correspond to (a) a diagram of a_1 rows and k columns, identical with the Ferrers-Sylvester graph of (a), in such a way that the summit of the j-th column coincides with the extreme right of the j-th row (the columns being numbered from left to right, and the rows from top to bottom). For example, the diagram corresponding to the partition (4, 2, 1) of 7 would be

$$\begin{array}{ccc} \cdot & & \\ \cdot & \cdot & \\ \cdot & \cdot & \cdot \\ \cdot & & \end{array}$$

If we replace the m points of the diagram by the m numbers $1, 2, \cdots, m$ we obtain a table (called by Thrall a labelling). Of the $m!$ tables corresponding to the same diagram (and therefore to the same partition), those in which the numbers are in ascending order of magnitude from left to right in each row and from top to bottom in each column are called *regular* tables. For example, the following table is regular for the partition (4, 2, 1)

$$\begin{array}{ccc} 1 & & \\ 2 & 5 & \\ 3 & 6 & 7. \\ 4 & & \end{array}$$

For any vector $x = (x_1, \cdots, x_k)$ with components in the field of complex numbers we define

$$\Delta(x) = \prod_{i<j} (x_i - x_j)$$

and

$$\nabla(x) = \prod_{i<j} (x_i + x_j).$$

Thrall [2] has proved the following theorem:

THEOREM. *Let $a_1, \cdots, a_k$ be natural numbers such that $a_1 > a_2 > \cdots > a_k > 0$. Then $g(a)$, the numbers of regular tables corresponding to the partition (a), is given by*

$$g(a) = \frac{(a_1 + \cdots + a_k)!\Delta(a)}{a_1! \cdots a_k!\nabla(a)}\cdot$$

The object of this note is to give a simple combinatorial proof of the above theorem using what we call strict-lattice permutations. Our method of proof depends heavily on results of MacMahon [1, Sect. III, Chap. V]. The notation we employ follows MacMahon.

2. Strict-lattice permutations. Consider an assemblage of letters $\alpha^p\beta^q\gamma^r \cdots$ in which the numbers $p, q, r, \cdots$ are in strictly descending order of magnitude. This particular permutation of the assemblage can be denoted by a strictly regular graph consisting of rows and columns. The successive rows will have $p, q, r, \cdots$ columns respectively; and the graph is the same as that which denotes the partition $(p, q, r, \cdots)$ of the number $p+q+r+ \cdots$. Such a graph for $p=4$, $q=2$, $r=1$, is

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \\ \cdot & & & \end{array}$$

The successive rows correspond to the letters $\alpha, \beta, \gamma, \cdots$ respectively.

If we take any permutation of $\alpha^p\beta^q\gamma^r \cdots$ we shall finally arrive at the same graph by proceeding from left to right of the permutation, and placing a point in the first row, or the second, or the third, according as we reach a letter α, or β, or γ, etc. Thus if we take the permutation $\alpha\alpha\beta\alpha\beta$ of the assemblage $\alpha^3\beta^2$ we obtain successively in this manner

I row	$\cdot\ \cdot$	$\cdot\ \cdot$	$\cdot\ \cdot\ \cdot$	$\cdot\ \cdot\ \cdot$
II row		$\cdot$	$\cdot$	$\cdot\ \cdot$
	$\alpha\ \alpha$	$\alpha\ \alpha\ \beta$	$\alpha\ \alpha\ \beta\ \alpha$	$\alpha\ \alpha\ \beta\ \alpha\ \beta$

and it will be observed that each of the four graphs thus reached is strictly regular, and is, in fact, the graph of a partition of a number. We are now ready to give the following definition.

DEFINITION. *Any permutation which possesses the property of yielding a succession of strictly regular graphs, in the manner described above, is called a strict-lattice permutation (s-l permutation).*

It is easily seen that *s-l* permutations are all lattice permutations in the sense of MacMahon; the converse does not hold.

Example. For the assemblage $\alpha^4\beta^2\gamma$ the following are *s-l* permutations:

$$\begin{array}{ll} \alpha\ \alpha\ \alpha\ \alpha\ \beta\ \beta\ \gamma & \alpha\ \alpha\ \beta\ \alpha\ \beta\ \alpha\ \gamma \\ \alpha\ \alpha\ \alpha\ \beta\ \alpha\ \beta\ \gamma & \alpha\ \alpha\ \beta\ \alpha\ \alpha\ \beta\ \gamma \\ \alpha\ \alpha\ \alpha\ \beta\ \beta\ \alpha\ \gamma & \alpha\ \alpha\ \beta\ \alpha\ \beta\ \gamma\ \alpha \\ \alpha\ \alpha\ \alpha\ \beta\ \beta\ \gamma\ \alpha & \end{array}$$

We now prove the following theorem

THEOREM. *The s-l permutations of the assemblage $\alpha^p\beta^q\gamma^r \cdots$ and the regular tables of the partition $(p, q, r, \cdots)$ are equinumerous.*

Consider any *s-l* permutation, say $\alpha\alpha\alpha\beta\beta\gamma\alpha(\alpha^3\beta^2\gamma\alpha)$, of the assemblage $\alpha^4\beta^2\gamma$. Write below the letters the first seven numbers in ascending order as follows:

$$\begin{array}{ccccccc} \alpha & \alpha & \alpha & \beta & \beta & \gamma & \alpha \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

Now, starting from the left, place each number in the first, second, or third column of the graph according as it falls below an α, a β, or a γ, at the same time observing the rule that the summit of the j-th column should coincide with the extreme right of the j-th row. Thus

$$\begin{array}{lll} 1 & & \\ 2 & 4 & \\ 3 & 5 & 6 \\ 7 & & \end{array}$$

The above graph is obviously regular. In fact, it is easily seen that all graphs formed by the above method from s-l permutations are regular in the sense of Thrall. Our method thus effects a one-to-one correspondence between the s-l permutations of the assemblage and the regular tables of the partition (4, 2, 1). This proves that they are equinumerous.

This procedure can be carried on to the general case without any difficulty and our assertion seen to be true. This proves our theorem.

From the above theorem it is clear that our problem, in its essentials, is to enumerate the s-l permutations of an assemblage. This, yielding the result of Thrall, can be easily done by adopting the method which MacMahon uses for enumerating the lattice permutations of an assemblage.

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MACLAURIN'S SECOND FORMULA AND ITS GENERALIZATION

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1. Introduction. In a recent paper [4] dealing with quadrature formulae which use values of the integrand and its derivatives, one of the present authors proposed a formula which can be written in the form

$$\int_a^b f(x)dx = \sum_{j=0}^{\infty} \frac{(b-a)^{2j+1}}{4^j(2j+1)!} f^{(2j)}\left(\frac{b-a}{2}\right). \tag{1.1}$$

From this quadrature formula a summation formula can be derived which differs somewhat from the Euler-Maclaurin formula (which we shall call Maclaurin's first formula)

$$\begin{aligned}\sum_{k=a}^{n} f(k) - \int_a^n f(x)dx &= \frac{f(n)+f(a)}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{f^{(2k-1)}(n) - f^{(2k-1)}(a)}{(2k)!} \\ &= \frac{f(n)+f(a)}{2} + \frac{1}{12}[f'(n) - f'(a)] - \frac{1}{720}[f'''(n) - f'''(a)] \\ &\quad + \frac{1}{30290}[f^{\mathrm{V}}(n) - f^{\mathrm{V}}(a)] + \cdots,\end{aligned} \tag{1.2}$$

where the B's are Bernoulli numbers. Here and later a denotes a positive integer.

In this paper we derive some related formulae and point out some historical connections.

The summation formula we have in mind can be obtained by writing (1.1) in the form

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(x)dx = \sum_{j=0}^{\infty} \frac{f^{(2j)}(k)}{4^j(2j+1)!} \tag{1.3}$$

and summing from $k=a$ to $k=n$ to obtain

$$\int_{a-\frac{1}{2}}^{n+\frac{1}{2}} f(x)dx = \sum_{k=a}^{n} f(k) + \sum_{j=1}^{\infty}\sum_{k=a}^{n} \frac{f^{(2j)}(k)}{4^j(2j+1)!}\,. \tag{1.4}$$

Similarly we find

$$\begin{aligned} \int_{a-\frac{1}{2}}^{n+\frac{1}{2}} f^{(2j)}(x)dx &= f^{(2j-1)}(n+\tfrac{1}{2}) - f^{(2j-1)}(a-\tfrac{1}{2}) \\ &= \sum_{k=a}^{n} f^{(2j)}(k) + \sum_{t=1}^{\infty}\sum_{k=a}^{n} \frac{f^{(2j+t)}(k)}{4^t(2t+1)!}\,. \end{aligned} \tag{1.5}$$

By using (1.5), the double summation in (1.4) can be eliminated to give the formula

$$\sum_{k=a}^{n} f(k) - \int_{a-\frac{1}{2}}^{n+\frac{1}{2}} f(x)dx = -\sum_{j=1}^{\infty} C_j[f^{(2j-1)}(n+\tfrac{1}{2}) - f^{(2j-1)}(a-\tfrac{1}{2})], \tag{1.6}$$

where the coefficients C_j are obtained recursively from the relations

$$C_j = \frac{1}{4^j(2j+1)!} - \sum_{t=1}^{j-1} \frac{C_t}{4^{j-t}(2j-2t+1)!} \tag{1.7}$$

with $C_1=1/24$.

This gives

$$\begin{aligned} \sum_{k=a}^{n} f(k) - \int_{a-\frac{1}{2}}^{n+\frac{1}{2}} f(x)dx &= -\frac{1}{24}[f'(n+\tfrac{1}{2}) - f'(a-\tfrac{1}{2})] \\ &\quad + \frac{7}{5760}[f'''(n+\tfrac{1}{2}) - f'''(a-\tfrac{1}{2})] - \frac{31}{967680}[f^{\mathrm{V}}(n+\tfrac{1}{2}) - f^{\mathrm{V}}(a-\tfrac{1}{2})] \\ &\quad + \cdots. \end{aligned} \tag{1.8}$$

This differs from the Euler-Maclaurin formula (1.2) in that the limits of integration and the points at which the derivatives are evaluated are at $n+\frac{1}{2}$ and $a-\frac{1}{2}$ instead of at n and a. Also there is no correction term involving the function, the weights of the derivative terms are different, and the signs re-

versed. The last fact suggests that at a given degree of approximation the two expressions will usually bracket the correct answer.

Examination of a number of current books failed to find any mention of formulae (1.6) or (1.8); an older text [1], however, gave an integration formula equivalent to (1.8). It is stated there that the formula is the chordal form of the Euler-Maclaurin formula originally given by Maclaurin [2] and was apparently not known to Euler. We shall therefore refer to it as Maclaurin's second formula. It is rather curious that this formula has been neglected since it is certainly as convenient to use as the conventional formula, and in many instances appears more accurate, as will be seen later in some examples. What is more, the formula may be generalized.

2. Generalization. In order to get a generalization of the second formula of Maclaurin, and at the same time determine the explicit form of the coefficients involved, we shall turn to the standard nomenclature of the Bernoulli polynomials and numbers as outlined by Nörlund in [3]. The Bernoulli polynomials are defined by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x) = \frac{te^{tx}}{e^t - 1}, \qquad |t| < 2\pi, \tag{2.1}$$

and, explicitly,

$$B_n(x) = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^n. \tag{2.2}$$

The Bernoulli numbers result when $x=0$, that is $B_n = B_n(0)$. We have $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, $\cdots$, and $B_{2m+1}=0$ ($m=1, 2, 3, \cdots$).

We shall need the relation (addition theorem)

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k(y), \tag{2.3}$$

and the fact that

$$B_n(x) = (-1)^n B_n(1-x). \tag{2.4}$$

Now to derive a generalized formula like (1.6)–(1.8) we shall begin with the summation formula for powers of natural numbers (see [5], 6.501)

$$\sum_{k=0}^{n} k^p = \frac{1}{p+1} \sum_{k=0}^{p} \binom{p+1}{k} (n+1)^{p+1-k} B_k = \frac{1}{p+1} [B_{p+1}(n+1) - B_{p+1}] \tag{2.5}$$

from which we see at once that

$$\sum_{k=a}^{n} k^p = \frac{1}{p+1} [B_{p+1}(n+1) - B_{p+1}(a)]. \tag{2.6}$$

Then we have

$$\begin{aligned}\sum_{k=a}^{n} k^p &= \frac{1}{p+1}\left[B_{p+1}(n-z+1+z) - B_{p+1}(a-z+z)\right]\\ &= \frac{1}{p+1}\sum_{k=0}^{p+1}\binom{p+1}{k} B_k(z)\left[(n+1-z)^{p+1-k} - (a-z)^{p+1-k}\right]\\ &= \sum_{k=0}^{p}\binom{p}{k} B_k(z)\,\frac{(n+1-z)^{p+1-k}-(a-z)^{p+1-k}}{p+1-k}\\ &= \sum_{k=0}^{p}\binom{p}{k} B_k(z)\int_{a-z}^{n+1-z} x^{p-k}dx.\end{aligned}$$

Now let $f(x)$ be a polynomial of degree s in x, that is,

$$f(x) = \sum_{p=0}^{s} a_p x^p.$$

Then we have

$$\sum_{k=0}^{n} f(k) = \sum_{p=0}^{s} a_p \sum_{k=0}^{p}\binom{p}{k} B_k(z)\int_{a-z}^{n+1-z} x^{p-k}dx = \sum_{k=0}^{s}\frac{B_k(z)}{k!}\int_{a-z}^{n+1-z} f^{(k)}(x)dx,$$

which is essentially the result we want. Writing the term when $k=0$ separately, we obtain the interesting exact formula for polynomials

$$\sum_{k=a}^{n} f(k) - \int_{a-z}^{n+1-z} f(x)dx = \sum_{k=1}^{s}\frac{B_k(z)}{k!}\left[f^{(k-1)}(n+1-z) - f^{(k-1)}(a-z)\right], \tag{2.7}$$

with s the degree of the polynomial $f(x)$.

To see that this general result contains the Maclaurin second quadrature formula as a special case, let $z=\frac{1}{2}$. It follows at once from (2.4) that $B_k(\frac{1}{2})=0$ if k is odd, so all terms for odd k drop out of the series. Moreover, it is evident that the coefficients in (1.6) defined by the recursive relation (1.7) are given explicitly by the formula

$$C_j = \frac{B_{2j}(\frac{1}{2})}{(2j)!} = \frac{1}{(2j)!2^{2j}}\sum_{k=0}^{2j}\binom{2j}{k} 2^k B_k, \tag{2.8}$$

which could be simplified still more since $B_{2m+1}=0$ for $m=1, 2, 3, \cdots$.

Moreover, if we choose $z=0$ it is evident that (2.7) becomes the Euler-Maclaurin formula (1.2), for we obtain

$$\begin{aligned}&\sum_{k=a}^{n} f(k) - \int_{a}^{n+1} f(x)dx\\ &\qquad = -\tfrac{1}{2}[f(n+1) - f(a)] + \sum_{k=1}^{[s/2]}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n+1) - f^{(2k-1)}(a)\right],\end{aligned}$$

which is indeed equivalent to (1.2). Thus our generalized formula contains both the Maclaurin formulae and many others. The result seems implicit in Nörlund in several places but does not seem to appear explicitly in this form.

There are a number of other formulae somewhat related to this general result. Thus, from the power sum formula (2.6) it is easily found in the same way that

$$\sum_{k=a}^{n} f(k) - \int_{a-z}^{n+z} f(x)dx = \sum_{k=0}^{s} \frac{B_k - (z-1)^k}{k!} \int_{a}^{n+1} f^{(k)}(x)dx. \tag{2.9}$$

In this case with $z=\frac{1}{2}$, it is clear that the terms corresponding to $k=0$ and 1 drop out but no others. Thus the difference between sum and integral is zero for any linear function $f(x)=mx+c$.

It may be of interest to point out that a simple linear change of variable allows us to rewrite our main formula (2.7) in the form

$$\sum_{k=a}^{n} f(k+z) - \int_{a}^{n+1} f(x)dx = \sum_{k=1}^{s} \frac{B_k(z)}{k!} [f^{(k-1)}(n+1) - f^{(k-1)}(a)], \tag{2.10}$$

with Maclaurin's first formula as the special case when $z=0$. Thus it is evident that the general formula could be obtained from the Euler-Maclaurin formula by a change of variable plus application of appropriate formulae for Bernoulli polynomials. It is also of interest to note that Maclaurin's first formula is again obtained when we let $z=1$ in (2.7).

We have remarked that formula (2.7) though not explicitly given in Nörlund [3] seems implicit. The formula (2.10) now allows us to show that (2.7) does follow from well-known results. Nörlund says (page 29) that the Euler-Maclaurin formula is commonly written in the form (z being any real number)

$$f(u+z) = \int_{u}^{u+1} f(x)dx + \sum_{j=1}^{s} \frac{B_j(z)}{j!} \mathop{\Delta}_{u,1} f^{(j-1)}(u), \tag{2.11}$$

where $f(x)$ is a polynomial of degree s in x. Summing both sides of this from $u=a$ to $u=n$ yields

$$\sum_{u=a}^{n} f(u+z) = \int_{a}^{n+1} f(x)dx + \sum_{j=1}^{s} \frac{B_j(z)}{j!} [f^{(j-1)}(n+1) - f^{(j-1)}(a)]$$

which is precisely relation (2.10) above. Thus the relation (2.7) follows from Nörlund's form of the Euler-Maclaurin formula by a change of variable. It follows that the treatment of error terms, remainder, and the like can be obtained from Nörlund's discussion by effecting the necessary linear change of variable.

It is the opinion of the present authors that even though the relation (2.7) is not novel or new yet it appears to have been neglected and seems to offer certain interesting advantages, in particular the special case $z=\frac{1}{2}$ with which we started. Nörlund has a chapter wherein the basic formula is extended to the so-

called Bernoulli polynomials of higher order so that a more general companion to (1.8) could be obtained in that context also.

Nörlund [3, p. 30] gives the remainder term for (2.11) in the following form. Let the function f have a continuous derivative $f^{(s)}(x)$ of order s in the interval $u \leqq x \leqq u+1$. Then the remainder term may be written as

$$R_s = -\int_0^1 \frac{\overline{B}_s(z-x)}{s!} f^{(s)}(u+x)dx, \tag{2.12}$$

where $\overline{B}_s(x) = B_s(x)$ for $0 \leqq x < 1$ and $\overline{B}_s(x)$ is periodic with period unity. Our relation (2.10) above is then essentially relation (45*) of Nörlund [3, p. 55], and a sharp estimate of the remainder term is given by Nörlund (p. 61) using the Tchebychef integral inequality. As Nörlund (p. 30) observes, the remainder (2.12) has been studied by Poisson, Jacobi, Malmsten, Darboux, Schendel, Sonine, and Lindelöf to name but a few.

The corresponding remainder term for (2.10) is then

$$R_s = -\int_0^1 \frac{\overline{B}_s(z-x)}{s!} \sum_{u=a}^{n} f^{(s)}(u+x)dx, \tag{2.13}$$

which Nörlund discusses under the assumption that the series of derivatives converges conditionally as $n \to \infty$.

3. Some examples. We should like now to examine three examples in order to compare the usual Euler-Maclaurin formula (1.2) with the second formula of Maclaurin (1.8).

Let us first choose a polynomial, $f(x) = x^3$. Using one term of the series involved we obtain the exact sum, but we wish to compare the respective correction terms.

By the Euler-Maclaurin formula (1.2) we have

$$\sum_{k=1}^{n} k^3 = \int_1^n x^3 dx + \frac{n^3+1}{2} + \frac{3n^2-3}{12} = \frac{n^4+2n^3+1}{4} + \frac{n^2-1}{4}, \tag{3.1}$$

whereas by the Maclaurin second formula (1.8) we have

$$\begin{aligned} \sum_{k=1}^{n} k^3 &= \int_{\frac{1}{2}}^{n+\frac{1}{2}} x^3 dx - \frac{1}{24}\left[3(n+\tfrac{1}{2})^2 - 3(\tfrac{1}{2})^2\right] \\ &= \frac{2n^4+4n^3+3n^2+n}{8} - \frac{n^2+n}{8}. \end{aligned} \tag{3.2}$$

The remainder terms correct the main term in opposite directions and moreover it is evident that

$$\frac{n^2+n}{8} < \frac{n^2-1}{4}$$

for $n \geqq 3$. In any event the correct answer lies between the two main terms.

As a second example, let us choose $f(x)=\exp x$. The exact formula of course is

$$\sum_{k=0}^{n} e^k = \frac{e^{n+1}-1}{e-1}. \tag{3.3}$$

By the Euler-Maclaurin formula (1.2) we have

$$\begin{aligned}\sum_{k=0}^{n} e^k &= e^n - 1 + \tfrac{1}{2}(e^n+1) + (e^n-1)\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\\ &= \left(\frac{3}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\right)e^n - \left(\frac{1}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\right).\end{aligned} \tag{3.4}$$

By the more general form of (1.8) which is (2.7) we have

$$\sum_{k=0}^{n} e^k = \frac{e^{n+1}-1}{e^z}\sum_{k=0}^{\infty}\frac{B_k(z)}{k!}, \tag{3.5}$$

so that the general formula even looks more like the exact formula (3.3) than the more well-known form (3.4).

As a third example, we choose $f(x)=x^{-2}$ and let $n\to\infty$. By the Euler-Maclaurin formula (1.2) we have

$$\sum_{k=a}^{\infty}\frac{1}{k^2} = \left(\frac{1}{a}+\frac{1}{2a^2}\right)+\frac{1}{6a^3} - \cdots \tag{3.6}$$

and by the second Maclaurin formula (1.8) we have

$$\sum_{k=a}^{\infty}\frac{1}{k^2} = \left(\frac{1}{a-\frac{1}{2}}\right)-\frac{1}{12(a-\frac{1}{2})^3}+\cdots \tag{3.7}$$

For various small values of a we give a table of numerical values to illustrate these two formulae.

a	exact sum	EM(1)	diff.	EM(2)	diff.
1	1.644934	1.500000	+.144934	1.666667	−.021733
2	.644934	.625000	.019934	.645833	−.000899
3	.394934	.388889	.006045	.395062	−.000128
4	.283823	.281250	.002573	.283823	.000000

a	exact sum	M2(1)	diff.	M2(2)	diff.
1	1.644934	2.000000	−.355066	1.333333	+.311601
2	.644934	.666667	−.021733	.641976	.002958
3	.394934	.400000	−.005066	.394667	.000267
4	.283823	.285714	−.001891	.283777	.000046

The notation EM(1) means numerical value using Euler-Maclaurin formula (3.6) taking into account only the dominant term, and EM(2) means the value from (3.6) taking into account the first correction term involving derivatives. Similarly the notation M2 refers to the second Maclaurin formula (3.7).

The above values indicate that apparently the usual Euler-Maclaurin formula is more accurate here, but the two summation formulae give upper and lower estimates for the desired sum which is useful.

4. An application to operational mathematics. We should like to conclude our present remarks by commenting on an interesting application to operational mathematics made recently by I. Frank [**6**].

Frank applied the first formula of Maclaurin to find an expression for the inverse Laplace transform of

$$F(s) = \frac{1}{s^{1/2}} \cdot \frac{1 + e^{-2q}}{1 - 2^{-2q}}, \tag{4.1}$$

where $q = s^{\frac{1}{2}} x_0$, valid for all values of t, where $f(t) = L^{-1}(F(s))$. This involved setting

$$\frac{1 + e^{-2q}}{1 - e^{-2q}} = 1 + \frac{2e^{-2q}}{1 - e^{-2q}} = 1 + 2 \sum_{n=1}^{\infty} e^{-2nq}.$$

Applying Maclaurin's first formula gives

$$\sum_{n=1}^{\infty} e^{-2nq} = e^{-2q} \left[\frac{1}{2q} + \frac{1}{2} + \frac{q}{6} - \frac{q^3}{90} + \cdots \right] \tag{4.2}$$

which makes possible a term by term inversion.

If we apply Maclaurin's second formula we find

$$\sum_{n=1}^{\infty} e^{-2nq} = e^{-q} \left[\frac{1}{2q} - \frac{1q}{12} + \frac{7q^3}{720} - \cdots \right] \tag{4.3}$$

which may also be inverted term by term. Because of the reversal in sign, the two series (4.2) and (4.3) will bound the correct answer.

In this simple case, however, the use of the summation formulae is not necessary. Frank's result may be obtained by expanding $(1 - e^{-2q})^{-1}$ in a power series, and the alternate form may be obtained by writing

$$\frac{e^{-2q}}{1 - e^{-2q}} = e^{-q}(e^q - e^{-q})^{-1}$$

and expanding the right-hand member in a power series.

Our formula (1.8) may be found in [**7**, p. 78], where it is obtained from a more general formula credited to Sonine and Hermite [**8**]; it is not mentioned there, however, that Maclaurin himself knew the formula.

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A CORRECTION FOR "NETS AND FILTERS IN TOPOLOGY"

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Professor A. Wilansky has pointed out an error in the writer's note entitled, "Nets and filters in topology," which appeared in volume 62 (1955), pages 551–557 of this MONTHLY. The error appears in the construction given in Proposition 2.5; the last line of the argument being false.

In fact, if the net $\mathfrak{x}$ is sufficiently degenerate, the construction fails. For example, take $\mathfrak{x}$ to be a constant net defined on the directed set of natural numbers, then the associated filter base $\mathfrak{B}(\mathfrak{x})$ consists of a single set. For a refinement $\mathfrak{D}$ of $\mathfrak{B}(\mathfrak{x})$ consisting of only a finite number of sets, the construction does not yield a subnet of $\mathfrak{x}$.

Since this expository paper has been found to be useful, we propose to supply a correct construction of the desired result. In doing so we retain the notations employed in the body of the note.

Let $\mathfrak{x}=\{x_\alpha\}_{\alpha\in A}$ be a net in X and let $\mathfrak{B}(\mathfrak{x})$ be the associated filter base. If $\mathfrak{D}$ is a refinement of $\mathfrak{B}(\mathfrak{x})$, consider the subset C of $A\times\mathfrak{D}$ consisting of all pairs $\gamma=(\alpha, F)$ such that $x_\alpha\in F\in\mathfrak{D}$. Order by requiring $\gamma=(\alpha, F)\leqq\gamma'=(\alpha', F')$ if and only if $\alpha\leqq\alpha'$ and $F\supseteq F'$. It is clear that C is partially ordered by this ordering; to see that it is directed, let $\gamma_i=(\alpha_i, F_i)$, $i=1, 2$, be given and let $\alpha_3\geqq\alpha_1, \alpha_2$. The set $E(\alpha_3)=\{x_\alpha: \alpha\geqq\alpha_3\}$ belongs to $\mathfrak{B}(\mathfrak{x})$ and hence contains a set F_3 in the refinement filter base $\mathfrak{D}$. Let G be any set in $\mathfrak{D}$ which is contained in $F_1\cap F_2\cap F_3$; since $G\subseteq F_3\subseteq E(\alpha_3)$, there exists an element $\beta\geqq\alpha_3$ with x_β in G. Therefore, the pair $\gamma=(\beta, G)$ belongs to C and follows γ_1, γ_2.

For an element $\gamma=(\alpha, F)$ in C, we define $y_\gamma=x_\alpha$. Then $\mathfrak{y}=\{y_\gamma\}_{\gamma\in C}$ is a net and the map $\pi: C\rightarrow A$ defined by $\pi(\alpha, F)=\alpha$ shows that $\mathfrak{y}$ is a subnet of $\mathfrak{x}$. Finally, let F_0 be an element of $\mathfrak{D}$ and select α_0 in A. The set $E(\alpha_0)$ contains some set F_1 in $\mathfrak{D}$ and there is an α_1 in A such that x_{α_1} belongs to $F_0\cap F_1$. Now let

$\gamma_0=(\alpha_1, F_0)$ be such that $\gamma_0 \in C$. Then if $\gamma_0 \leqq \gamma=(\alpha, F)$, it follows that $x_\alpha \in F \subseteq F_0$ so that $F(\gamma_0)=\{y_\gamma: \gamma_0 \leqq \gamma\} \subseteq F_0$. This proves that the associated filter base $\mathfrak{B}(\mathfrak{y})$ is a refinement of $\mathfrak{D}$.

We have established the following replacement for Proposition 2.5.

2.5′. PROPOSITION. *Let* $\mathfrak{x}=\{x_\alpha\}_{\alpha\in A}$ *be a net in* X *and let* $\mathfrak{B}(\mathfrak{x})$ *be the associated filter base. Let* $\mathfrak{D}$ *be a refinement of* $\mathfrak{B}(\mathfrak{x})$. *Then there exists a subnet* $\mathfrak{y}=\{y_\gamma\}_{\gamma\in C}$ *of* $\mathfrak{x}$ *which is such that its associated filter base* $\mathfrak{B}(\mathfrak{y})$ *is a refinement of* $\mathfrak{D}$.

In case $\mathfrak{D}$ is an ultrafilter base refining $\mathfrak{B}(\mathfrak{x})$, then it is clear that $\mathfrak{y}$ is a universal subnet of $\mathfrak{x}$. Hence we can use Proposition 2.5′ to establish Proposition 3.3 in the note.

MATHEMATICAL NOTES

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DIAGONALIZATION OF QUADRATIC FORMS WITH MATRIX COEFFICIENTS

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Let A_{ij}, $1 \leqq i, j \leqq n$ be $n \times n$ matrices with complex entries such that $A_{ij}^* = A_{ji}$ where * denotes conjugate transpose. Suppose further that for every $z=\{z_1, \cdots, z_n\}$ in complex n-dimensional space the matrix

$$U(z) = \sum_{i,j=1}^{n} A_{ij} z_i \bar{z}_j \tag{1}$$

is nonnegative. The following question was posed by P. Lax (private communication): Does there exist a representation of $U(z)$ as a sum of nonnegative matrices? More specifically, does there exist an integer m and $n \times n$ matrices C_{ik}, $1 \leqq i \leqq m$, $1 \leqq k \leqq n$ such that if

$$C_i(z) = \sum_{k=1}^{n} C_{ik} z_k \tag{2}$$

then

$$U(z) = \sum_{i=1}^{m} C_i(z) C_i^*(z)? \tag{3}$$

The "sum of squares" representation (3) is not valid in general. We shall give a counterexample for $n=2$. In this note we shall prove the following result.

THEOREM 1. *$U(z)$ has the representation* (3) *if and only if the $n^2 \times n^2$ matrix*

$$A = (A_{ij}) \tag{4}$$

is nonnegative.

Proof. If the representation (3) is valid for all z then

$$A_{ij} = \sum_{s=1}^{m} C_{si} C_{sj}^{*}. \tag{5}$$

Let t^k, $k=1, \cdots, n$ be any n complex n-dimensional vectors and set

$$t = \{t^1, \cdots, t^n\}. \tag{6}$$

From (4), (5) and (6) it follows that

$$t^* A t = \sum_{s=1}^{m} \left\| \sum_{j=1}^{n} C_{sj}^{*} t^j \right\|^2 \geqq 0$$

and A is nonnegative.

If A is nonnegative then $A = RR^*$ where R is an $n^2 \times n^2$ matrix. If we partition R into $n \times n$ matrices R_{ij} then

$$A_{ij} = \sum_{s=1}^{n} R_{is} R_{js}^{*},$$

from which the representation (3) follows. (We wish to thank the referee for his recommendation concerning notation which simplified the proof of Theorem 1.)

A counterexample for $n=2$ is obtained by setting

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad 0 < \lambda < 1,$$

where the 2×2 matrices A_{ij} are defined in the obvious way. Then A is not nonnegative while the nonnegativeness of $U(z)$ is a consequence of the inequality valid for any two complex numbers a, b

$$2 \operatorname{Re} (ab) \leqq |a|^2 + |b|^2.$$

If the only restriction on $U(z)$ as defined by (1) is $A_{ij}^* = A_{ji}$ then A is still hermitian and hence

$$A = R_1 R_1^* - R_2 R_2^*$$

where R_1, R_2 are $n^2 \times n^2$ matrices. From this observation we obtain the following result:

THEOREM 2. *If* A_{ij}, $1 \leqq i, j \leqq n$, *are any* $n \times n$ *matrices such that* $A_{ij}^* = A_{ji}$ *then there always exist matrices* C_{ik}, D_{ik}, $1 \leqq i, k \leqq n$ *such that for every* $z = \{z_1, \cdots, z_n\}$

$$\sum_{i,j=1}^{n} A_{ij} z_i \bar{z}_j = \sum_{i=1}^{n} C_i(z) C_i^*(z) - \sum_{i=1}^{n} D_i(z) D_i^*(z),$$

where

$$C_i(z) = \sum_{k=1}^{n} C_{ik} z_k, \qquad D_i(z) = \sum_{k=1}^{n} D_{ik} z_k.$$

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A NOTE ON BOREL SETS

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The class of *Borel* sets in a topological space, according to [1, §51], is the σ-ring generated by the class of compact sets. In [2], and elsewhere, one considers the σ-ring (algebra) generated by the class of closed sets; let us call these *weak Borel* sets. Recall that a set is *σ-bounded* if it is contained in a countable union of compact sets.

THEOREM. *In any Hausdorff space, the class of Borel sets coincides with the class of σ-bounded weak Borel sets.*

Proof. Every Borel set is a weak Borel set (compact sets are closed), and is σ-bounded by [1, 51A].

If C is compact and F is closed, then $C \cap F$ is compact, hence is a Borel set. If E is a Borel set and F is closed, then $E \cap F$ is a Borel set; for, the class of all sets S such that $S \cap F$ is a Borel set, is evidently a σ-ring containing every compact set. If E is a Borel set and A is a weak Borel set, then $E \cap A$ is a Borel set; for, the class of all sets T, such that $E \cap T$ is a Borel set, is a σ-ring containing every closed set. In the language of ring theory, the Borel sets form an ideal in the ring of weak Borel sets.

Suppose now that A is a σ-bounded weak Borel set. Let C_n be a sequence of compact sets such that $A \subset \bigcup C_n$. Then $E = \bigcup C_n$ is a Borel set, hence so is $E \cap A = A$.

The proof works as long as compact sets are closed; the conclusion fails for the space X consisting of two points, with open sets $\varnothing$ and X.

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AN INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS

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Let $p(\theta)=\sum_{\nu=-n}^{n} a_\nu e^{i\nu\theta}$ be a trigonometric polynomial of degree n. It is known [3] that for every $\delta \geqq 1$

$$\left(\int_0^{2\pi} |p'(\theta)|^\delta d\theta\right)^{1/\delta} \leqq n\left(\int_0^{2\pi} |p(\theta)|^\delta d\theta\right)^{1/\delta}. \tag{1}$$

By Minkowski's inequality we can easily deduce that for $\delta \geqq 2$

$$\left(\int_0^{2\pi} [\{|p'(\theta)|^2+n^2|p(\theta)|^2\}^{1/2}]^\delta d\theta\right)^{1/\delta} \leqq \sqrt{2}\, n\left(\int_0^{2\pi} |p(\theta)|^\delta d\theta\right)^{1/\delta}. \tag{2}$$

In (2) equality holds, for example, for $p(\theta)=e^{in\theta}$.

The example $p(\theta)=\sin n\theta$ shows that (1) is the best possible result even for real trigonometric polynomials. We can however prove the following refinement of (2) in this case.

THEOREM. *If $p(\theta)$ is a real trigonometric polynomial of degree n, then for every $\delta \geqq 1$*

$$\left(\int_0^{2\pi} [\{|p'(\theta)|^2+n^2|p(\theta)|^2\}^{1/2}]^\delta d\theta\right)^{1/\delta} \leqq (C_\delta)^{1/\delta} 2n\left(\int_0^{2\pi} |p(\theta)|^\delta d\theta\right)^{1/\delta}, \tag{3}$$

where

$$C_\delta = 2\pi \Big/ \int_0^{2\pi} |1+e^{i\eta}|^\delta d\eta = 2^{-\delta}\sqrt{\pi}\,\Gamma(\tfrac{1}{2}\delta+1)/\Gamma(\tfrac{1}{2}\delta+\tfrac{1}{2}).$$

As $\delta \to \infty$ we get the following result of van der Corput and Schaake [2].

COROLLARY. *If $p(\theta)$ is a real trigonometric polynomial of degree n, then*

$$\{|p'(\theta)|^2+n^2|p(\theta)|^2\}^{1/2} \leqq n \max_{0\leqq\theta<2\pi} |p(\theta)|.$$

Proof of the theorem. If $p(\theta)=\sum_{\nu=-n}^{n} a_\nu e^{i\nu\theta}$ then $P(z)=z^n\sum_{\nu=-n}^{n} a_\nu z^\nu$ is a polynomial of degree $2n$. By a known result [1, p. 1271, see line 11] it follows that for every $\delta \geqq 1$ and real η

$$\int_0^{2\pi}\left|P(e^{i\theta})-e^{i\theta}\frac{P'(e^{i\theta})}{2n}+e^{i(\eta+\theta)}\frac{P'(e^{i\theta})}{2n}\right|^\delta d\theta \leqq \int_0^{2\pi} |P(e^{i\theta})|^\delta d\theta. \tag{4}$$

If $P(e^{i\theta}) \neq 0$ then

$$e^{i\theta}\frac{P'(e^{i\theta})}{P(e^{i\theta})} = \sum_{\nu=1}^{2n}\frac{e^{i\theta}}{e^{i\theta}-z_\nu}, \tag{5}$$

where $z_1, z_2, \cdots, z_{2n}$ are the zeros of $P(z)$.

Since $a_{-\nu}=\bar{a}_\nu$ it follows that if $z_\nu=R_\nu e^{i\theta_\nu}$ is a zero of $P(z)$ then $e^{i\theta_\nu}/R_\nu$ is also a zero of $P(z)$. If $|z_\nu|=1$ then clearly $\mathrm{Re}(e^{i\theta}/(e^{i\theta}-z_\nu))\equiv\frac{1}{2}$. If however $|z_\nu|=R_\nu\neq 1$ then

$$\mathrm{Re}\left(\frac{e^{i\theta}}{e^{i\theta}-R_\nu e^{i\theta_\nu}}+\frac{e^{i\theta}}{e^{i\theta}-e^{i\theta_\nu}/R_\nu}\right)=1.$$

Consequently

$$\mathrm{Re}\left\{e^{i\theta}\,\frac{P'(e^{i\theta})}{P(e^{i\theta})}\right\}=n.$$

It follows that if $P(e^{i\theta})\neq 0$, then

$$\left|2n-e^{i\theta}\,\frac{P'(e^{i\theta})}{P(e^{i\theta})}\right|=\left|e^{i\theta}\,\frac{P'(e^{i\theta})}{P(e^{i\theta})}\right|.$$

Therefore

$$\left|P(e^{i\theta})-e^{i\theta}\,\frac{P'(e^{i\theta})}{2n}\right|=\left|e^{i\theta}\,\frac{P'(e^{i\theta})}{2n}\right|,$$

except possibly for those values of θ for which $P(e^{i\theta})$ vanishes. By continuity it holds for $0\leqq\theta<2\pi$.

Thus from (4) we have

$$\int_0^{2\pi}\left|\frac{P'(e^{i\theta})}{2n}\right|^\delta d\theta\int_0^{2\pi}|1+e^{i\eta}|^\delta d\eta\leqq 2\pi\int_0^{2\pi}|P(e^{i\theta})|^\delta d\theta$$

or

$$\left(\int_0^{2\pi}|P'(e^{i\theta})|^\delta d\theta\right)^{1/\delta}\leqq 2n(C_\delta)^{1/\delta}\left(\int_0^{2\pi}|P(e^{i\theta})|^\delta d\theta\right)^{1/\delta},$$

where

$$C_\delta=2\pi\bigg/\int_0^{2\pi}|1+e^{i\eta}|^\delta d\eta=2^{-\delta}\sqrt{\pi}\,\Gamma(\tfrac{1}{2}\delta+1)/\Gamma(\tfrac{1}{2}\delta+\tfrac{1}{2}).$$

Since $|P(e^{i\theta})|=|p(\theta)|$ and $|P'(e^{i\theta})|=\{|p'(\theta)|^2+n^2|p(\theta)|^2\}^{1/2}$, the theorem follows.

The equality holds in (3) if, for example, $p(\theta)=\sin n\theta$.

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ON SEVERAL APPLICATIONS OF THE FIRST DERIVATIVE OF A DETERMINANT

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Recently, Vein [1] proved a lemma concerning the zero sum of n determinants, each of order n, derived (by a rule involving cyclic dislocations of columns) from a given determinant of order n, which we denote by $D_n = |C_1C_2 \cdots C_n|$, where C_i represents the ith column containing n elements, a_{si}, $s=1, 2, \cdots, n$. It should be pointed out that the lemma of [1] is a special case of a more general theorem which has been established by Deruyts [2] and Muir [3, vol. 4, pp. 25–26], namely:

THEOREM 1. *Let δ_p denote a permutation operator to be applied to the elements of a column C_i leaving p elements of C_i fixed (so that C_i and δ_pC_i have exactly p equal elements in corresponding places). Then*

$$
\begin{aligned}
(1)\qquad & |\delta_pC_1C_2 \cdots C_n| + |C_1\delta_pC_2 \cdots C_n| + \cdots + |C_1C_2 \cdots \delta_pC_n| \\
& \qquad = p\,|C_1C_2 \cdots C_n| .
\end{aligned}
$$

We note that $p=0$ for cyclic dislocations of order $1, 2, \cdots,$ or $n-1$. The formal analogy of (1) with the well-known differentiation formula,

$$
\begin{aligned}
(2)\qquad D_n' = {} & |C_1'C_2 \cdots C_n| + |C_1C_2' \cdots C_n| + |C_1C_2C_3' \cdots C_n| + \cdots \\
& + |C_1C_2 \cdots C_{n-1}C_n'| ,
\end{aligned}
$$

where $' \equiv d/dx$, is apparent, and effective use of this analogy will be made in the proof of the following generalization of Theorem 1:

THEOREM 2. *Let*

$$
C_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \cdot \\ \cdot \\ \cdot \\ a_{ni} \end{bmatrix} \quad \textit{and} \quad \delta C_i = \begin{bmatrix} a_{1i}^* \\ a_{2i}^* \\ \cdot \\ \cdot \\ \cdot \\ a_{ni}^* \end{bmatrix} \qquad i = 1, 2, \cdots, n,
$$

where δ indicates one of the following five column processes:

P-1. *A cyclic dislocation of C_i of order* $1, 2, \cdots,$ *or* n;

P-2. *Set* $a_{ni}^*=0$, $a_{ki}^*=a_{ni}$, $k=1, 2, \cdots, n-1$;

P-3. *Set* $a_{1i}^*=a_{2i}+a_{3i}+ \cdots +a_{ni}$, $a_{ki}^*=0$, $k=2, 3, \cdots, n$;

P-4. *Set* $a_{si}^*=a_{si}$, $s=1, 2, \cdots, m$, $(m<n)$, $a_{ki}^*=a_{1i}$, $k=m+1, \cdots, n$;

P-5. *Set* $a_{1i}^*=a_{ni}$, $a_{ni}^*=a_{1i}$, $a_{ki}^*=a_{ki}$, $k=2, 3, \cdots, n-1$.

If p is the number of elements of the column δC_i coinciding with the corresponding elements of C_i, then (1) is valid (for the δ operator).

Remark 1. Our proof of Theorem 2, for column *processes*, uses (2), as well as the following two well-known properties of determinants: (i) D_n is unchanged in value if to the elements of any row are added m times the corresponding elements

of any other row, and (ii) D_n is multiplied by m if all the elements of one row are multiplied by a number m. Obviously the above enumeration of column *processes* is not exhaustive, and it will soon be apparent how other column *processes* may be defined.

Proof. The proof of the theorem depends on the *process* selected. Specifying, say P-5 as the column *process* to be used on C_i of D_n, $n \geqq 3$, we now proceed to multiply rows 2, 3, $\cdots$, and $n-1$ by e^x, multiply the elements of the nth row by x and add these to the corresponding elements in row 1, and then multiply the elements of the first row by x and add these to the corresponding elements in row n. Thus, the following identity in x is obtained:

$$\begin{vmatrix} a_{11} + a_{n1}x & \cdots a_{1n} + a_{nn}x \\ a_{21}e^x & \cdots a_{2n}e^x \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{n-1,1}e^x & \cdots a_{n-1,n}e^x \\ a_{n1} + a_{11}x + a_{n1}x^2 & \cdots a_{nn} + a_{1n}x + a_{nn}x^2 \end{vmatrix} \equiv D_n e^{(n-2)x}, \tag{3}$$

where (3) yields $D_n = |C_1C_2 \cdots C_n|$ for $x=0$. If we differentiate (3) with respect to x (using (2)) and then set $x=0$, we obtain our result, specified by P-5 and Theorem 2, where $p=n-2$, $n \geqq 3$. On the other hand in the case P-1 our proof of Theorem 2 would involve only the addition of multiples of rows to other rows, and since no row would be multiplied by e^x, Theorem 2 would then be established where $p=0$.

It should be noted that column *processes* must be defined so as to take into account the changes (or no change) that can occur in the element value, a_{ij}, through algebraic additions (or none) of row multiples to other rows of D_n. If a column *process* leaves p elements of a column in their original position, then the proof of Theorem 2 would require the multiplication of e^x into p rows, each of which contains the original element specified by the *process*.

Remark 2. A row *process* is a procedure for specifying the elements of a given row. A proof of Theorem 2 for a row *process* would involve the algebraic addition of column multiples to other columns, as well as the multiplication of columns by e^x. The first derivative of this resulting determinant, as a function of x, must be taken using rows, not columns! Theorem 2 would then follow by setting $x=0$ in the resulting identity.

Additional theorems. The relation (2) may be applied to give a new proof of the following theorem, which is due to Deruyts [3, vol. 4, p. 15] and restated by Muir [3, vol. 4, p. 38]:

THEOREM 3. *If any two determinants A and B of the nth order are given, and from them two sets of determinants are formed, namely, first a set of n determinants each of which is in one row identical with A and in the remaining rows with B, and secondly, a set of n determinants each of which is in one column identical with A*

and in the remaining columns with B, then the sum of the first set of determinants is equal to the sum of the second set.

Proof. Let $A = |a_{ij}|$, $B = |b_{ij}|$, $i, j = 1, 2, \cdots, n$, and set

$$h(x) \equiv |a_{ij}x + b_{ij}|, \qquad i, j, = 1, 2, \cdots, n. \tag{4}$$

Let $h_c'(x)$ denote $h'(x)$, as determined from (4), using columns, and let $h_r'(x)$ denote $h'(x)$, as determined from (4), using rows. Since $h_c'(x) \equiv h_r'(x)$, $h_c'(0) \equiv h_r'(0)$ is the statement of Theorem 3.

(2) may also be used to give a new proof of the following theorem, which is due to Le Paige [3, vol. 4, p. 4]:

THEOREM 4. *If A, Y, and Z are the determinants obtained by leaving out from an n by $(n+1)$ array the 1st, the nth, and the $(n+1)$th columns respectively, and if Δ_r is the determinant whose rth row agrees with the rth row of A and whose other rows agree with those of Z, then*

$$\Delta_1 + \Delta_2 + \cdots + \Delta_n = Y. \tag{5}$$

Proof. Let $C_i = [a_{1i}, a_{2i}, \cdots, a_{ni}]^T$, $i = 1, 2, \cdots, n+1$, denote the given array so that

$$A = |C_2C_3 \cdots C_{n+1}|, \quad Y = |C_1C_2 \cdots C_{n-1}C_{n+1}|, \text{ and } Z = |C_1C_2 \cdots C_n|.$$

Set

$$g(x) \equiv |C_1 + C_2x, C_2 + C_3x, \cdots, C_k + C_{k+1}x, \cdots, C_n + C_{n+1}x|. \tag{6}$$

Since $g_c'(x) \equiv g_r'(x)$, $g_c'(0) \equiv g_r'(0)$ yields the result, (5).

Acknowledgment. I want to thank the referee for several notational suggestions which enabled me to simplify the statement of Theorems 1 and 2, and to reduce the length of the original paper.

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AN APPLICATION OF STIELTJES INTEGRATION TO THE POWER SERIES COEFFICIENTS OF THE RIEMANN ZETA FUNCTION

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Let $f(x)$ be continuous on the interval (a, b), let $[x]$ denote the greatest integer in x, and let the integrals be Stieltjes integrals. We examine the following integral,

$$\int_a^b ([x] + 1 - x)df(x).$$

Integrating by parts [1]

$$\int_a^b ([x] + 1 - x)df(x)$$

$$(1) \qquad = \int_a^b d\{([x] + 1 - x)f(x)\} - \int_a^b f(x)d[x] + \int_a^b f(x)dx$$

$$= - \sum_{n=[a]+1}^{[b]} f(n) + \int_a^b f(x)dx + ([b] + 1 - b)f(b) - ([a] + 1 - a)f(a).$$

If in (1) we set $a=1$, $b=N$, where N is a positive integer, and $f(x)=x^{-s}$ with $s>1$, and subsequently permit $N\to\infty$, we have an integral form of the Riemann zeta function [2]

$$(2) \qquad \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{s-1} - \int_1^{\infty} ([x] + 1 - x)d(x^{-s}).$$

Since $|[x]+1-x| \leqq 1$, $\lim_{s\to 1} (s-1)\zeta(s)=1$, and we may write

$$(3) \qquad h(s) = \zeta(s) - \frac{1}{s-1} = \sum_{k=0}^{\infty} A_k(s-1)^k$$

for the Taylor expansion of $h(s)$. In fact,

$$k!A_k = h^{(k)}(1) = (-1)^{k+1} \lim_{N\to\infty} \int_1^N ([x] + 1 - x)d\left(\frac{\ln^k x}{x}\right).$$

If we set $f(x)=(\ln^k x)/x$ in equation (1) with $a=1$ and $b=N$, we obtain

$$k!A_k = (-1)^k \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{\ln^k n}{n} - \int_1^N \frac{\ln^k x}{x} dx \right\}.$$

Thus we have the following result [3].

THEOREM. *If*

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} A_k(s-1)^k$$

then

$$A_k = \frac{(-1)^k}{k!} \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{\ln^k n}{n} - \frac{\ln^{k+1} N}{k+1} \right\}.$$

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NOTE ON PREFERENTIAL ARRANGEMENTS

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In a recent article on Preferential Arrangements, (this MONTHLY, January 1962, page 4), O. A. Gross showed that

$$f_{n+4} \equiv f_n \pmod{10}, \qquad n \geqq 1,$$

where

$$f_n = 1 + \sum_{j=1}^{n-1} \binom{n}{j} f_{n-j}.$$

This periodicity can be extended to more than just the units digit by use of Euler's theorem (if m is any positive integer and if r is any integer prime to m, $r^{\phi(m)} \equiv 1 \pmod{m}$) and the following two properties:

1. The product of two consecutive even integers is divisible by eight.
2. If r is odd, $r^{2n}+1$ is twice an odd integer. These periodicities are:

$$\begin{aligned} f_{n+20} &\equiv f_n \pmod{100}, && n \geqq 2, \\ f_{n+100} &\equiv f_n \pmod{1000}, && n \geqq 3, \\ f_{n+500} &\equiv f_n \pmod{10{,}000}, && n \geqq 4. \end{aligned}$$

The proof of the final congruence is typical. Using Gross's notations, it must be shown that

$$f_{n+500} - f_n \equiv \sum_{r=0}^{\infty} \nabla^r \{ r^n (r^{500} - 1) \} \pmod{10{,}000}, \qquad n \geqq 4,$$

where

$$\nabla^r r^n = \sum_{j=0}^{r} \binom{r}{j} (-1)^j (r-j)^n.$$

By Euler's theorem $r^{500}-1$ is divisible by 625 for all r prime to 5, since $\phi(625) = 500$; and $r^n(r^{500}-1)$ is divisible by 625 for all r, since $n \geqq 4$. If r is even, r^n is divisible by 16 for $n \geqq 4$. Hence

$$r^n(r^{500} - 1) \equiv 0 \pmod{10{,}000}.$$

If r is odd,

$$r^{500} - 1 = (r^{250} + 1)(r^{125} + 1)(r^{125} - 1)$$

is also divisible by 16 by the two properties stated above.

The first two congruences were verified on the IBM 7090. Periodicities in the digits beyond the thousands may be found by repeated application of Euler's theorem.

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RATIONAL ORTHOGONAL MATRICES

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M. Sholander [1] has presented a construction of rational orthogonal matrices of the third order, giving rational expressions for the elements in terms of 4 parameters a, b, c, d. Except for the case $b=0$ these expressions are essentially given by Cayley in 1846 ([2], formulas (19)). The formulas of Sholander are obtained from the formulas of Cayley by setting $\lambda=a/b$, $\mu=c/b$, $\nu=d/b$. Cayley gives the corresponding formulas for orthogonal matrices of the fourth order ([2], formulas (22), (24) and the following formula). The last formula contains an error. A term $-\theta^2$ is missing in each diagonal element. The method of derivation used by Cayley is applicable to matrices of any order. It is based on a connection between skew-symmetric and orthogonal matrices. Cayley did not at that time use the term "matrix" or matrix notation. The main contents of Cayley's paper were given in matrix notation by Turnbull ([3], Chapter 9, Section 4). Turnbull explicitly points out the use of the formulas for construction of rational orthogonal matrices. The Cayley formulas in matrix form are given in other books (for instance [4], Chapter 6, Theorem 10, and [5], Chapter 9, formulas (123)–(126)) without any mention of this application.

We shall present the formulas in a form which is more suited for calculations of explicit expressions for the elements of the matrices and for numerical computations. Let K be a skew-symmetric (real or complex) matrix of arbitrary order, and let I be the identity matrix of the same order. If K has no characteristic root equal to -1, then the matrix

$$P = 2(I + K)^{-1} - I \tag{1}$$

exists and is orthogonal. If K is real, the matrix (1) always exists, since the nonzero characteristic roots of a real skew-symmetric matrix are pure imaginaries. Solving (1) with respect to K we get

$$K = 2(I + P)^{-1} - I. \tag{2}$$

When P is an orthogonal matrix with no characteristic root equal to -1, the right hand side of (2) exists and is a skew-symmetric matrix.

In constructing a skew-symmetric matrix the elements above the main diagonal may be chosen arbitrarily. If we choose rational values for all these elements, the elements of P will all be rational. From (2) it is evident that any rational orthogonal matrix which has no characteristic root equal to -1 may be written in the form (1) with rational skew-symmetric K. In particular any real rational orthogonal matrix which has no characteristic root equal to -1 may be written in the form (1) with real rational skew-symmetric K.

Similarly any (real) rational orthogonal matrix which has no characteristic root equal to 1 may be written in the form

$$P = -2(I - K)^{-1} + I \tag{3}$$

with (real) rational skew-symmetric K.

The problem of constructing rational orthogonal matrices which have both 1 and -1 as characteristic roots does not seem to have been considered in the literature in the general case. We shall give a method of constructing such matrices.

Let R_n be the set of all rational orthogonal matrices of order n. Let R_{n1} be the subset of all matrices of R_n having no characteristic root equal to -1, and let R_{n2} denote the set of all matrices of R_n having a characteristic root equal to -1, in other words, the set of all matrices of R_n which do not belong to R_{n1}.

We have seen that all matrices of R_{n1} may be obtained by (1). We shall show that

(4) *Any matrix of R_{n2} may be obtained by a change of signs of rows of a matrix of R_{n1}.*

Let E be the set of all diagonal matrices X each of whose diagonal elements is 1 or -1. The statement (4) is equivalent to

(5) *Any matrix Q of R_n may be written in the form $Q=XP$ where X belongs to E and where P belongs to R_{n1}.*

In order to prove (5) we shall prove this more general result.

(6) *For any square matrix Q there exists a matrix X belonging to the set E such that $Q+X$ is nonsingular.*

We shall prove (6) by induction. Let Q and X be matrices of order n. Denoting the determinant value of a matrix by *det* we have

$$det(Q + X) = x_1\, det(Q_1 + X_1) + D, \tag{7}$$

where x_1 is the first diagonal element of X, where Q_1+X_1 is the matrix obtained by deleting the first row and the first column of $Q+X$, and where D is the value of $det(Q+X)$ when $x_1=0$, so that D does not depend on x_1. Suppose that (6) holds for all square matrices of order less than n. Since Q_1 and X_1 are of order $n-1$ there exists an X_1 belonging to the set E such that $\det(Q_1+X_1)\neq 0$. With such an X_1, the right hand side of (7) must be different from zero for $x_1=1$ or for $x_1=-1$. Hence (6) holds for matrices of order n. It is trivial that (6) holds for matrices of order 1. Hence we conclude that it holds in general.

When $Q+X$ is nonsingular, $XQ+I$ is also nonsingular, hence XQ does not have -1 as a characteristic root, and the statement (5) follows.

We shall return to the particular case of matrices of the third order. Let S denote the matrix of Sholander with integral values of a, b, c, d. In the case $b\neq 0$, the matrix S is obtained from (1). According to (4) any rational orthogonal matrix of the third order may be written in the form S or may be obtained from a matrix S by a change of signs of one or more of the rows. Let us consider, for instance, the matrix S_1 obtained by a change of signs of the elements of the first row of S. If in the matrix S_1 we replace a by b, b by $-a$, c by d, d by $-c$, we get the matrix $-S$. Similarly we may show that any matrix obtained from S by a change of signs of one or two rows may be written in the form S or $-S$ after

interchanges among the symbols a, b, c, d and changes of signs of some of these symbols. We have thus shown that any rational orthogonal matrix of the third order is given by the matrix of Sholander with a plus or minus sign before it. We know that the set of all matrices of the form S and $-S$ with $b\neq 0$ is identical with the set of all rational orthogonal matrices of third order which do not have both 1 and -1 as characteristic roots. Hence the set of matrices S and $-S$ with $b=0$ is identical with the set of all rational orthogonal matrices of the third order which have both 1 and -1 as characteristic roots.

Acknowledgement. The proof of (6) given here represents a simplification, suggested by the referee, of my original proof.

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UTILIZING GREEN'S FUNCTION TO SOLVE NONHOMOGENEOUS DIFFERENTIAL SYSTEMS

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In this paper Green's function is utilized to solve systems consisting of n linear first order nonhomogeneous ordinary differential equations and n linearly independent boundary conditions that are not necessarily homogeneous.

We begin by recalling some properties of the Green's function of the completely homogeneous system,

$$[D + F(x)]\overrightarrow{y(x)} = \overrightarrow{0}, \qquad a \leqq x \leqq b; \tag{1a}$$

$$A\overrightarrow{y(a)} + B\overrightarrow{y(b)} = \overrightarrow{0}, \tag{1b}$$

where D is the n by n diagonal matrix operator $[d/dx]$; $F(x)$ is an n by n matrix, continuous in x; A, B are n by n matrices whose elements are constants; and $\overrightarrow{y(x)}$ is an n-dimensional column vector. The Green's function, $G(x, s)$, of system (1) is an n by n matrix defined as follows:

(i) the columns of $G(x, s)$ satisfy (1) except at $x=s$, $a<s<b$, so that $[D+F(x)]G(x, s)=0$, $x\neq s$, and $AG(a, s)+BG(b, s)=0$;

(ii) $G(x, s)$ is continuous in x except for the diagonal elements, each of which has a single discontinuity at $x=s$, in such a way that $G(s+, s)-G(s-, s)=I$, where I is the $n\times n$ identity matrix.

It is known [1] that Green's function exists and is unique if and only if system (1) is incompatible.

Suppose that (1) is incompatible and let $Y(x)$ denote a fundamental matrix of (1a), that is, $Y(x)$ is nonsingular and $[D+F(x)]Y(x)\equiv 0$ on $a\leqq x\leqq b$. Then $AY(a)+BY(b)$ is nonsingular and $G(x, s)$ is expressible in the form:

$$(2) \qquad G(x, s) = Y(x)R(s) \quad (a \leqq x < s), \qquad G(x, s) = Y(x)T(s) \quad (s < x \leqq b),$$

where

$$(3) \quad R(s) = -[AY(a) + BY(b)]^{-1}BY(b)Y^{-1}(s); \qquad T(s) = R(s) + Y^{-1}(s);$$

and from (ii)

$$(4) \qquad Y(x)[T(x) - R(x)] = I.$$

The definition of $G(x, s)$ may be extended so that $G(x, a)=Y(x)T(a)$; $G(x, b)=Y(x)R(b)$ where $T(a)$ and $R(b)$ are determined by (3). Since the boundary conditions (1b) are linearly independent, the n by $2n$ matrix $[A, B]$ has at least one nonsingular n by n submatrix C whose successive columns are those, say, in $[A, B]$ having column indices $k_1, k_2, \cdots, k_n$. It is possible to set $C=[A, B]M$ where the only nonzero elements of the $2n$ by n matrix M occur in its n by n submatrix having row indices $k_1, k_2, \cdots, k_n$ and this submatrix is the identity matrix of order n.

Our principal result may now be stated as follows:

THEOREM. *The nonhomogeneous system*

$$(5a) \qquad [D + F(x)]\vec{y}(x) = \vec{f}(x), \qquad a \leqq x \leqq b;$$

$$(5b) \qquad A\vec{y}(a) + B\vec{y}(b) = \vec{c};$$

where $\vec{f}(x)$ is a column vector, continuous in x, and $\vec{c}$ is a constant vector, has the solution

$$(6) \qquad \vec{y}(x) = \int_a^b G(x, s)\vec{f}(s)ds + [G(x, a); -G(x, b)]MC^{-1}\vec{c}.$$

Proof. It is well known [1] that the solution of (5) when $\vec{c}=\vec{0}$ is

$$\vec{y}(x) = \int_a^b G(x, s)\vec{f}(s)ds.$$

To show that (6) is the solution of (5), in general, we therefore need only show that

$$(7) \qquad \vec{y}(x) = [G(x, a), -G(x, b)]MC^{-1}\vec{c}$$

is the solution when $\vec{f}(x)=\vec{0}$, because the system is linear.

Recalling that $Y(x)$ is a fundamental matrix of (1a), it is readily seen that (7) satisfies (5a) when it is homogeneous, for

$$\begin{aligned}[D + F(x)]\vec{y}(x) &= [(D + F(x))Y(x)T(a), -(D + F(x))Y(x)R(b)]MC^{-1}\vec{c} \\ &= [0, 0]MC^{-1}\vec{c} = \vec{0}.\end{aligned}$$

It remains to verify that (7) also satisfies (5b)

To this end we observe, in view of (2) and property (i), that

$$AY(a)R(s) + BY(b)T(s) = 0;$$

hence,

$$-AY(a)R(b) = BY(b)T(b) \quad \text{and} \quad BY(b)T(a) = -AY(a)R(a).$$

We utilize these relations, together with (4) and the definition of M, to write

$$\begin{aligned}&A\vec{y}(a) + B\vec{y}(b) \\ &\quad = [AY(a)T(a), -AY(a)R(b)]MC^{-1}\vec{c} + [BY(b)T(a), -BY(b)R(b)]MC^{-1}\vec{c} \\ &\quad = [AY(a)[T(a) - R(a)], BY(b)[T(b) - R(b)]]MC^{-1}\vec{c} \\ &\quad = [A, B]MC^{-1}\vec{c} = CC^{-1}\vec{c} = \vec{c},\end{aligned}$$

and the theorem is established.

Note that this theorem may also be employed to solve systems consisting of an nth order linear differential equation and n linear boundary conditions.

Example. By means of the substitutions $y_1 = u$, $y_2 = u'$, the system

$$u''(x) - (1/x)\,u'(x) = 3x, \qquad 1 \leqq x \leqq 2, \tag{8a}$$

$$u'(1) = 3, \qquad u'(1) - 2u(2) + 2u'(2) = 11, \tag{8b}$$

transforms into the equivalent system

$$\begin{bmatrix} d/dx & -1 \\ 0 & (d/dx - 1/x) \end{bmatrix}\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 3x \end{bmatrix}, \qquad 1 \leqq x \leqq 2, \tag{9a}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} y_1(2) \\ y_2(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}. \tag{9b}$$

A fundamental matrix of the homogeneous system corresponding to (9a) is

$$Y(x) = \begin{bmatrix} x^2 & 1 \\ 2x & 0 \end{bmatrix}, \quad \text{and its inverse is} \quad Y^{-1}(x) = \begin{bmatrix} 0 & 1/2x \\ 1 & -x/2 \end{bmatrix}.$$

Since

$$AY(1) + BY(2) = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}$$

is nonsingular, the homogeneous system corresponding to (9) is incompatible and therefore has a Green's function. A simple computation reveals that

$$R(s) = \begin{bmatrix} 0 & 0 \\ -1 & s/2 \end{bmatrix}, \qquad T(s) = \begin{bmatrix} 0 & 1/2s \\ 0 & 0 \end{bmatrix},$$

and thus

$$G(x, s) = \begin{bmatrix} -1 & s/2 \\ 0 & 0 \end{bmatrix}, \ 1 \leqq x < s; \qquad G(x, s) = \begin{bmatrix} 0 & x^2/2s \\ 0 & x/s \end{bmatrix}, \ s < x \leqq 2.$$

Selecting the second and third columns of $[A, B]$ as the columns of C and utilizing (6), we obtain

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \int_1^x \begin{bmatrix} 0 & x^2/2s \\ 0 & x/s \end{bmatrix}\begin{bmatrix} 0 \\ 3s \end{bmatrix} ds + \int_x^2 \begin{bmatrix} -1 & s/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3s \end{bmatrix} ds$$

$$+ \begin{bmatrix} 0 & x^2/2 & 1 & -1 \\ 0 & x & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} x^3 \\ 3x^2 \end{bmatrix}$$

and hence the solution of (8) is $u(x) = y_1(x) = x^3$.

Reference

1. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955, p. 296.

ON THE QUADRATIC CHARACTER OF CERTAIN QUANTITIES

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Let a, b be integers, not both zero (mod p), p an odd prime, and x, x' inverse elements (mod p). Let (n/p) be the Legendre symbol. Then the congruence

$$ax - b \equiv -x(bx' - a) \pmod{p}$$

implies

$$\left(\frac{(ax - b)(bx' - a)}{p}\right) = (-1)^{(p-1)/2}\left(\frac{x}{p}\right).$$

But since

$$(ax - b)(bx' - a) \equiv 2ab - (a^2x + b^2x') \pmod{p}$$

we have proved

THEOREM 1. *If a, b are integers, not both zero* (mod p), *p is an odd prime, and x, x' are inverse elements* (mod p); *then*

$$\left(\frac{2ab-(a^2x+b^2x')}{p}\right)=(-1)^{(p-1)/2}\left(\frac{x}{p}\right).$$

From Theorem 1 we can derive some results relating the quadratic character of certain quantities and the quadratic character of two. In this note we shall obtain two such results. First, a connection between the quadratic characters of 2 (mod p) and the product ab, where a and b are integers and p is an odd prime such that p divides a^2+b^2. Second, we deduce a quadratic character relationship between 2 (mod p) and x, where x and x' are inverse elements (mod p) such that $x+x'\equiv 0$ (mod p).

THEOREM 2. *If p is an odd prime such that p divides a^2+b^2, where a and b are integers, neither of which is zero* (mod p), *then the product ab must satisfy* $(ab/p)=(2/p)$.

Proof. In Theorem 1 put $x=x'=1$. Then we have

$$\left(\frac{2ab-(a^2+b^2)}{p}\right)=(-1)^{(p-1)/2}\left(\frac{1}{p}\right)=(-1)^{(p-1)/2},$$

but $2ab-(a^2+b^2)\equiv 2ab$ (mod p) so that

$$\left(\frac{ab}{p}\right)=(-1)^{(p-1)/2}\left(\frac{2}{p}\right).$$

Now, p can be a prime divisor of the sum of two squares only if $p\equiv 1$ (mod 4). Hence it follows that

$$\left(\frac{ab}{p}\right)=\left(\frac{2}{p}\right).$$

THEOREM 3. *If x and x' are inverse elements* (mod p), *p an odd prime, such that* $x+x'\equiv 0$ (mod p), *then x must satisfy* $(x/p)=(2/p)$.

Proof. In Theorem 1 put $a=b=1$, then we have

$$\left(\frac{2-(x+x')}{p}\right)=(-1)^{(p-1)/2}\left(\frac{x}{p}\right).$$

But $x+x'\equiv 0$ (mod p) so that this becomes

$$\left(\frac{2}{p}\right)=(-1)^{(p-1)/2}\left(\frac{x}{p}\right).$$

Now, inverse elements x and x' whose sum is divisible by p exist only if $p\equiv 1$ (mod 4). Therefore we must have

$$\left(\frac{2}{p}\right)=\left(\frac{x}{p}\right).$$

THE REMAINDER TERMS IN NUMERICAL INTEGRATION FORMULAS

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1. Introduction. Let $f \in C^{n+2}$ on an interval of real numbers containing the sequence $a, x_0, \cdots, x_n, b$. We are concerned with the remainder terms associated with a numerical integration formula for estimating $\int_a^b f(t)dt$ in terms of the values of f at $x_0, \cdots, x_n$. Such a remainder may often be expressed in the form

$$R(f) = Af^{(m+2)}(\xi) \tag{1.1}$$

with ξ intermediate to the numbers $a, x_0, \cdots, x_n, b$ and A a constant related to the polynomial

$$P_m(x) = \prod_{j=0}^{m} (x - x_j). \tag{1.2}$$

The value of m is generally n or $n-1$, whichever is even. Estimates of the form (1.1) are available for the Newton-Cotes formulas ([1], p. 73). For n even, this result is generally attributed to Steffensen ([3], p. 154 f) and for n odd, to Walter [4].

The present note is intended to extend the results of Steffensen and Walter by relaxing the equal interval condition. We will require that $x_0 \leqq x_1 \leqq \cdots \leqq x_m$ (if n is odd, x_n is arbitrary in the remainder estimates), that the points be symmetrically distributed about the midpoint, $x_{m/2}$, and that the lengths $x_{i+1} - x_i$ of the intervals be a nonincreasing function of i for $i \leqq m/2 - 1$. In the case that n is even, we will also require that a and b be symmetric about $x_{m/2}$, and either $a = x_0$ or $x_0 - a \geqq x_1 - x_0$. In the odd case, we will assume that $a = x_0$ and $b = x_n$.

The proof that an estimate of the form (1.1) is available depends upon showing that the function

$$Q_m(x) = \int_a^x P_m(t)dt \tag{1.3}$$

is consistently positive or negative throughout (a, b). Then Steffensen's reasoning may be used to obtain the desired estimate for $R(f)$.

Properties of $Q_m(x)$ are derived in Section 2. In the particular case of equally spaced points, a simpler inductive proof of the essential property of $Q_m(x)$ is given. These properties, together with elementary properties of divided differences are used in Section 3 to obtain remainder estimates of the form (1.1), under the conditions described above. The concluding section gives a few examples, including a symmetrically distributed sequence x_0, x_1, x_2, x_3, x_4 for which no remainder estimate of the form (1.1) exists.

2. Inequalities and relations on $Q_m(x)$. Let us now consider a given sequence $x_0, x_1, \cdots, x_n$ and $m=n$ or $n-1$, whichever is even, and assume that it has the properties

$$(2.1)\qquad \text{(a)}\ \frac{x_i+x_{m-i}}{2}=x_{m/2},$$

$$\text{(b)}\ x_i-x_{i-1}\geqq x_{i+1}-x_i>0, \qquad i=1,2,\cdots,m/2.$$

(The essential content of our conclusion is unaltered if we admit the possibility that $x_{i+1}-x_i=0$ for some i's at the center of the range. We also give an example with completely different conditions on $\{x_i\}$.)

If $P_m(x)$ and $Q_m(x)$ are defined by (1.2) and (1.3), then $Q_m'(x)=P_m(x)$, so that $Q_m'(x)$ is positive or negative in the interval (x_i, x_{i+1}) according as i is even or odd. It follows that $Q_m(x)$ is alternately increasing or decreasing in these intervals. In particular, if $x\in(x_{m-1}, x_m)$ then

$$(2.2)\qquad Q_m(x_m)<Q_m(x)<Q_m(x_{m-1}).$$

Since m is even, $P_m(x)$ is an odd function of $x-x_{m/2}$, so $Q_m(x)$ is an even function of $x-x_{m/2}$ and by (1.3):

$$(2.3)\qquad Q_m(b)=Q_m(a)=0, \quad \text{if}\quad \frac{a+b}{2}=x_{m/2}.$$

We will now give a sequence of lemmas leading to the main theorem about the behavior of $Q_m(x)$.

LEMMA 2.1. *Let i be an integer such that $0\leqq i\leqq m/2-1$. Let $x=x_i-u$, $y=x_{i+1}-u$, with $0<u<x_{i+1}-x_i$. Then, if $0\leqq v\leqq i-1$, $(x-x_v)(x-x_{m-v-1})\leqq(y-x_{v+1})(y-x_{m-v})$; if $i\leqq v\leqq m/2-1$, the inequality is reversed. In either case,*

$$|(x-x_v)(x-x_{m-v-1})|\geqq|(y-x_{v+1})(y-x_{m-v})|.$$

Proof. The assertion about absolute values follows from the preceding inequality, for $x-x_{m-v-1}$ and $y-x_{m-v}$ are always negative, while $x-x_v$ and $y-x_{v+1}$ are positive for the range $0\leqq v\leqq i-1$ and negative for $i\leqq v\leqq m/2-1$. Now:

$$(2.4)\qquad \begin{aligned}&(x-x_v)(x-x_{m-v-1})-(y-x_{v+1})(y-x_{m-v})\\&=x^2-y^2-x(x_v+x_{m-v-1})+y(x_{v+1}+x_{m-v})+x_vx_{m-v-1}-x_{v+1}x_{m-v}.\end{aligned}$$

Let $\sigma=x_{v+1}-x_v=x_{m-v}-x_{m-v-1}$ (by the symmetry condition (2.1(a))). When x_{v+1} and x_{m-v-1} are eliminated by these relations, the last expression (2.4) becomes:

$$
\begin{aligned}
&x^2 - y^2 - x(x_v + x_{m-v} - \sigma) + y(x_v + x_{m-v} + \sigma) - \sigma(x_v + x_{m-v}) \\
&\quad = (x + y - x_v - x_{m-v})(\sigma - (y - x)) \qquad (2.5)\\
&\quad = (x + y - 2x_{m/2})((x_{v+1} - x_v) - (x_{i+1} - x_i));
\end{aligned}
$$

we have used the symmetry condition and the equations defining x and y in the last step. The first factor in the last member of (2.5) is always negative in the range we are considering. The desired inequality is then an immediate consequence of the relation (2.1(b)), from which it is clear that the second factor is nonnegative if $0 \leqq v \leqq i-1$, zero if $v=i$, and nonpositive if $i+1 \leqq v \leqq (m/2)-1$.

LEMMA 2.2. *Under the conditions of Lemma* 2.1, $|P_m(x)| > |P_m(y)|$.

Proof. We may compare factors of $P_m(x)$ two-by-two with factors of $P_m(y)$, by using the preceding lemma, with $v=0, 1, \cdots, m/2-1$, and note that all of the factors $(x-x_0), \cdots, (x-x_{m-1})$ are compared to the factors $(y-x_1), \cdots, (y-x_m)$. We need only show that $|x-x_m| > |y-x_0|$. But this is equivalent to $x_m - x > y - x_0$, or $x_m + x_0 > x + y$. Since the left hand side is $2x_{m/2}$, this last inequality holds by our choice of x and y.

LEMMA 2.3. *For any pair of numbers* (c, d), *let* $I(c, d) = \int_c^d P_m(t)dt$. *Then under the conditions of the preceding lemmas:*

(a) $I(x_i, x_{i+1})$ *is positive if i is even, and negative if i is odd.*

(b) $|I(x_0, x_1)| > |I(x_1, x_2)| > \cdots > |I(x_{m/2-1}, x_{m/2})|$.

(c) *If* $x_0 - a \geqq x_1 - x_0$, $|I(a, x_0)| > |I(x_0, x_1)|$.

Proof. (a) is an immediate consequence of the sign of $P_m(x)$ as given by (1.2). For (*b*),

$$
\begin{aligned}
|I(x_{i-1}, x_i)| &= \int_{x_{i-1}}^{x_i} |P_m(t)| \, dt = \int_0^{x_i - x_{i-1}} |P_m(x_i - u)| \, du \\
&\geqq \int_0^{x_{i+1}-x_i} |P_m(x_i - u)| \, du > \int_0^{x_{i+1}-x_i} |P_m(x_{i+1} - u)| \, du \\
&= \int_{x_i}^{x_{i+1}} |P_m(t)| \, dt = |I(x_i, x_{i+1})| .
\end{aligned}
$$

The first inequality in the above argument invokes the condition (2.1(b)) and the second involves the inequality of Lemma 2.2. The proof of (c) is exactly similar; we note that the range of the variable in $I(a, x_0)$ is included in the case $i=0$ in the preceding lemmas.

We also remark that in case (c), $I(a, x_0) < 0$, consistent with the alternating sign pattern given in (a) of this lemma. We now come to the main theorem of this section.

THEOREM 2.1. *Let* $Q_m(x) = \int_a^x P_m(t)dt$, *as in* (1.3). *Then if* $x_0, \cdots, x_m$ *satisfies* (2.1):

(a) *If* $a = x_0$, $Q_m(x) > 0$ *for all x except* $x = a$ *or* $2x_{m/2} - a$.

(b) *If* $x_0 - a \geqq x_1 - x_0$, $Q_m(x) < 0$ *in the range* $a < x < 2x_{m/2} - a$.

Proof. By the evenness of $Q_m(x)$ in the factor $x-x_{m/2}$, it is sufficient to consider the case $x \leqq x_{m/2}$, say $x \in (x_i, x_{i+1})$. Then $Q_m(x)$ is related to the series $I(x_0, x_1)+I(x_1, x_2)+ \cdots +I(x_i, x)$ in case (a), or to the series $I(a, x_0)+I(x_0, x_1) + \cdots +I(x_i, x)$ in case (b). $Q_m(x)$ will consist of a partial sum of one of these two series through the term $I(x_{i-1}, x_i)$ plus a fractional part of the term $I(x_i,x_{i+1})$. In either case, a finite alternating series with decreasing terms results. Its sign must be that of the first term of the series, and the conclusion follows from the last lemma. For $x<x_0$ in case (a), we simply use the fact that $Q_m'(x)=P_m(x)$ is negative in this range, so $Q_m(x)$ is decreasing, and has the value 0 at the upper bound x_0. Then $Q_m(x)$ must be positive for $x<x_0$.

Remark. In the case of equal intervals, a simpler proof of the behavior of $Q_m(x)$ may be obtained by allowing m to take on any integer value $\geqq 1$. It can be shown inductively that if $x_0 \leqq x \leqq x_m$, then $Q_m(x) \leqq 0$ or $Q_m(x) \geqq 0$, according as m is odd or even. To show this, since $P_m(x)=(x-x_m)P_{m-1}(x)$, we have on integrating by parts:

$$Q_m(x) = (x - x_m)Q_{m-1}(x) - \int_a^x Q_{m-1}(t)dt. \tag{2.6}$$

Then if $x_0 \leqq x \leqq x_{m-1}$, the sign of $Q_m(x)$ will be positive or negative according as the sign of $Q_{m-1}(x)$ is negative or positive in that range, since $x-x_m<0$. To extend to the range $x_{m-1}<x\leqq x_m$, we use (2.2). In the odd case $Q_m(x)<Q_m(x_{m-1}) \leqq 0$, and in the even case $Q_m(x)>Q_m(x_m)=0$, because of the condition (2.3). The case $m=1$ which starts the induction is trivial since $(x-x_0)(x-x_1)$ is negative for $x\in(x_0, x_1)$. Equal intervals are needed so the symmetry may be preserved as two upper end points are omitted to give $P_{m-2}(x)$ and $Q_{m-2}(x)$. Equation (2.6) is valid in general, however, and if n is an odd integer, $m=n-1$, $x=a$ and $b>a$, then:

$$Q_n(b) = -\int_a^b Q_{n-1}(t)dt \tag{2.7}$$

and is negative, provided the conditions of Theorem 2.1 are satisfied.

3. The remainder estimates. Let $f(x)\in C^{n+2}$ as before. The *divided difference* $f(y_0, y_1, \cdots, y_m)$ of $f(x)$ associated with any number sequence $y_0, y_1, \cdots, y_m$, where $m\leqq n+2$, is defined to be the coefficient of x^m in the unique polynomial $p(x)$ of degree $\leqq m$ such that $f(x)-p(x)$ has a zero at each point y of its domain of multiplicity at least equal to the number of times y occurs in the sequence $y_0, y_1, \cdots, y_m$. The symmetry of $f(y_0, y_1, \cdots, y_m)$ in its arguments is immediately obvious, and it is easy to derive from this definition the usual properties:

$$\begin{aligned} &f(y_0, \cdots, y_{m-1}, y_m) - f(y_0, \cdots, y_{m-1}, y_m') \\ &\qquad = (y_m - y_m')f(y_0, \cdots, y_{m-1}, y_m, y_m'), \end{aligned} \tag{3.1}$$

$$\partial/\partial y_m f(y_0, \cdots, y_m) = f(y_0, \cdots, y_m, y_m), \tag{3.2}$$

$$f(y_0, \cdots, y_m) = \frac{1}{m!} f^{(m)}(\xi), \tag{3.3}$$

ξ intermediate to the range of values $y_0, \cdots, y_m$. If $g(x) \in C^{n+2}$ is a function that already has zeros of the indicated orders at the points $y_0, \cdots, y_m$, then for any y not in the sequence $y_0, y_1, \cdots, y_m$,

$$h(x) = \left(g(y) \prod_{j=1}^{m} (x - y_j)\right) \Big/ \prod_{j=0}^{m} (y - y_j)$$

is the polynomial that "fits" with $g(x)$ on the sequence $y_0, \cdots, y_m, y$, so that

$$g(y_0, \cdots, y_m, y) = g(y) \Big/ \prod_{j=0}^{m} (y - y_j)$$

and

$$g(y) = \left\{\prod_{j=0}^{m} (y - y_j)\right\} g(y_0, \cdots, y_m, y). \tag{3.4}$$

But (3.4) is also valid even if $y = y_i$ for any i since both sides of that relation then reduce to zero.

To estimate the remainder term in an integration formula involving the values of $f(x)$ (or its derivatives if a point is repeated) at the points $x_0, \cdots, x_n$, let $p(x)$ be the polynomial of degree $\leqq n$ that fits $f(x)$ at the points $x_0, \cdots, x_n$ and let $g(x) = f(x) - p(x)$. Then $R(f) = R(g) + R(p)$. In most cases, the integration formula is defined to be $\int_a^b p(t)dt$ so that $R(p) = 0$, and hence $R(f) = R(g)$. Using (3.4), this implies for any $m \leqq n$ that

$$R(g) = \int_a^b P_m(t) g(x_0, \cdots, x_m, t) dt, \tag{3.5}$$

with $P_m(t)$ as given by (1.2). We may integrate by parts, using (3.2), to obtain

$$R(g) = Q_m(b) g(x_0, \cdots, x_m, b) - \int_a^b Q_m(t) g(x_0, \cdots, x_m, t, t) dt. \tag{3.6}$$

The first term drops out if either $Q_m(b) = 0$ or $g(x_0, \cdots, x_m, b) = 0$. The following theorem gives some cases in which a mean value estimate of the form (1.1) may be found.

THEOREM 3.1. *Let $m = n$ or $n - 1$, whichever is even, and let the sequence $x_0, \cdots, x_m$ satisfy the conditions* (2.1) *of the preceding section. Then the remainder $R(g)$ has an estimate of the form:*

$$R(g) = \frac{-1}{(m+2)!} f^{(m+2)}(\xi) \int_a^b Q_m(t) dt, \tag{3.7}$$

with ξ intermediate to the numbers $a, x_0, \cdots, x_m, b$ in both of the following cases:

(i) $n=m$, $x_{m/2}=(a+b)/2$ *and either* $a=x_0$ *or* $x_0-a\geqq x_1-x_0$.
(ii) $n-1=m$, $a=x_0$, $x_n=b$, *which may be arbitrary.*

Proof. In case (i), $Q_m(b)=Q_m(a)=0$ by (2.3) and in case (ii), $g(x_0,\cdots,x_m,b)=g(x_0,\cdots,x_n)=0$ by the definition of $g(x)$. In either situation, the first term of (3.6) drops out. The conditions of Theorem 2.1 are satisfied, and so $Q_m(t)$ is either consistently positive or consistently negative throughout (a,b). The first mean value theorem for integrals then gives an estimate of the form:

$$R(g) = -g(x_0,\cdots,x_m,\eta,\eta)\int_a^b Q_m(t)dt, \qquad \text{with } \eta\in(a,b). \tag{3.8}$$

The divided difference $g(x_0,\cdots,x_m,\eta,\eta)$ may be replaced by

$$\frac{1}{(m+2)!}g^{(m+2)}(\xi)$$

by (3.2). Finally, $g^{(m+2)}(\xi)$ may be replaced by $f^{(m+2)}(\xi)$ because $p^{(m+2)}(x)=0$, since $p(x)$ is a polynomial of degree $\leqq m+1$.

4. Some examples. In the following, we give some examples of integration formulas with remainder using the theory of Section 3 along with examples indicating limitations on other extensions of this theory. The first two give generalizations of the usual Simpson's rule and the three-eighths rule. The third example considers repeated points in the sequence $x_0, x_1,\cdots,x_n$ and the final example gives a sequence for which no remainder estimate of the form (1.1) is available.

(a) *Generalized Simpson's Rule.* Let $a\leqq x_0<x_1<x_2\leqq b$ with $x_0-a=b-x_2=s$ and $x_2-x_1=x_1-x_0=t$. From the theory of Section 3,

$$\begin{aligned}\int_a^b f(x)dx = \frac{(s+t)}{3t^2}\{(s+t)^2f_0+2[3t^2-(s+t)^2]f_1+(s+t)^2f_2\}\\ -\frac{1}{180}(s+t)^3[5t^2-3(s+t)^2]f^{(4)}(\xi)\end{aligned} \tag{4.1}$$

if $s=0$ or $s\geqq t$. Here, $f_i=f(x_i)$. Actually (4.1) holds if $s\geqq(\sqrt{2}-1)t$, even though the hypotheses of Theorem 3.1 are not satisfied for the entire range of s.

(b) *The Generalized Three-Eighths Rule.* Let x_0, x_1, x_2, x_3 denote the four points to be considered and let f_0, f_1, f_2, f_3 denote the corresponding values of $f(x)$. We take $x_1-x_0=x_2-x_1=h$ and, for the moment, $x_3-x_2=sh$, $s>0$. Then

$$\begin{aligned}&\int_{x_0}^{x_3} f(x)dx\\ &=\frac{h}{24}\left\{(s+2)(s^2-2s+4)f_0+\frac{2(s+2)^3(2-s)}{s+1}f_1+(s+2)^3f_2+\frac{6s(s+2)}{s+1}f_3\right\}\\ &\qquad\frac{-h^5}{1440}(s+2)^3(3s^2-3s+2)f^{(4)}(\xi).\end{aligned} \tag{4.2}$$

If $s=1$, (4.2) reduces to the usual three-eighths rule for equal intervals and if $s=2$, it reduces to Simpson's rule. Theorem 3.1 shows that (4.2) is also valid for $s=0$ in which case it again reduces to Simpson's rule.

(c) *Coincident Points.* Consider the sequence a, x_0, x_0, x_1, x_1, b with $a<x_0<x_1<b$. We take $x_1-x_0=s$ and $b-a=2\mu s$, $\mu>1$ with $b-x_1=x_0-a$. Now $P_4(x)=(x-x_0)^2(x-x_1)^2\geqq 0$ so that $Q_4(x)>0$, $a<x<b$. Then $\int_a^b f(x)dx$ may be expressed in the form

$$W_0f_0+W_1f_1+W_2f'(x_0)+W_3f'(x_1)+Af^{(4)}(\xi), \tag{4.3}$$

in agreement with the expected form for repeated points. Upon applying (4.3) successively to the functions 1, x, x^2, x^3, and x^4, we obtain

$$\begin{aligned} \int_a^b f(x)dx &= \mu s\{f_0+f_1\} - \frac{\mu(\mu^2-3)}{6}s^2\{f'(x_0)-f'(x_1)\} \\ &\quad + \frac{2\mu s^5}{4!}\left\{\frac{\mu^4}{5}-\frac{2}{3}\mu^2+1\right\}f^{(4)}(\xi). \end{aligned} \tag{4.4}$$

(d) *A Poor Distribution.* If we consider the sequence $a=-(7/3)^{1/2}$, -1, 0, 1, $(7/3)^{1/2}=b$, we obtain

$$P_4(x) = x(x^2-1)(x^2-7/3) \text{ and}$$

$$Q_4(x) = \frac{1}{6}\left\{x^6-5x^4+7x^2-\frac{49}{27}\right\}.$$

Finally,

$$\int_a^b Q_4(x)dx = \frac{4}{105}b^5(3b^2-7)=0.$$

Since evidently, $R(f)$ is not always 0, a remainder estimate of the form (1.1) is not available for a numerical integration of the type considered for this sequence.

References

1. F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill, New York, 1956.
2. K. Kunz, Numerical Analysis, McGraw-Hill, New York, 1957.
3. J. F. Steffensen, Interpolation, Williams and Wilkins, Baltimore, 1927.
4. A. Walter, Zur Numerischen Integration, Skand. Aktuarietidskr., 1925.

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Allen L. Shields, Department of Mathematics, University of Michigan, Ann Arbor, Michigan.

LOGICAL QUANTIFIERS: AN AID TO CLEAR THINKING

WASFI A. HIJAB, American University of Beirut, Lebanon

The universal quantifier $\forall$, "for every," and the existential quantifier $\exists$, "there exists," have not as yet been generally recognized as part and parcel of the standard mathematical terminology employed in the exposition of material offered at the undergraduate or graduate levels. Even in those mathematical texts where quantifiers are introduced and explained, one is generally left with the feeling that this is being done for ornamental purposes, since their usefulness is not substantiated by subsequent use. The aim of this note is to indicate a familiar situation that is frequently met in undergraduate (and more frequently in more advanced) courses, a situation where the logical quantifiers, notationally modified, offer a valuable aid to clear thinking. This situation arises when one considers *what exactly is involved in the denial of a complicated statement that includes several clauses beginning with "for every" or "there exists."*

Let us consider an example. Take the definition that A is the limit of the sequence $\{a_n\}$, "A is said to be the limit of $\{a_n\}$ if and only if for each $\epsilon>0$ there exists a positive integer N such that, for all $n>N$, $|a_n-A|<\epsilon$." Suppose now that we need to show, in the course of proving a theorem or solving a problem, that a given A is *not* the limit of a given sequence $\{a_n\}$. Even an experienced person, let alone an undergraduate, will hesitate for a moment before stating what exactly is involved in this. What do we have to show about A that will justify the claim that A is *not* a limit of the sequence? But before one is clear about this, it would be difficult to think of a line of attack. However, if we employ the quantifier language and apply the rule for their denials, we are able with the utmost ease and almost mechanically to arrive at what we are looking for.

For this purpose let us introduce the following notation for the logical quantifiers. This notation is different from the usual one in that it puts the variable as a subscript to, rather than in parentheses after, the two main symbols. This makes them more manageable for the present purpose.

DEFINITION. $\forall_{x\in S}(P)\equiv$ *For every x in S, P is true.*

DEFINITION. $\exists_{x\in S}(P)\equiv$ *There exists an x of S such that P is true.*

Here P is a propositional form that becomes a proposition whenever x is re-

placed by one of its particular values. Reference to set S may be suppressed whenever the context is sufficiently clear.

The rule for getting the contradictories of quantified propositions is

$$\sim \forall_{x \in S}(P) \equiv \exists_{x \in S}(\sim P),$$
$$\sim \exists_{x \in S}(P) \equiv \forall_{x \in S}(\sim P).$$

"$\sim$" stands for the contradictory operator "Not" or "It is not the case that." In words, to get the contradictory (denial, or negation) of a quantified proposition replace the universal (existential) quantifier by the existential (universal) quantifier and replace the associated propositional form by its contradictory.

Let us apply this to our example. The definition of the limit can be stated in quantifier language as follows:

DEFINITION. *A is said to be the limit of a sequence $\{a_n\}$ if and only if*

$$\forall_{\epsilon>0}(\exists_{N \epsilon I}(\forall_{n>N}(|A - a_n| < \epsilon))).$$

This gives us a neater formulation of the definition, reflecting at the same time the relations among the various clauses involved. This in itself should invite us to make more use of quantifier language. The decided advantage of such a language is apparent, however, when we try to formulate exactly what is involved in saying that A is not the limit of the sequence $\{a_n\}$. This is now straight forward. All we have to do is to apply the above rule to the definition:

COROLLARY. *A is not the limit of a sequence $\{a_n\}$ if and only if*

$$\exists_{\epsilon>0}(\forall_{N \epsilon I}(\exists_{n>N}(|A - a_n| \geqq \epsilon))).$$

Now, this can be translated into ordinary language, if desired: "A is not the limit of a sequence $\{a_n\}$ if and only if there exists an $\epsilon>0$ such that, for every positive integer N, there exists an $n>N$ for which $|A-a_n| \geqq \epsilon$".

The same situation arises frequently when we want to prove a theorem by contradiction. The first step in such a proof is to pose the contradictory of what the theorem asserts. If the theorem is a complicated statement involving several "for every" and "there exists" clauses, then the quantifier language provides us with an invaluable aid for carrying out this first step in an almost mechanical manner.

As an example of this kind, consider the following

LEMMA. *If K is a closed and bounded set of real numbers and $\{G_\alpha\}$ is an open covering of K, then there exists a fixed $\delta>0$ such that, for any point x of K, we can find an index α such that the open interval $(x-\delta, x+\delta)$ is contained in G_α.*

The proof by contradiction begins by assuming the contradictory of the lemma's conclusion. Let us first put the lemma's conclusion in quantifier language:

$$\exists_{\delta>0}(\forall_{x \epsilon K}(\exists_\alpha((x - \delta, x + \delta) \subset G_\alpha))).$$

Now it is a mechanical matter to apply the above rule: Contradictory of lemma's conclusion in quantifier language:

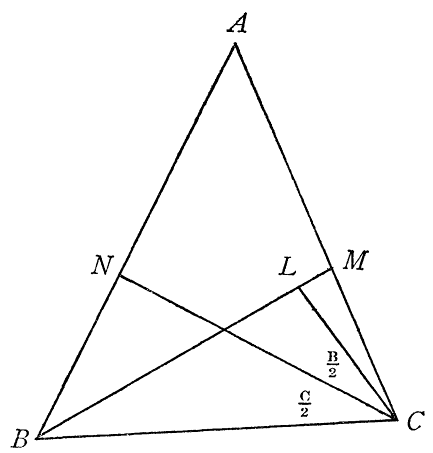
$$\forall_{\delta>0}(\exists_{x\epsilon K}(\forall_{\alpha}((x-\delta,\ x+\delta) \not\subset G_{\alpha}))).$$

This can be rendered into ordinary language. "For every $\delta>0$, there exists an x of K such that, for every index α, the open interval $(x-\delta,\ x-\delta)$ is not included in G_{α}."

Conclusion. The language of logical quantifiers, as modified above, provides not only a much neater formulation, that reveals the relationships among the various "for every" and "there exists" clauses involved in the more complicated mathematical definitions and theorems, but also an invaluable aid for writing down in an almost mechanical manner exactly what is involved in denying such definitions or theorems. The author feels that more frequent employment of the quantifier notation in texts and classroom discussion will contribute to clearer writing and thinking in mathematics.

THE STEINER-LEHMUS THEOREM

G. Gilbert, B.I.C.C. Research Organization, London and
D. MacDonnell, Fenlow Electronics, Weybridge, England



Theorem. *Any triangle having two equal internal angle bisectors (each measured from a vertex to the opposite side) is isosceles.*

Proof. Let ABC be the triangle with equal angle bisectors BM and CN, as in the figure. If the angles B and C are not equal, one must be less, say $B<C$. Take L on BM so that $\angle LCN=\frac{1}{2}B$. Since this is equal to $\angle LBN$, the four points L, N, B, C are concyclic (lie on a circle). Since

$$B<\tfrac{1}{2}(B+C)<\tfrac{1}{2}(A+B+C),$$

$\angle CBN < \angle LCB < 90°$. Since smaller chords of a circle subtend smaller acute angles, and $BL < CN$,

$$\angle LCB < \angle CBN.$$

We thus have a contradiction.

Editorial note. Martin Gardner, in his review of Coxeter's *Introduction to geometry* (Scientific American, 204 (1961) 166–168) described this famous theorem in such an interesting manner that hundreds of readers sent him their own proofs. He took the trouble to refine this massive lump of material until only the above gem remained. This theorem was proposed in 1840 by C. L. Lehmus, and proved by Jacob Steiner. For its history until 1940 see J. A. McBride, Edinburgh Math. Notes, 33 (1943) 1–13.

THE CANTOR EXPANSION OF REAL NUMBERS

STEFAN DROBOT, University of Notre Dame

The Cantor expansion of a real number α in a given base-sequence $\{b_n\}$ of natural numbers $b_n \geqq 2$ is

$$\alpha = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \cdots b_n} \tag{1}$$

with a_0 an integer and nonnegative integers (digits) $a_n \leqq b_n - 1$, $n \geqq 1$.

The following formula proves to be useful in establishing irrationality of some numbers:

$$a_n = [b_n b_{n-1} \cdots b_2 b_1 \alpha] - b_n [b_{n-1} \cdots b_2 b_1 \alpha] \tag{2}$$

in which $[\xi]$ denotes the greatest integer not exceeding ξ. Here are some examples.

1. If $b_n = n+1$, $a_0 = 2$, $a_n = 1$, the Cantor expansion (1) represents the number e, the irrationality of which follows by (2) immediately: if $e = r/q$ take $n = q$ to get the contradiction $1 = 0$.

2. In an analogous way one can prove the irrationality of the numbers: sinh 1, cosh 1, and $I_k(1)$ and $I_k(2)$ $(k = 0, 1, 2, \cdots)$ for the Bessel functions

$$I_k(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{2^n n!(n+k)!}.$$

3. If b_n is the nth prime p_n in the natural sequence and $a_n = 1$, the irrationality of the number

$$\epsilon = \sum_{n=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_n}$$

can be proved from (2). If $\epsilon = r/q$ choose n in (2) so that $q \leqq p_{n-1}$. If all the prime

factors of q occur with exponent 1 only, formula (2) gives the contradiction $1=0$. If some prime factors occur with exponent higher than 1, write

$$p_n p_{n-1} \cdots p_2 p_1 \frac{r}{q} = A + \frac{a}{s}, \qquad p_{n-1} \cdots p_2 p_1 \frac{r}{q} = B + \frac{b}{s}$$

with natural numbers A, B, $1 \leqq a < s$, $1 \leqq b < s$. It follows that

$$\text{(i)} \qquad 2s < q$$

because not all primes would cancel with the prime factors of q. Thus, formula (2) would give $1 = A - p_n B$ whence, in view of $A + (a/s) = p_n(B + (b/s))$ it would follow that $p_n = (a+s)/b < 2s$, in contradiction to (i).

It is well known (see, e.g. [1]) that a sufficient condition for an infinite Cantor expansion to represent an irrational number is that each prime divides infinitely many of the b_n's. The number e gives an extreme example showing that the condition is not necessary. If a Cantor expansion of π were known it would yield an elementary proof (without using integrals) that π is irrational.

Reference

1. I. Niven, Irrational Numbers, Carus Mathematical Monograph no. 11, 1956, pg. 10–11.

A NOTE ON THE DERIVATION OF RODRIGUES' FORMULAE

JAMES M. HORNER, University of Alabama

In the study of special functions, solutions to differential equations in the form of Rodrigues' Formulae are of considerable interest. The following elementary method provides the derivation of these formulae for a particular class of second order differential equations.

Suppose we have the differential equation

$$\text{(1)} \qquad (Ax^2 + Bx + C)y'' + (Dx + E)y' + Fy = 0,$$

where A, 5, C, D, E, and F are independent of x, and the associated equation

$$\text{(2)} \qquad As^2 + (A - D)s + F = 0$$

has a positive integral root, say $s = j$.

Construct a second differential equation

$$\text{(3)} \qquad (Ax^2 + Bx + C)z'' + [(D - 2Aj)x + (E - Bj)]z' = 0,$$

whose solution is

$$\text{(4)} \qquad z' = K(Ax^2 + Bx + C)^j \exp\left\{-\int \frac{Dx + E}{Ax^2 + Bx + C}\, dx\right\},$$

where K is an arbitrary constant.

Now differentiate (3) j times. It is easily verified that this result is

$$(Ax^2 + Bx + C)z^{(j+2)} + (Dx + E)z^{(j+1)} \tag{5}$$
$$+ [j(j-1)A + j(D - 2jA)]z^{(j)} = 0.$$

Simplification gives exactly (1), with y replaced by $z^{(j)}$, since j is a root of (2). Therefore

$$y = K\frac{d^{j-1}}{dx^{j-1}}(Ax^2 + Bx + C)^j \exp\left\{-\int \frac{Dx + E}{Ax^2 + Bx + C}\, dx\right\} \tag{6}$$

is a solution to (1).

This result may also be used to prove the orthogonality of a general class of polynomials. Suppose the quadratic $Ax^2+Bx+C=0$ has distinct real roots $x=a$ and $x=b$. By the above results, when n is a nonnegative integer, it follows that

$$y_n = K_n \frac{d^n}{dx^n}(Ax^2 + Bx + C)^n \tag{7}$$

is a solution to

$$(Ax^2 + Bx + C)y'' + (2Ax + B)y' - n(n+1)Ay = 0. \tag{8}$$

Using the usual proof it is easily shown from the differential equation that the y_n form an orthogonal set on $[a, b]$ when A, B, and C are independent of n.

MATRICES APPLIED TO RELATIVE MOTION

Captains R. L. EISENMAN AND D. R. BARR, Air Force Academy

In vector analysis the student often is troubled by the Coriolis and centripetal terms which occur in the formula relating accelerations in two bases. The standard derivation begins with a lemma relating first derivatives in relative motion and reapplies that lemma to relate second derivatives.

This note uses elementary matrix ideas to derive the following strengthened form of the relation between first derivatives in case of relative motion of bases: *The difference between derivatives of a vector in two bases is the image of that vector under a linear transformation. The matrix of the linear transformation is the so-called "product derivative," $\dot{M}M^{-1}$, of the matrix, M, of the change of basis. In order that the product derivative be skew-symmetric, it is necessary as well as sufficient that $M=TC$ where T is orthogonal and C is a constant nonsingular matrix.*

Remarks on Notation.

(a) The vector $\mathbf{F}=f_1\mathbf{a}_1+f_2\mathbf{a}_2+\cdots+f_n\mathbf{a}_n$ may be written as a formal matrix product $\mathbf{F}=f\mathfrak{A}$, where $f=(f_1f_2\cdots f_n)$, $\mathfrak{A}=(\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n)^t$ is a fixed basis, and the superscript t denotes transpose. If $\mathfrak{B}$ is another basis for the same n-dimensional space, then $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ may be expressed as linear combinations of the vec-

tors of $\mathcal{B}$ and this system of simultaneous linear vector equations may be written as $\mathcal{A}=M\mathcal{B}$ where M is a nonsingular n by n matrix which we assume to be continuously differentiable. Then $\mathbf{F}=f\mathcal{A}=f(M\mathcal{B})=(fM)\mathcal{B}$.

(*b*) The derivative of $\mathbf{F}$ in the basis $\mathcal{A}$ is found by differentiating the matrix f of scalar components. Thus, by definition, $\dot{\mathbf{F}}_{\mathcal{A}}=\dot{f}\mathcal{A}$ and $\dot{\mathbf{F}}_{\mathcal{B}}=\dot{\overline{(fM)}}\mathcal{B}$.

(c) If $\mathcal{A}$ is an orthonormal right-handed basis in three dimensions, the cross product of $(f_1f_2f_3)\mathcal{A}$ with $(\omega_1\omega_2\omega_3)\mathcal{A}$ is

$$\mathbf{F}\times\mathbf{\Omega} = (f_1f_2f_3)\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\mathcal{A}.$$

Thus note that for orthonormal right-handed bases in three dimensions, cross-product with a fixed vector is equivalent with a skew-symmetric linear transformation, since the fixed vector and the skew-symmetric linear transformation are directly deducible from each other.

LEMMA 1. *If* $\mathbf{F}=f\mathcal{A}$ *and* $\mathcal{A}=M\mathcal{B}$, *then* $\dot{\mathbf{F}}_{\mathcal{B}}-\dot{\mathbf{F}}_{\mathcal{A}}=f(\dot{M}M^{-1})\mathcal{A}$.

Proof. $\dot{\mathbf{F}}_{\mathcal{B}}-\dot{\mathbf{F}}_{\mathcal{A}}=(fM)^{\boldsymbol{\cdot}}\mathcal{B}-\dot{f}\mathcal{A}=\dot{f}M\mathcal{B}+f\dot{M}\mathcal{B}-\dot{f}\mathcal{A}=f\dot{M}\mathcal{B}=f(\dot{M}M^{-1})\mathcal{A}$.

LEMMA 2. *If M is orthogonal (or $M=TC$ where T is orthogonal and C is a nonsingular constant matrix), then $\dot{M}M^{-1}$ is skew-symmetric.*

Proof. Differentiating the identity $MM^t=I$ gives $\dot{M}M^t+M\dot{M}^t=0$, whence $\dot{M}M^{-1}=-(\dot{M}M^{-1})^t$ or $\dot{M}M^{-1}$ is skew-symmetric. The case $M=TC$ follows as a corollary.

LEMMA 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be orthonormal right-handed bases in three dimensions and let $\mathcal{A}$ and $\mathcal{B}$ be related by rotation. Then there is an angular velocity vector $\mathbf{\Omega}$ for which* $\dot{\mathbf{F}}_{\mathcal{B}}-\dot{\mathbf{F}}_{\mathcal{A}}=\mathbf{F}\times\mathbf{\Omega}$.

Proof. We are given $\mathcal{A}=M\mathcal{B}$ with M orthogonal. Lemma 1 shows that the difference in derivatives is a linear transformation, Lemma 2 shows that $\dot{M}M^{-1}$ is skew-symmetric, and Remark (c) shows that skew-symmetry of $\dot{M}M^{-1}$ makes $f(\dot{M}M^{-1})\mathcal{A}=\mathbf{F}\times\mathbf{\Omega}$, where $\mathbf{\Omega}$ is formed from $\dot{M}M^{-1}$. ($\mathbf{F}\times\mathbf{\Omega}$ is the same vector in both frames because both frames are right-handed orthonormal.)

Remarks. Lemma 3 is the relation between first derivatives usually used to derive the well-known relation between second derivatives:

$$\ddot{\mathbf{F}}_{\mathcal{B}}-\ddot{\mathbf{F}}_{\mathcal{A}} = 2(\dot{\mathbf{F}}_{\mathcal{A}}\times\mathbf{\Omega}) + [(\mathbf{F}\times\mathbf{\Omega})\times\mathbf{\Omega}] + \mathbf{F}\times\dot{\mathbf{\Omega}}.$$

Without restricting oneself to rotations of orthonormal bases, the analogous more general relation between second derivatives,

$$\ddot{\mathbf{F}}_{\mathcal{B}}-\ddot{\mathbf{F}}_{\mathcal{A}} = 2\dot{f}(\dot{M}M^{-1})\mathcal{A}+f(\dot{M}M^{-1})^2\mathcal{A}+f(\dot{M}M^{-1})^{\boldsymbol{\cdot}}\,\mathcal{A},$$

can be found directly by reapplication of Lemma 1 and the fact that differentiation of the identity $MM^{-1}=I$ gives $(M^{-1})^{\cdot}=-M^{-1}\dot{M}M^{-1}$.

Lemmas 1 and 2 apply in any number of dimensions. In three dimensions orthogonality leads to a crossproduct. In general, orthogonality of M is sufficient for skew-symmetry of $\dot{M}M^{-1}$. Is it also necessary?

The following is a converse of Lemma 2.

LEMMA 4. *If $\dot{M}M^{-1}$ is skew-symmetric and continuous, then $M=TC$ with T orthogonal and C constant.*

Proof. By standard theory of matrix differential equations any two solutions T and T^* of $\dot{M}M^{-1}=K$ are related by $T=T^*C$ where C is constant. Thus it suffices to show that $\dot{M}M^{-1}=K$ has one orthogonal solution in case K is skew-symmetric. For this purpose let T be the solution satisfying the initial condition $T(0)=I$, i.e., T is the so-called matrizant whose existence is proved in the elementary theory of matrix differential equations. We will show that this T remains orthogonal for all values of time. If we show that $(T^t)^{-1}$ is also a solution of $\dot{M}M^{-1}=K$, it will follow that $T=(T^t)^{-1}C$, and since $C(0)=I$, $C=I$, hence $TT^t=I$. Thus the proof will be completed by showing that if $\dot{T}T^{-1}=K$ and K is skew-symmetric, then $((T^t)^{-1})^{\cdot}(T^t)^{-1})^{-1}=K$. But

$$\begin{aligned}((T^t)^{-1})^{\cdot}((T^t)^{-1})^{-1} &= ((T^{-1})^{\cdot})^tT^t = (T(T^{-1})^{\cdot})^t = (T(-T^{-1}\dot{T}T^{-1}))^t \\ &= (-\dot{T}T^{-1})^t = -K^t = K.\end{aligned}$$

Concluding Remarks. Lemmas 1, 2 and 4 provide the theorem stated in the introductory paragraph of this note. The number of dimensions is arbitrary, the bases need not be orthonormal or right-handed, and the relation between orthogonality and skew-symmetry is shown to be necessary as well as sufficient.

For several semesters students at the Air Force Academy have responded favorably to direct uses of matrices in vector analysis such as in Lemmas 1–3. In fact the ideas developed from a student's concern over the apparent breach between matrix theory and classical vector analysis.

Reference

1. R. L. Eisenman, Matrix Vector Analysis, McGraw-Hill (forthcoming).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED by HOWARD EVES, University of Maine
Collaborating Editor: C. W. DODGE, University of Maine.

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1556. *Proposed by R. A. Cunninghame-Green, Leo Computers Ltd., England*

A man throws a rough stick 13 inches long into an upright cylindrical hatbox a foot high and 10 inches in diameter. What is the probability that he cannot now close the lid?

E 1557. *Proposed by D. R. Hayes, Duke University*

If $S_n = (1/n^2)\sum_{k=0}^{n} \log \binom{n}{k}$, evaluate $S = \lim_{n\to\infty} S_n$.

E 1558. *Proposed by M. V. Subbarao, University of Missouri*

A positive integer n is called a *balanced number* if it satisfies the equation $\sigma(n)/t(n) = n/2$, where $\sigma(n)$ and $t(n)$ denote as usual the sum and number of divisors of n respectively. Prove that 6 is the only balanced number.

E 1559. *Proposed by Omar Khayyam, Jr., University of California at Berkeley*

Prove there exists no finite group every nonidentity element of which commutes with exactly half of the elements of the group.

E 1560. *Proposed by Helen M. Marston, Douglass College*

Equilateral triangle PQR of side s has vertex Q on the x-axis and R on the y-axis. What is the locus of P?

E 1561. *Proposed by R. B. Killgrove, San Diego State College*

Find an explicit example of a positive nth order determinant whose elements are ones and minus ones and which has minimum possible value.

E 1562. *Proposed by D. Pedoe, Purdue University*

If ABC and $A'B'C'$ are any two triangles, a, b, c and a', b', c' their respective sides, and K, K' their respective areas, prove that

$$a^2(-a'^2 + b'^2 + c'^2) + b^2(a'^2 - b'^2 + c'^2) + c^2(a'^2 + b'^2 - c'^2) \geqq 16KK',$$

with equality if and only if the two triangles are similar.

SOLUTIONS

The Great Mathematician and the Poor Nut

E 1511 [1962, 311]. *Proposed by Underwood Dudley and Arnold Lebow, University of Michigan*

"This system of n linear equations in n unknowns," said the Great Mathematician, "has a curious property."

"Good heavens!" said the Poor Nut, "What is it?"

"Note," said the Great Mathematician, "that the constants are in arithmetic progression."

"It's all so clear when you explain it!" said the Poor Nut. "Do you mean like $6x+9y=12$ and $15x+18y=21$?"

"Quite so," said the Great Mathematician, pulling out his bassoon. "Indeed, the system has a unique solution. Can you find it?"

"Good heavens!" cried the Poor Nut, "I am baffled."

Are you?

Solution by David Rothman, Electronic Specialty Co., Los Angeles, Calif. If $n \geqq 3$, the system is dependent and the solution is not unique. Hence $n<3$. But the term "system" implies $n>1$. Hence $n=2$. If the equations are

$$ax + (a+d)y = a + 2d,$$
$$(a+3d)x + (a+4d)y = a + 5d,$$

then $x=-1$, $y=2$.

Also solved by Jack Abad, Louise R. Alger, E. R. Barnes, P. B. Baumer, Joseph Beer and Perry Scheinok (jointly), E. D. Bender, D. M. Bloom, J. L. Botsford, Robert Bowen, Brother Joseph Heisler, Brother T. C. Wesselkamper, Brother Louis Zirkel, C. Y. Chao and A. G. Konheim (jointly), D. I. A. Cohen, R. J. Cormier, T. R. Curry, Frank Dapkus, Monte Dernham, H. J. de St. Germain, Jane Di Paola, C. W. Dodge, R. E. Edwards, J. W. Ellis, Jane Evans, D. O. Faus, Seymour Geisser, D. P. Giesy, Todd Gitlin, Michael Goldberg, L. D. Goldstone, R. M. Grassl, Ralph Greenberg, J. C. Hickman, A. R. Hyde, R. A. Jacobson, Alan Judson, Erwin Just and Norman Schaumberger (jointly), A. J. Keeping, E. S. Keeping, T. E. Lewis, Julius Lieblein, G. E. Lindamood, Esther A. Linfield, A. M. Linn, W. R. McEwen, Robert Maas, D. C. B. Marsh, E. V. Martin, G. M. Merriman, D. A. Moran, J. B. Muskat, D. E. Myers, R. J. Oberg, C. S. Ogilvy, Brian O'Mahoney, Barbara L. Osofsky, Walter Penney, J. L. Pietenpol, C. F. Pinzka, Anatol Rapoport, G. S. Rogers, C. M. Sandwick, Sr., E. M. Scheuer, D. L. Silverman, Barry Simon, D. R. Simpson, Arnold Singer, R. P. Soni, R. L. Syverson, R. P. Tapscott, Gomer Thomas, C. J. Vanderlin, Jr., S. Vatriquant, J. E. Vinson, G. W. Walker, W. C. Waterhouse, William Webb, Harry Weingarten, Charles Wetherell II, J. S. W. Wong, J. E. Yeager, and the proposers.

Probability that a Determinant Be Even

E 1512 [1962, 312]. *Proposed by C. W. Trigg, Los Angeles City College*

(a) If the nine positive digits are arranged at random in a square array, what is the probability that the determinant of the array will be an even number?

(b) Will the probability be the same if the first nine nonnegative digits are similarly arranged and the determinant evaluated?

Solution by D. C. B. Marsh, Colorado School of Mines. (a) There are, of course, six terms of three factors each in the determinant's expansion. With 5 odd digits available, the determinant will be *odd* if and only if precisely one of these terms has all three factors odd. We may choose the term in 6 ways, its factors in ${}_5P_3$ ways, the remaining pair of odd digits may be placed in 24 ways, and the even digits may be assigned in 4! ways to the residual places. The probability of an odd value is thus

$$6(5\cdot 4\cdot 3)(24)(4\cdot 3\cdot 2\cdot 1)/9! = 4/7,$$

and the probability that the value be even is 3/7.

(b) For $0, 1, \cdots, 8$, an *odd* value may occur in $6({}_4P_3)(6!)$ ways, giving the probability of an even value as 5/7.

Also solved by Jack Abad, Joseph Beer and Perry Scheinok (jointly), Marjorie Bicknell, Robert Bowen, Brother T. C. Wesselkamper, J. H. Butchart and Frederick Merkle (jointly), Dan Cohen, D. I. A. Cohen, David Forslund, C. M. Frye, Michael Goldberg, Sidney Heller, A. R. Hyde, A. J. Keeping, Helen M. Marston, D. A. Moran, C. F. Pinzka, C. M. Sandwick, Sr., D. L. Silverman, W. B. Stovall, Jr., S. Vatriquant, and the proposer. Some of these solutions were only partially correct.

An Application of Rolle's Theorem

E 1513 [1962, 312]. *Proposed by Seymour Kass, Illinois Institute of Technology*

Let $P(x)$ and $Q(x)$ be differentiable on an open interval I. Show that between any two real zeros of $P(x)$ on I there exists a real zero of $P'(x)+P(x)Q'(x)$.

Solution by A. R. Bednarek, Goodyear Aircraft Corporation. The function $Z(x)=P(x)\exp Q(x)$ vanishes at the real roots of $P(x)$, and

$$Z'(x) = [P'(x) + P(x)Q'(x)] \exp Q(x).$$

By Rolle's theorem $Z'(x)$, and therefore $P'(x)+P(x)Q'(x)$, must vanish between any two real roots of $P(x)$ on I.

Also solved by Jerrold Bebernes, Joseph Beer and Perry Scheinok (jointly), D. A. Breault, Brother T. C. Wesselkamper, J. L. Brown, Jr., P. R. Chernoff and W. C. Waterhouse (jointly), Y. B. Chun, Sherwood Ebey, Emory University Honors Freshman Calculus Class, G. P. Farrell and J. L. Shores (jointly), Ralph Greenberg, Norman Greenspan, A. J. Gross, Sidney Heller, V. E. Hoggatt and I. D. Ruggles (jointly), R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), A. J. Keeping, L. L. Keller and C. Vaseekaran (jointly), A. G. Konheim, G. E. Lindamood, Jiang Luh, C. R. MacCluer, P. B. Manchester, D. C. B. Marsh, M. L. Monahan, D. A. Moran, R. M. More, Barbara L. Osofsky, G. B. Parrish, M. Perisastri and M. S. R. K. Sastry (jointly), J. L. Pietenpol, B. E. Rhoades, G. S. Rogers, C. M. Sandwick, Sr., Dennis Sentilles, Jr., Oved Shisha, R. P. Soni, Dennis Travis, S. Vatriquant, J. E. Vinson, C. K. Woo, J. S. W. Wong, J. E. Yeager, David Zeitlin, and the proposer.

A Function of the Digit in the *n*th Decimal Place of a Number

E 1514 [1962, 312]. *Proposed by R. P. Current and J. F. Leetch, Bowling Green State University*

Let n be a positive integer. For each real x, let $d(x)$ be the digit in the nth decimal place of the infinite decimal expansion of x (assume no expansion terminates with repeated 9's). Define $f(x)=1$ if $d(x)$ is odd, and $f(x)=0$ if $d(x)$ is even. Prove that f has intervals of continuity for every n.

Solution by R. J. Oberg, University of California at Berkeley. Let $I_k = (k, k+10^{-n})$, where k is a positive digit. For $x \in I_k$, $d(x)=0$, $f(x)=0$, so f is continuous in I_k.

Also solved by Joseph Beer and Perry Scheinok (jointly), Robert Bowen, Brother U. Alfred, Brother Joseph Heisler, D. I. A. Cohen, Michael Gemignani, Michael Goldberg, Ralph Greenberg, R. A. Jacobson, Erwin Just, A. J. Keeping, D. C. B. Marsh, R. M. More, Jim Morrow, Barbara L. Osofsky, J. L. Pietenpol, Peter Rosenthal, C. M. Sandwick, Sr., D. L. Silverman, Steve Stephen, Gomer Thomas, J. E. Vinson, George Walker, W. C. Waterhouse, J. S. W. Wong, J. E. Yeager, and the proposers.

A Decomposition of the Euclidean Plane

E 1515 [1962, 312]. *Proposed by Joachim Lambek, McGill University, and Leo Moser, University of Alberta*

Show that for every positive integer n, the Euclidean plane, considered as a point set, can be decomposed into n congruent connected subsets.

Solution by L. F. Meyers, The Ohio State University. First we define the following point sets I, B, C of the Euclidean plane:

$$I = \{(x, y): 0 \leqq x < 1,\ 0 \leqq y < 1\},$$

$$B = \bigcup_{m=2}^{+\infty} \{(x, y): 0 < x < 1,\ y = (1-x)/m\} \cup \{(0, 0)\},$$

$$C = I \backslash B.$$

Thus I is the half-closed unit square; B (a subset of I) is a union of certain open segments together with the limit point $(0, 0)$; C is the complement of B with respect to I. Next we introduce a convenient notation; if T is a point set of the Euclidean plane, and a and b are integers, then by $T_{a,b}$ we mean the point set obtained by translating the set T through the vector whose components are a and b. Now we define

$$R = \left(\bigcup_{k=-\infty}^{0} I_{0,k}\right) \cup \left(\bigcup_{k=1}^{n-1} C_{0,k}\right) \cup \left(\bigcup_{k=n}^{+\infty} I_{0,k}\right) \cup \left(\bigcup_{k=1}^{n-1} B_{k,0}\right),$$

and then

$$S^{(0)} = \bigcup_{k=-\infty}^{+\infty} R_{kn,-kn},$$

and finally

$$S^{(i)} = S^{(0)}_{i,-i} \qquad \text{for } i = 0, 1, \cdots, n - 1.$$

It is not a difficult matter to show that the n congruent sets $S^{(i)}$ are individually connected and mutually disjoint, and that together they exhaust the whole Euclidean plane.

If $n=2$, then a simple solution (polygonally connected) can be obtained if congruence by reflection is permitted. The two sets are the upper and lower open half-planes, respectively, together with portions of their boundary. The portions of the boundary will be intervals open on the left and closed on the right, all of equal length, and belonging alternately to the two sets.

Also solved, for the case $n=2$ (as above), by V. E. Hoggatt.

ADVANCED PROBLEMS AND SOLUTIONS

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor.

PROBLEMS FOR SOLUTION

5065. *Proposed by Robert Spira, University of California, Berkeley*

For distinct integers k and l let

$$\int_0^{\pi} f(kx)f(lx)dx = 0, \qquad \int_0^{\pi} [f(kx)]^2 dx = \tfrac{1}{2}\pi.$$

What additional restrictions on f are needed to insure that $f(x) = \sin x$?

5066. *Proposed by P. J. Sally, Jr., Boston College*

In an article in this Monthly (Oct. 1961, p. 795) the authors have posed the following question. If G is a semigroup with local left identities and inverses, that is, writing the semigroup operation multiplicatively,

1) for each a in G, there is an element $e=e(a)$ such that $ea=a$,
2) for each e in G such that $ea=a$, there is an element $a'=a'(e)$ in G such that $a'a=e$,

is G then a group? This question is answered negatively with a simple example. A theorem is stated to the effect that a commutative semigroup with properties

1) and 2) is a group. The restriction to commutativity is not essential and precludes the application of these conditions to the noncommutative case. Show that the necessary and sufficient condition for a semigroup G satisfying 1) and 2) to be a group is that every local left identity is in $C(G)$, the center of G with respect to the semigroup structure.

5067. *Proposed by Leonard Carlitz, Duke University*

I. Given $a^2=4p^2b^2+c^2$, $(a, 2p)=1$, $a>0$, where p is an odd prime. Show that a is a quadratic residue, mod p.

II. Given $a^2=p^2b^2+4c^2$, $(a, 2p)=1$, $a>0$, where p is an odd prime. Show that $2a$ is a quadratic residue, mod p.

5068. *Proposed by Peter Flor, Mathematics Institute of the University of Vienna*

Let K be a quadratic field over the rationals with discriminant d. For any irrational number $t\in K$, let $at^2+bt+c=0$, $(a, b, c)=1$ (a, b, c rational integers) and let $D(t)=b^2-4ac$. Let (α, β) be any pair of numbers of K which are linearly independent over the rationals. Then it can be shown easily that (α, β) is the basis of some (integral or fractional) ideal in K if and only if $D(\alpha/\beta)=d$. Can this result be extended in any way to characterize ideal bases in algebraic number fields of arbitrary degree?

5069. *Proposed by D. R. Andrew, University of Southwestern Louisiana*

Prove or disprove the following statement. If S is a Hausdorff space and f: $S\to T$ is a weakly continuous one to one mapping of S onto the space T such that f^{-1}: $T\to S$ is weakly continuous, then T is necessarily a Hausdorff space. (See Norman Levine, *A Decomposition of Continuity in Topological Spaces*, this MONTHLY, 1961, pp. 44–46, for the definition of a weakly continuous function.)

5070. *Proposed by S. Birnbaum, New York City*

Let

$$A, B, \cdots, K \text{ be } k \text{ events in a probability space;} \tag{1}$$

$$P(A \cup B \cup \cdots \cup K) + P(A \cup B \cup \cdots \cup \overline{K}) + \cdots + P(A \cup \overline{B} \cup \cdots \cup \overline{K}) \tag{2}$$

be denoted by $\sum P$. Show that $\sum P=2^k-1$.

5071. *Proposed by Peter Ungar, New York University*

Let $F(u, u_{x_1}, \cdots, \partial^k u/\partial x_n^k)$ depend continuously on its arguments. Assume that for every region R

$$I(R, u) = \int \cdots \int_R F(u, \cdots)dx_1 \cdots dx_n$$

depends only on the boundary values of u and its derivatives of order $\leq k$, i.e.,

$I(R, u)=I(R, v)$ whenever $u-v$ vanishes on the boundary of R together with its derivatives of order $\leqq k$. Then F is a divergence, i.e., there exist expressions $F_i(u, u_{x_1}, \cdots)$ such that

$$F(u, \cdots) = F(0, 0, \cdots, 0) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(u, u_{x_1}, \cdots).$$

SOLUTIONS

Locally-compact Space

4980 [1961, 673]. *Proposed by G. H. Meisters, RIAS, Baltimore, Md.*

If P is a topological property, we call a topological space X *locally-P* if and only if every neighborhood N of every point x contains a neighborhood N^* of x which has the property P in its relative topology. Otherwise our terminology is that of J. L. Kelley: *General Topology.* Prove (or disprove) the following statement: If X is a compact, locally-Hausdorff topological space, then X is locally-compact.

Solution by Manuel Berri, University of California, Los Angeles. The proposed statement is not correct as a counterexample will show. We remark first that a space X is locally-compact if and only if, for each point $x \in X$ and for each neighborhood N of X, there exists a compact neighborhood V of x such that $V \subset N$.

Now let $X=\{a_{ij}, c_k, a, b \mid i=1, 2, \cdots; j=1, 2, \cdots; k=1, 2, \cdots\}$. Define the following neighborhood systems on X, where each a_{ij} is isolated:

$$U_n(a) = \{a, a_{ij} \mid i = n, n+1, \cdots; j = 1, 2, \cdots;\}, \qquad n = 1, 2, \cdots,$$
$$U_n(c_i) = \{c_i, a_{ij} \mid j = n, n+1, \cdots\}, \qquad n = 1, 2, \cdots,$$
$$U_n(b) = \{b, c_i, a_{ij} \mid i = n, n+1, \cdots; j = n, n+1, \cdots\}, \quad n = 1, 2, \cdots.$$

With the topology determined by the above neighborhood system, one easily observes that X is compact, locally-Hausdorff, but not Hausdorff. Indeed, the only non-Hausdorff pair of elements are a and b.

We claim that X is not locally-compact at a. In fact, it suffices to show that $U_1(a)$ contains no compact neighborhood of a.

Let V be any neighborhood of a such that $V \subset U_1(a)$. To prove V is not compact, it suffices to show that V contains an infinite sequence of points with no limit point in V. Since $\{U_n(a)\}$ is a complete system of neighborhoods of a, there exists an integer p such that $U_p(a) \subset V$. Thus the infinite sequence $\{a_{pj}\}_{j=1}^{\infty}$ is a set of points contained in $U_p(a)$ and thus in V. In X, it has a unique limit point, namely c_p. Since $c_p \notin V$, then this sequence has no limit point in V. Thus V is not compact.

Also solved by W. G. Brown, Fred Galvin, Barbara L. Osofsky, S. M. Robinson, and the proposer.

Semigroups

4992 [1961, 934]. *Proposed by Seth Warner, Duke University*

Let $(E, +, \leqq)$ be an infinite totally ordered semigroup. Prove that $(E, +, \leqq)$ is isomorphic to the ordered semigroup of nonnegative integers if there exist elements 0 and $1 \in E$ such that $0 < 1$, and such that if S is any subset of E containing 0, and containing $x+1$ whenever it contains x, then $S=E$.

Solution by W. C. Waterhouse, Harvard University. The sequence S: $a_0=0$, $a_1=0+1$, $a_2=0+1+1$, $\cdots$, satisfies the inductive hypothesis and hence $S=E$. Since E is infinite, no two of the a_i can be the same. If $0+0=a_n$ and $1=a_m$, then $a_1=0+1=a_{n+m}$; since $0\neq 1$, we must have $n=0$ and $m=1$. The proof of isomorphism is now immediate.

Also solved by R. A. Cunninghame-Green, Grace Murray, Barbara L. Osofsky, P. P. Sawarotow, and the proposer.

Special Sequences

4993 [1961, 934]. *Proposed by A. G. Konheim and R. A. Willoughby, IBM Research, Yorktown Heights, N. Y.*

Find all regular sequences $\{a_i\}_{i=-\infty}^{\infty}$ such that, for a given positive integer N,

$$\text{(i)} \qquad a_{i+N} = a_i \qquad (i = 0, \pm 1, \pm 2, \cdots)$$

$$\text{(ii)} \qquad \sum_{i=0}^{N-1} a_i a_{i+\mu} = \begin{cases} 0 & \text{if } \mu \not\equiv 0 \pmod N \\ 1 & \text{if } \mu \equiv 0 \pmod N. \end{cases}$$

Solution by Leonard Carlitz, Duke University. Put $\epsilon = e^{2\pi i/N}$. Because of the periodicity $(a_r = a_{r+N})$ we may put

$$\text{(1)} \qquad a_r = \frac{1}{N}\sum_s b_s\epsilon^{rs} \qquad (b_s = b_{s+N}),$$

where the summation is over a complete residue system (mod N). Indeed (1) is equivalent to

$$\text{(2)} \qquad b_r = \sum_s a_s\epsilon^{-rs}.$$

Using (1) we get

$$\begin{aligned}\sum_r a_r a_{r+j} &= \frac{1}{N^2}\sum_r\sum_s b_s\epsilon^{rs}\sum_t b_t\epsilon^{(r+j)t} \\ &= \frac{1}{N^2}\sum_{s,t} b_s b_t\epsilon^{jt}\sum_r \epsilon^{r(s+t)} = \frac{1}{N}\sum_s b_s b_{-s}\epsilon^{-js},\end{aligned}$$

where we have used the well-known formula

$$\sum_r \epsilon^{rs} = \begin{cases} N & (s \equiv 0 \pmod N) \\ 0 & (s \not\equiv 0 \pmod N). \end{cases}$$

Thus

$$\sum_s b_s b_{-s}\epsilon^{-is} = \begin{cases} N & (s \equiv 0 \pmod{N}) \\ 0 & (s \not\equiv 0 \pmod{N}). \end{cases}$$

It follows that

$$(3) \qquad b_s b_{-s} = 1 \qquad \text{(for all } s).$$

Since a_r is real, (2) implies $b_{-r} = \sum_s a_s \epsilon^{rs} = \bar{b}_r$, so that (3) becomes $|b_s| = 1$ (for all s).

Therefore all real sequences satisfying the several hypotheses are of the form (1), where the coefficients b_s are complex numbers of absolute value 1 such that $b_{-s} = \bar{b}_s$.

Alternately we may put the general solution in the following form. If N is odd then

$$a_r = \frac{u_0}{N} + \frac{2}{N}\sum_{s=1}^{\frac{1}{2}(n-1)}\left(u_s \cos\frac{2rs\pi}{N} + v_s \sin\frac{2rs\pi}{N}\right),$$

where u_s, v_s are real and $u_s^2 + v_s^2 = 1$. If N is even there is an additional term

$$\frac{2}{N} u_n \cos r\pi \qquad (u_n = \pm 1,\ N = 2n).$$

Also solved by the proposers.

Term by Term Differentiation without Uniform Convergence

4994 [1961, 934]. *Proposed by Peter Ungar, New York University*

For $a < x < b$, let

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \lim_{n\to\infty} f_n'(x) = \phi(x).$$

Prove. If all the functions named above and also $f'(x)$ are continuous, then $f'(x) = \phi(x)$.

Solution by R. D. McWilliams, Florida State University. Let $[c, d] \subset (a, b)$, and for each positive integer n let H_n be the set of all $t \in [c, d]$ such that $|f_m'(t) - \phi(t)| < 1$ for all $m \geqq n$. Since $[c, d]$ is a set of the second category in itself, some H_n must be dense in a subinterval $[p, q]$ of $[c, d]$, and hence $\{f_n'\}$ is uniformly bounded on $[p, q]$. By the Lebesgue bounded convergence theorem, for each $x \in (p, q)$,

$$\int_p^x \phi(t)dt = \lim_{n\to\infty}\int_p^x f_n'(t)dt = f(x) - f(p),$$

and hence $f'(x) = \phi(x)$. Since $[c, d]$ is arbitrary, it follows that $f'(x) = \phi(x)$ for all x in a dense subset of (a, b) and hence, by continuity, for all $x \in (a, b)$.

Also solved by Robert Breusch, J. Czipszer, N. G. de Bruijn, David Greenstein, J. B. Linder, Norman Meyers, R. W. Newcomb, D. J. Newman, G. B. Parrish, Barbara L. Osofsky, W. C. Waterhouse, and the proposer.

Editorial Note. The theorem as usually given carries the additional hypothesis that f' be uniformly convergent. See, *e.g.*, A. Taylor, *Advanced Calculus*, Theorem 5, p. 601.

The proof given above is an illustration of Osgood's theorem, see Riesz and Nagy, *Functional Analysis*, New York 1955, p. 63.

Newcomb remarks that there is a similar theorem regarding distributions, given as Theorem IV, p. 20 of Mikusiński and Sikorski, *The Elementary Theory of Distributions* (I), Rozprawy Matematyczne XII, Warsaw, 1957.

Euler's Function

4995 [1961, 1010]. *Proposed by Oystein Ore, Yale University*

When $\phi(x)$ denotes Euler's function it is readily verified that $n=14$ is the smallest even number such that the equation $\phi(x)=n$ has no solution. Prove that for each exponent α there is a smallest odd integer k_α such that the equation $\phi(x)=2^\alpha k_\alpha$ has no solution. Determine k_2, k_3, k_4. Try to find bounds for k_α.

I. *Solution by J. L. Selfridge, University of Washington and University of California at Los Angeles.* If $k\cdot 2^n+1$ is prime then $\phi(x)=2^\alpha k$ has solutions for any $\alpha\geqq n$. If 2^s+1 is prime then $\phi(x)=2^\alpha(2^s+1)$ has solutions for any $\alpha\geqq s$. If p is prime and $p\cdot 2^n+1$ is composite for every $n\leqq\alpha$, and $p\neq 2^s+1$ for any $s\leqq\alpha$, then $\phi(x)=2^\alpha p$ has no solution. These statements are easy to verify. (See Solution II.)

It is known (Sierpiński, *Elem. Math.* 15 (1960) p. 73) that if k belongs to certain arithmetic progressions then any term of the sequence $k+1, 2k+1, \cdots, 2^nk+1, \cdots$ is divisible by one of a set of 6 or 7 fixed primes. For example, if p is prime, $p\equiv 1 \pmod{(2^{32}-1)\cdot 641}$, and $p\equiv -1 \pmod{(2^{32}+1)/641}$, then $p\cdot 2^n+1$ is composite for every n, so then $\phi(x)=2^\alpha p$ has no solution for any α, and $k_\alpha\leqq p$ for all α. For the prime 271129, the sequence $271129\cdot 2^n+1$ has the covering set 3, 5, 7, 13, 17, 241. Thus $k_\alpha\leqq 271129$ for all α.

Armed with the statements above, a glance at Robinson's table (*Proc. Amer. Math. Soc.* 9(1958), p. 674) shows that $k_0=3$, $k_1=7$, $k_2=17$, $k_3=k_4=k_5=19$, $k_6=k_7=31$, and $k_\alpha=47$ for $8\leqq\alpha<512$. Computations on the SWAC show that $47\cdot 2^n+1$ is composite for $n<583$, that $47\cdot 2^{583}+1$ is prime, that for each odd $k\neq 257$ such that $47<k<383$ there is a prime $k\cdot 2^n+1$ with $n\leqq 66$, and that $383\cdot 2^n+1$ is composite for every $n<2313$. Hence $k_\alpha=47$ for $8\leqq\alpha<583$ and $k_\alpha=383$ for $583\leqq\alpha<2313$.

II. *Solution by P. T. Bateman, University of Illinois and University of Pennsylvania.* We begin with several lemmas.

Lemma 1. *Suppose α is a given positive integer. Then there are infinitely many primes p such that $2p+1, 2^2p+1, \cdots, 2^\alpha p+1$ are all composite.*

Proof. By the Chinese Remainder Theorem there is an integer c_α such that $2^j c_\alpha+1\equiv 0 \pmod{p_{j+1}}$ for $j=1, 2, \cdots, \alpha$, where p_i denotes the ith odd prime. By Dirichlet's theorem there are infinitely many primes p such that $p\equiv c_\alpha (\text{mod } 3\cdot 5 \cdots p_{\alpha+1})$. If p is such a prime, then for $j=1, 2, \cdots, \alpha$ we have $2^jp+1\equiv 0 \pmod{p_{j+1}}$ and $2^jp+1>2^{j+1}>p_{j+1}$, so that 2^jp+1 is composite.

LEMMA 2. *Suppose α is a given positive integer. Let l_α be the smallest prime number such that*

(i) $$l_\alpha \neq 2^1+1,\ 2^2+1, \cdots, 2^\alpha+1 \text{ and}$$

(ii) $$2l_\alpha+1,\ 2^2l_\alpha+1, \cdots, 2^\alpha l_\alpha+1 \text{ are all composite.}$$

Then there is no integer x such that $\phi(x)=2^\alpha l_\alpha$. Thus k_α exists and $k_\alpha\leqq l_\alpha$.

Proof. Suppose $\phi(x)=2^\alpha l_\alpha$. Then x must be divisible either by l_α^2 or by a prime q such that $q-1=dl_\alpha$ for some integer d. Thus $\phi(x)$ is divisible either by $l_\alpha(l_\alpha-1)$ or by $l_\alpha d$. But by assumption (i) either $l_\alpha-1$ has an odd prime factor or $l_\alpha-1>2^\alpha$, and by assumption (ii) either d has an odd prime factor or $d>2^\alpha$. In either case we get a contradiction to the assumed relation $\phi(x)=2^\alpha l_\alpha$.

LEMMA 3. *Suppose α is a given positive integer. Let h_α be the smallest odd positive integer such that $2h_\alpha+1, 2^2h_\alpha+1, \cdots, 2^\alpha h_\alpha+1$ are all composite. Then $k_\alpha\geqq h_\alpha$.*

Proof. If m is an odd positive integer less than h_α, there exists a positive integer j such that $1\leqq j\leqq\alpha$ and 2^jm+1 is prime. Then $\phi(2^jm+1)=2^jm$, so that

$$\phi(2^{\alpha-j+1})(2^jm+1) = 2^\alpha m \qquad \text{if } j<\alpha$$

and

$$\phi(2^jm+1) = 2^\alpha m \qquad \text{if } j=\alpha.$$

Thus for any odd positive integer m less than h_α we see that $2^\alpha m$ is a value taken on by the Euler function. Thus $k_\alpha\geqq h_\alpha$.

Collecting the results of these three lemmas, we see that k_α exists and satisfies the inequalities $h_\alpha\leqq k_\alpha\leqq l_\alpha$. Now it turns out that $h_\alpha=l_\alpha$ for the first few values of α. This empirical fact provides us with the determination of k_α for such values of α. In particular, $h_1=k_1=l_1=7$, $h_2=k_2=l_2=17$, $h_\alpha=k_\alpha=l_\alpha=19$ for $3\leqq\alpha\leqq 5$, and $h_\alpha=k_\alpha=l_\alpha=31$ for $6\leqq\alpha\leqq 7$. (It is conceivable that $h_\alpha=l_\alpha$ for all values of α, but this would be difficult to decide.)

Also solved by W. J. Blundon, Robert Bowen, Robert Breusch, L. Carlitz, and A. Makowski.

Partial Differentiation. By Hugh A. Thurston. Prentice-Hall, Englewood Cliffs, N. J., 1961. ix+160 pp. \$7.50.

The book covers the standard material on partial derivatives and implicit functions. Part I, about 100 pages, is dedicated to the development of the subject, while Part II contains proofs of some of the basic theorems. The book presupposes an introductory course in calculus, and at least an intuitive knowledge of some basic facts of three-dimensional analytic geometry. According to preface and introduction, the book was written largely because of the author's unhappiness about confusing and frequently inconsistent notations, such as $f_x(a, b)$; he prefers $f_1(a, b)$, with good reasons. The book does not contain enough material for a full-length course, but it might profitably be used for self-study, or be assigned as collateral reading in a course such as thermodynamics; many examples involving the gas laws, show that the author had this particular situation in mind. The exposition is somewhat uneven. Most of the material in Part I, is presented carefully and slowly, with lucidity, pedagogical skill, and an exemplarily clear notation; and so are most of the proofs in Part II. On the other hand, the two chapters on "Integration" and "Taylor's theorem," are so short (about two pages each, incl. exercises,) as to be practically meaningless. On page 116, the word "continuity" is employed a few lines before it is defined; on page 123, the conditions for differentiability of a function of more than two variables, are stated incorrectly; and on page 140 (boundedness of a function continuous in a closed rectangle), the omission of potential "equal" signs in the inequalities, makes for a "proof" which would apply just as well to open rectangles. The use of terms such as "singulary" for a function of one variable, and "interval" for "rectangle," may bother some other readers too. There are numerous exercises, many of them imaginative, unusual, and challenging; they are perhaps the best feature of the book. A last remark: \$7.50 seems like an outrageously high price for a book of this size.

ROBERT BREUSCH
Amherst College

BRIEF MENTION

Philosophy of Science. By Philipp Frank. Prentice-Hall, Englewood Cliffs, New Jersey, 1962. xxii+394 pp. \$2.45 (paper).

Subtitled "The Link Between Science and Philosophy," this well-known book was first published in 1957.

Differential and Integral Calculus. (Sixth Edition.) By the late C. E. Love and E. D. Rainville. Macmillan, New York, 1962. xvii+579 pp. \$7.50.

Contains some new material both theoretical (e.g., proof of the existence of e) and applied (e.g., vibration of a spring), and 4000 exercises of which 2000 are new in this edition.

An Introduction to the Theory of Newtonian Attraction. By A. S. Ramsey. Cambridge University Press, New York, 1962. ix+184 pp. $1.75 (paper).

First published in 1940 and reviewed in this MONTHLY in 1942, Aug.–Sept. issue.

Numerical Mathematical Analysis. (Fifth Edition.) By James B. Scarborough. The Johns Hopkins Press, Baltimore, 1962. xxi+594 pp. $7.00.

Reviewed in this MONTHLY in 1931, Aug.–Sept. issue. The fifth edition includes, as important additions, "a chapter on interpolation with unequal intervals of the argument by means of Newton's general formula of interpolation, the derivation of all central-difference interpolation formulas by means of divided differences and methods of investigating the errors in the solutions of single equations and systems of linear equations when the coefficients are subject to errors."

High School Mathematics, Unit 9. By University of Illinois Committee on School Mathematics. University of Illinois Press, Urbana, 1962. xv+369 pp. $2.00.

"Elementary Functions: Power, Exponentials, and Logarithms."

Set Theory. (Second English Edition.) By Felix Hausdorff. Chelsea, New York, 1962. 352 pp. $6.50.

A translation of the third (1937) edition of *Mengenlehre*, this second edition contains two short appendices written by R. L. Goodstein: on the contradictions in naive set theory, and on the axiom of choice.

Dispersion Relations. By G. R. Screaton, ed. Interscience Publishers, New York, 1961. xiii+290 pp. $9.25.

A report of the 1960 Scottish Universities' Summer School, this volume contains articles on theoretical physics by eight distinguished individuals from Austria, England, Italy, Switzerland (one each), and the U.S.A. (four).

Systems Philosophy. By David O. Ellis and Fred J. Ludwig. Prentice-Hall, Englewood Cliffs, New Jersey, 1962. xl+387 pp. $13.00 (trade).

"The major concepts involved in the engineering and management of man-made systems and man/machine systems." Two-thirds of the book is devoted to appendices, ranging from a precise definition of "systems" to discussion of "relativistic Doppler effect, a new approach to space navigation."

Uncertainty and Structure as Psychological Concepts. By Wendell R. Garner. Wiley, New York, 1962. ix+369 pp. $7.50 (coll.); $8.95 (trade).

Studies in the Theory of Numbers. By Leonard Eugene Dickson. Chelsea, New York, n.d. viii+230 pp. $3.95.

Reprint of the first (1930) edition.

Lectures on the Calculus of Variations. By Oskar Bolza. Dover, New York, 1961. xi+271 pp. $1.65 (paper back).

"Unabridged and corrected republication . . . first published by the University of Chicago Press in 1904." A further correction is to be noted on p. 265, line 7: Add: "unless $u \equiv 0$ in (ab)."

A Treasury of World Science. By Dagobert D. Runes, ed. Philosophical Library, New York, 1962. xxi+978 pp. $15.00.

Selections from the works of 100 scientists, from Agricola to Volta, with brief biographical sketches of each.

Forces and Fields. By Mary B. Hesse. Philosophical Library, New York, 1962. x+318 pp. $10.00.

Subtitled "The concept of Action at a Distance in the history of physics."

Network Analysis and Synthesis. By Franklin F. Kuo. Wiley, New York, 1962. xiv+413 pp. $9.25.

"This book is an introduction to the study of electric networks based upon a *system theoretic* approach."

Basic Principles of the Tracer Method. By C. W. Sheppard. Wiley, New York, 1962. xviii+282 pp. $8.00.

Subtitled "Introduction to Mathematical Tracer Kinetics," this is a research monograph on radioactive isotope tracers, with the theory "necessarily presented in mathematical terms, but in such a way that those without a very high level of mathematical sophistication may be able to follow the analysis."

Electromagnetic Waves. By Rudolph E. Langer, ed. The University of Wisconsin Press, Madison, 1962. xii+396 pp. $6.00.

Contains 16 papers presented at a symposium conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, April 10–12, 1961.

Asymptotic Estimates and Entire Functions. By M. A. Evgrafov, Gordon and Breach, New York, 1962. x+181 pp. $8.00.

Russian Tracts on Advanced Mathematics and Physics, vol. 4. Translated by Allen L. Shields. From the translator's preface: (This book) is not a compendium of results. Rather it attempts to explain certain general methods by working out a number of examples.

"The first chapter introduces the principal methods to be used, and the second chapter discusses the theory of entire functions of finite order. The third chapter contains the deepest results of the book: the application of the methods of Chapter 1 to the functions of Chapter 2. Here the reader will find much material that has not previously appeared in book form."

The Logic of Chance. (Fourth Edition.) By John Venn. Chelsea, New York, 1962. xxix+508 pp. $4.95 (cloth), $2.25 (paper).

A reprint of the third (1888) edition. A charming, discursive, nonmathematical exposition.

Mathematics for Pleasure. By Oswald Jacoby, with William H. Benson. McGraw-Hill, New York, 1962. x+191 pp. $4.95.

About 160 problems and puzzles, of various types, with solutions.

Graphics. By A. S. Levens. Wiley, New York, 1962. x+743 pp. $9.50.

Part 1 contains traditional material of engineering drawing and elementary descriptive geometry, with up-to-date examples and conventions; Part 2 presents graphical solutions and computations, including nomography; Part 3 is entitled "Introduction to Conceptual Design."

Basic College Algebra. By J. D. Mancill and M. O. Gonzalez. Allyn and Bacon, Boston, 1962. xi+458 pp. $6.75.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor F. A. Ficken, New York University, represented the Association at the inauguration of Dr. James McN. Hester as President of New York University on October 25, 1962.

Professor Evan Johnson, Pennsylvania State University, represented the Association at the academic convocation honoring F. D. Rossini, J. A. Cassidy, and Hon. J. F. Fogarty and the dedication of the Father John P. J. Sullivan, T.O.R., Science Hall of St. Francis College.

Professor G. P. Johnson, Wesleyan University, represented the Association at the inauguration of Dr. Charles E. Shain as Sixth President of Connecticut College on October 19, 1962.

Professor C. T. Salkind, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Dr. John H. Fischer as President of Teachers College, Columbia University on November 13, 1962.

Professor R. R. Stoll, Oberlin College, represented the Association at the inauguration of Dr. G. L. McConagha as President and R. N. Montgomery as First Chancellor of Muskingum College on November 2, 1962.

University of Akron: Dr. A. R. Bednarek, Goodyear Aircraft, Akron, Ohio, has been appointed Assistant Professor; Associate Professor Margaret E. Mauch has been promoted to Professor.

University of Alabama: Mrs. Ayrlene Jones has been promoted to Assistant Professor; Associate Professor R. W. Bagley has been promoted to Professor.

Brown University: Professor David Gale is on leave of absence for the academic year 1962–63 at the University of Osaka, Japan; Professor W. H. Fleming is on leave of absence for the academic year 1962–63 at the U. S. Army Mathematics Research Center, University of Wisconsin; Professor F. M. Stewart has been appointed Acting Chairman of the Mathematics Department for the academic year 1962–63.

Central State College: Assistant Professor LaVern Loman has been promoted to Associate Professor; Mr. H. H. Hunt has been promoted to Assistant Professor.

College of the Holy Cross: Rev. J. J. MacDonnell, S.J. has been promoted to Assistant Professor; Professor D. G. Dewey has been awarded a NSF Science Faculty Fellowship and is on leave for the academic year 1962–63 at Brown University.

Cornell University: Assistant Professor Anil Nerode has been promoted to Associate Professor; Dr. C. R. Curjel has been promoted to Assistant Professor.

Dartmouth College: Mr. W. E. Slesnick, St. Paul's School, Concord, New Hampshire, has been appointed Assistant Professor; Associate Professor J. L. Snell has been promoted to Professor; Professor B. H. Brown retired June 1962 with the title of Professor Emeritus.

University of Detroit: Mr. J. R. Gillis and Miss Maryjo Nichols have been promoted to Assistant Professors.

Dickinson College: Associate Professor W. M. Miller has been promoted to Professor; Associate Professor C. E. Kerr has been appointed Acting Chairman of the Mathematics Department.

East Tennessee State College: Assistant Professor S. H. Johnson has been promoted to Associate Professor; Assistant Professor Henry Linsert has been promoted to Professor; Professor T. C. Carson retired June 1, 1962.

University of Idaho: Assistant Professors D. L. Boyer and H. W. Crowley have been promoted to Associate Professors; Associate Professor H. W. Crowley has been appointed Director of the Computing Center.

University of Illinois, Chicago Undergraduate Division: Dr. I. K. Feinstein has been promoted to Professor of Mathematics Education; Mr. N. C. Scholomiti has been promoted to Assistant Professor; Assistant Professor H. J. Curtis, Acting Head of the Division of Mathematics, has been appointed Head.

Indiana University: Professor S. G. Ghurye, University of Minnesota, has been appointed Professor; Drs. R. J. Troyer and W. P. Ziemer have been promoted to Assistant Professors.

Kansas State University: Associate Professors J. M. Marr and W. L. Stamey have been promoted to Professors.

Louisiana Polytechnic Institute: Assistant Professors J. D. Gilbert and J. B. Garner have been promoted to Associate Professors.

McNeese State College: Associate Professor P. L. Ford has been promoted to Professor and appointed Head of the Department of Mathematics; Dr. S. M. Spencer, Jr. has been appointed Dean of the Division of Pure and Applied Sciences.

University of Maryland: Dr. Abraham Sinkov, National Security Agency, Washington, D. C., has been appointed Associate Professor; Associate Professor Seymour Goldberg, New Mexico State University, has been appointed Visiting Associate Professor; Drs. G. R. Lehner and Rose W. Sedgwick have been promoted to Assistant Professors.

*Michigan State University:*Assistant Professor D. E. Sanderson, Iowa State University, has been appointed Visiting Associate Professor; Professors W. E. Deskins and J. G. Hocking are on leave for the academic year 1962–63 at the University of Tübingen, Germany; Professor R. H. Oehmke is on leave for the academic year 1962–63 at the Institute of Defense Analyses.

University of Nebraska: Dr. A. M. Fink, University of Virginia, has been appointed Assistant Professor; Assistant Professor G. H. Meisters has been promoted to Associate Professor.

Pratt Institute: Assistant Professors D. G. Mead and B. D. Seckler have been promoted to Associate Professors.

Reed College: Assistant Professor T. P. Dennehy, John Carroll University, has been appointed Assistant Professor; Associate Professor J. B. Roberts has been promoted to Professor; Assistant Professor Dorothy J. Williams has been promoted to Associate Professor.

State University of New York at Buffalo (formerly University of Buffalo): Dr. B. L. Chilton, University of Toronto, and Mr. B. B. Sharpe, Kenmore High School, Kenmore, New York, have been appointed Assistant Professors; Mrs. Judith Blankfield has been promoted to Assistant Professor.

Syracuse University: Drs. A. A. Sagle, University of Chicago, and J. H. Stoddard, Michigan State University Oakland, have been appointed Assistant Professors; Assistant Professors P. T. Church and Mark Mahowald have been promoted to Associate Professors.

Tulane University: Assistant Professor F. T. Birtel, Yale University, has been appointed Assistant Professor; Assistant Professor J. G. Horne, Jr., University of Georgia, has been appointed Visiting Associate Professor; Associate Professor F. B. Wright has been promoted to Professor.

Washburn University: Assistant Professor Laura Z. Greene and Miss Margaret E. Martinson have been promoted to Associate Professors; Mr. T. D. McAdam has been promoted to Assistant Professor.

University of Waterloo: Associate Professor W. T. Tutte, University of Toronto, has been appointed Professor; Associate Professor D. A. Sprott has been promoted to Professor.

Dr. Oliver Aberth, University of Pennsylvania, has been appointed Assistant Professor at the University of Illinois.

Assistant Professor J. W. Addison, Jr., University of Michigan, has been appointed Associate Professor at the University of California, Berkeley.

Dr. J. M. Anderson, RCA Laboratories, Princeton, New Jersey, has been appointed Assistant Professor at Iowa State University.

Professor J. W. Andrushkiw, Seton Hall University, has been appointed Chairman of the Mathematics Department.

Assistant Professor P. L. Armstrong, Clemson College, retired on June 30, 1962.

Dr. F. W. Ashley, Jr., Oklahoma State University, has been appointed Assistant Professor at Southwest Missouri State College.

Mr. D. M. Bloom, Harvard University, has been appointed Assistant Professor at the University of Massachusetts.

Dr. D. M. Brown, Remington Rand, St. Paul, Minnesota, has been appointed Professor and Head of the Department of Mathematics at Norwich University.

Dr. Eugene Butkov, St. John's University, has been appointed Assistant Professor at Hunter College.

Professor Evelyn W. Cassner, Randolph Macon Woman's College, has been appointed Acting Chairman of the Department of Mathematics.

Mr. W. A. Christenson, University of North Dakota, has accepted a position as Mathematician with Phillips Petroleum Company, Idaho Falls, Idaho.

Mr. W. H. Churchill, Miami University, has accepted a position as Associate Engineer with Sperry Gyroscope, Great Neck, New York.

Professor John DeCicco, De Paul University, has been appointed Professor at the Illinois Institute of Technology.

Mr. W. P. Devers, Gleason Works, Rochester, New York, has accepted a position as Project Engineer with Curtiss-Wright, Caldwell, New Jersey.

Assistant Professor A. J. DiPetro, Eastern Illinois University, has been promoted to Associate Professor.

Professor M. D. Donsker, University of Minnesota, has been appointed Professor at the Courant Institute of Mathematical Sciences of New York University.

Assistant Professor Jacqueline P. Evans, Wellesley College, has been promoted to Associate Professor.

Mr. D. L. Fien, Dartmouth College, has been appointed Assistant Professor at Alfred University.

Mr. J. B. Fraleigh, Dartmouth College, has been appointed Assistant Professor at the University of Rhode Island.

Dr. T. S. Frank, Syracuse University, has been appointed Assistant Professor at Le Moyne College.

Associate Professor A. M. Garsia, University of Minnesota, has been appointed Associate Professor at the California Institute of Technology.

Mr. S. K. Gibson, Florida Southern College, has accepted a position as Engineer with General Electric, Utica, New York.

Mr. L. S. Goldsmith, Arizona State College, has accepted a position as Mathematician at the Picatinny Arsenal, Dover, New Jersey.

Mr. J. J. Goode, University of North Carolina, has been appointed Assistant Professor at Georgia Institute of Technology.

Dr. K. K. Gorowara, University of Delhi, India, has been appointed Assistant Professor at Montana State University.

Mr. G. C. Graff, University of Illinois has been appointed Assistant Professor at Ohio State University.

Mr. E. H. Greene, University of Virginia, has been appointed Assistant Professor at Beloit College.

Associate Professor H. W. Guggenheimer, University of Minnesota, has been promoted to Professor.

Assistant Professor W. J. Hardell, Worcester Polytechnic Institute, has been promoted to Associate Professor.

Dr. H. S. Hayashi, University of Southern California, has been appointed Assistant Professor at the University of Hawaii.

Assistant Professor E. W. Holt, Monmouth College, has been promoted to Associate Professor.

Mr. Brindell Horelick, Lafayette College, has been promoted to Assistant Professor.

Professor R. C. Huffer, Beloit College, has been appointed Professor and Head of the Mathematics Department at Tougaloo College.

Professor M. Gweneth Humphreys, Randolph Macon Woman's College is on sabbatical leave and has been appointed Visiting Professor at the University of British Columbia.

Dr. R. W. Hunt, Marshall Space Flight Center, Huntsville, Alabama, has been appointed Associate Professor at Southern Illinois University.

Mr. W. H. Jamison, Montana State College, has been appointed Assistant Professor at Rocky Mountain College.

Mr. W. R. Jones, Rutgers, the State University, has been appointed Assistant Professor at the University of Massachusetts.

Mr. R. A. Jorgensen, Andrews University, has been promoted to Assistant Professor.

Mr. L. F. Kinch, University of Kentucky, has been appointed Assistant Professor at the University of Alaska.

Assistant Professor Evelyn K. Kinney, Kansas State University, has been appointed Associate Professor at the University of Mississippi.

Dr. R. R. Korfhage, North Carolina State College, has been appointed Assistant Professor at Purdue University.

Mr. H. L. Kruse, Jr., Shippensburg State College, has been appointed Assistant Professor at Bloomfield College.

Assistant Professor R. G. Kuller, Dartmouth College, has been appointed Assistant Professor at the University of Colorado.

Associate Professor George Laush, University of Pittsburgh, has been promoted to Professor.

Professor E. H. Lee, Brown University, has been appointed Professor of Engineering Mechanics at Stanford University.

Associate Professor J. R. Lee, College of William and Mary, has promoted to Professor.

Mr. D. R. Lichtenberg, University of Wisconsin, has been appointed Assistant Professor at the University of South Florida.

Professor J. M. Long, Frederick College, has been appointed Professor of Biometrics and Division Head at the School of Medicine at the University of Arkansas.

Mr. A. J. Lucchesi, Cooper Union, has been promoted to Assistant Professor.

Associate Professor Edith H. Luchins, University of Miami, has been appointed Associate Professor at Rensselaer Polytechnic Institute.

Dr. R. M. McConnel, Duke University, has been appointed Assistant Professor at the University of Arizona.

Assistant Professor F. H. McGar, Jr., Douglass College, has been appointed Assistant Professor of Physics at Harpur College.

Mr. N. L. Massey, Seattle Public Schools, Seattle, Washington, has been appointed Director of Mathematics.

Dr. R. A. Melter, University of Missouri, has been appointed Assistant Professor at the University of Rhode Island.

Mr. M. H. Millar, University of Chicago, has been appointed Assistant Professor at the State College of Iowa.

Professor S. B. Murray, Mississippi State University, has been appointed Acting Head of the Department of Mathematics.

Professor E. P. Northrop, on leave from the University of Chicago for the past two years, has resigned from the University and will continue indefinitely as the Representative in Turkey of the Ford Foundation.

Associate Professor W. M. Perel, Lousiana State University, New Orleans, has been appointed Visiting Associate Professor at Randolph Macon Woman's College.

Associate Professor D. A. Pope, University of California, Los Angeles, has accepted a position as a member of the Technical Staff of the Space Technology Laboratories, Redondo Beach, California.

Mrs. Shu-Shen Sah, University of Illinois, has been appointed Assistant Professor at Lycoming College.

Dr. R. L. San Soucie, Sylvania Electronic Systems, Batavia, New York, has accepted a position as Vice President and Technical Director of the Electronics and Avionics Division of Emerson Electric, St. Louis, Missouri.

Assistant Professor N. S. Scarritt, Jr., Purdue University, has been appointed Assistant Professor at New Mexico State University.

Assistant Professor Ernest Schlesinger, Wesleyan University, has been appointed Assistant Professor at Connecticut College.

Dr. Donald Schmidt, Iowa State University, has been appointed Assistant Professor at Colorado State College.

Mr. Arnold Seiken, University of Michigan, has been appointed Assistant Professor at Michigan State University Oakland.

Mr. F. C. Sherburne, Jr., Hope College, has been promoted to Assistant Professor.

Dr. Oved Shisha, National Bureau of Standards, Washington, D. C., has accepted a position as Research Mathematician at the Applied Mathematics Research Laboratory, Wright-Patterson Air Force Base, Ohio.

Professor W. O. Shriner, Indiana State College, retired August 22, 1962 with the title of Professor Emeritus.

Dr. Harold Shulman, Service Bureau Corporation, New York, New York, has accepted a position as Mathematican with Republic Aviation, Farmingdale, New York.

Professor W. A. Small, Tennessee Polytechnic Institute, has been appointed Professor at the State University of New York College at Geneseo.

Assistant Professor C. S. Smith, Drury College, has been appointed Assistant Professor at Lake Forest College.

Assistant Professor W. E. Smith, Occidental College, has been appointed Assistant Professor at the University of Colorado.

Dr. R. S. Spira, University of California, Berkeley, has been appointed Visiting Assistant Professor at Duke University.

Dr. S. T. Stern, State University of New York at Buffalo, has been appointed Assistant Professor at the State University College at Buffalo.

Assistant Professor C. S. Sutton, The Citadel, has been promoted to Associate Professor.

Miss Frances L. Torgerson, North Dakota State University, has been appointed Assistant Professor at Texas Lutheran College.

Associate Professor B. L. Turney, McNeese State College, has been appointed Associate Professor at Northeastern State College.

Dr. M. E. Waddill, University of Pittsburgh, has been appointed Assistant Professor at Wake Forest College.

Dr. E. L. Walter, New Mexico State University, has been appointed Associate Professor at Arizona State College.

Miss V. Agnes Wessling, College of St. Theresa, has been appointed Assistant Professor at Wisconsin State College.

Assistant Professor A. M. White, U. S. Army Mathematics Research Center, University of Wisconsin, has been appointed Assistant Professor to Harvey Mudd College.

Mr. E. R. Young, University of Miami, has accepted a position as Numerical Analyst with Pratt and Whitney Aircraft, West Palm Beach, Florida.

Dr. David Zeitlin, Remington Rand Univac, St. Paul, Minnesota, has accepted a position as Senior Mathematician in the Scientific Computing Section of the Data Processing and Computing Department of Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota.

Professor W. H. Bussey, University of Minnesota, died on June 6, 1962. He was a Charter Member of the Association. During 1916–1918 he served as a member of the Editorial Board of the MONTHLY, and was Editor-in-chief of the MONTHLY for 1927–1931.

Mr. Louis Child, New Mexico State University, died on September 9, 1962. He was a member of the Association for 7 years.

Mrs. Nancy M. Dismuke, Oak Ridge National Laboratory, Oak Ridge, Tennessee, died on August 30, 1962. She was a member of the Association for 6 years.

Mr. O. E. Eggert, Morton, Pennsylvania, died on August 26, 1962. He was a member of the Association for 28 years.

Mr. C. W. Peck, University of Tennessee, died on June 16, 1962. He was a member of the Association for 6 years.

Professor L. R. Van Deventer, Eastern Illinois University, died on July 8, 1962. He was a member of the Association for 12 years.

Dr. Arthur Wormser, Infilco, Inc., Tucson, Arizona, died on October 4, 1962. He was a member of the Association for 14 years.

BIOMATHEMATICS TRAINING AT NORTH CAROLINA STATE COLLEGE

In 1961 a training program in biomathematics was initiated in the Institute of Statistics at North Carolina State College. This program, made possible with the aid of a grant from the National Institutes of Health, offers training at both the doctoral and postdoctoral levels. It is the purpose of the biomathematics program to promote competence in the applications of mathematics to biology and in the development of biological theory by integrated study of biology, mathematics, statistics and the physical sciences.

The program is quite flexible and the Ph. D. degree may be granted in any one of several areas. Graduate assistantships and post-doctoral fellowships are available within the program, but persons anticipating support from outside sources are also invited to apply. Information concerning the program may be obtained by writing to: Dr. James H. Meade, Jr., Institute of Statistics, P. O. Box 5457, Raleigh, North Carolina.

CANADIAN MATHEMATICAL CONGRESS

The ninth biennial seminar of the Canadian Mathematical Congress will take place at the University of Saskatchewan, Saskatoon, Canada, from August 12 to 30, 1963.

The main theme of the seminar will be combinatorial mathematics, but two of the series of lectures will be noncombinatorial. There will be four series of six research lectures and four series of fifteen instructional lectures. The program is not yet complete and the following is only a partial list.

Research lecturers are R. H. Bruck, U. S. A., Existence problems for classes of finite planes; G. Pickert, Germany, Projective planes; E. M. Wright, Great Britain, Partitions; and K. Kuratowski, Poland, Semicontinuity in topology.

Instructional lecturers (all from Canada) are W. J. Tutte, Graph theory; A. L. Dulmage, Combinatorial problems related to graph theory and linear programming; and A. P. Guinand, Fourier transforms and summation formulae.

Besides the lecture series there will be a colloquium which will meet once or twice a week, at which one-half hour and one hour papers on combinatorial topics will be presented.

Dormitory accommodation for both single and married participants will be available at the University of Saskatchewan residences. Inquires should be addressed to the Executive Director of the Canadian Mathematical Congress, Professor L. F. S. Ritcey, 985 Sherbrooke Avenue West, Montreal, Canada.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met on April 27, 1962, at the Missouri School of Mines and Metallurgy, Rolla, Missouri. Fifty-eight members of the Association were in attendance. The meeting was presided over by Professor R. M. Rankin, Chairman of the Mathematics Department at the host institution and also Chairman of the Missouri Section.

The following officers were elected for the coming year: Chairman, Professor Carl V. Fronabarger, Southwest Missouri State College, Springfield; Vice-Chairman, Professor Charles A. Johnson, Missouri School of Mines; Secretary-Treasurer, Professor Edward Z. Andalafte, Southwest Missouri State College, Springfield.

The following papers were presented at the morning session:

1. *Convex sets*, by Professor J. N. Younglove, University of Missouri.
2. *Factoring Mersenne numbers*, by Professor Edgar Karst, Evangel College.
3. *Sums of powers of integers*, by Professor E. G. Eigel, Jr., St. Louis University.
4. *Bell-shaped functions*, by Professor I. I. Hirschman, Washington University.
5. *Developments in computer programming techniques*, by Mr. R. F. Keller, University of Missouri.

The luncheon held jointly with the Missouri Council of Teachers of Mathematics was addressed by Professor F. E. Hohn of the University of Illinois, who is an MAA Visiting Lecturer. The topic of Dr. Hohn's address was "Frames, Games, and Mathematics," and dealt primarily with development of the concepts of modern mathematics at the elementary, high school and college level.

An afternoon panel discussed "Applied Mathematics in the University, Government, and Industry" and was presided over by Dr. Hohn with Mr. Andrew Cochran of the U. S. Bureau of Mines, Mr. R. J. Katzman of IBM, and Professor C. A. Johnson of the Missouri School of Mines as panelists. Tours of the Nuclear Reactor Building and of the Missouri School of Mines Computer Center were also conducted in the afternoon for members and guests.

C. A. Johnson, *Secretary*

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THE AMERICAN MATHEMATICAL MONTHLY

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Sections of the MONTHLY (Mathematical Notes, Classroom Notes, Mathematical Education Notes Elementary Problems, Advanced Problems Recent Publications, News and Notices) Editor of Section (Address at head of Section)

Other editorial correspondence F. A. FICKEN, New York University, New York 53, N. Y.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

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THE EULER CHARACTERISTIC IN COMBINATORIAL GEOMETRY

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0. Introduction. A familiar tool in algebraic topology is the Euler-Poincaré characteristic, defined as an alternating sum of ranks of homology groups [1]. Hadwiger [6] has observed that for the class $\mathfrak{U}$ of all finite unions of compact convex sets in R^n, the existence of an Euler characteristic χ can be established in an elementary way, independent of algebraic topology; χ is a *valuation* on $\mathfrak{U}$ such that $\chi\varnothing=0$ ($\varnothing$ being the empty set) and $\chi C=1$ for each nonempty convex $C\in\mathfrak{U}$. As Hadwiger showed [5, 6], χ is useful in connection with many questions from combinatorial geometry, especially those concerning intersection properties of convex sets.

The primary purpose of this note is to call attention to Hadwiger's approach, which seems quite suitable for inclusion in college geometry courses. The basic notion is presented in a way which exhibits its lattice-theoretic nature, and some additional applications to combinatorial geometry are described. The exposition is self-contained except for some elementary notions from lattice theory [2] and the geometry of convex bodies [1, 3, 8].

An important role is played by the well-known arithmetical identity,

$$(\dagger)\qquad \sum_{i=0}^{m}(-1)^i\binom{n}{i}=(-1)^m\binom{n-1}{m},$$

which is evident for $m=0$. If (†) is known for $m=j$, then

$$\sum_{i=0}^{j+1}(-1)^i\binom{n}{i}=(-1)^j\binom{n-1}{j}+(-1)^{j+1}\binom{n}{j+1}$$
$$=(-1)^j\left[\binom{n-1}{j}-\binom{n}{j+1}\right]=(-1)^j\left[-\binom{n-1}{j+1}\right],$$

whence (†) holds for $m=j+1$ and is thus proved by mathematical induction. (Note that (†) holds for all nonnegative integers m and n, under the standard conventions $\binom{-1}{0}=1=\binom{n}{0}$ and $\binom{n}{i}=0$ for $i>n$.)

1. Lattice-theoretic setting for the Euler characteristic. Throughout this section, L will denote a fixed distributive lattice, which has an initial element θ. Thus L is a partially ordered set with θ as first element, each pair x and y of elements of L admits a least upper bound $x\smile y\in L$ and a greatest lower bound $x\frown y\in L$ and for all x,y and $x\in L$ it is true that $(x\smile y)\frown z=(x\frown z)\smile(y\frown z)$. (It is known that such a lattice must be isomorphic with a ring of sets.) Each nonempty finite subset A of L admits a least upper bound $\bigvee A\in L$ and a greatest lower bound $\bigwedge A\in L$. Further, $\bigvee\varnothing=\theta\in L$. For each subset S of L we define

$$S^{\smile} = \{\vee A\colon \text{nonempty finite } A \subset S\},$$
$$S^{\frown} = \{\wedge A\colon \text{nonempty finite } A \subset S\},$$

and

$$S^{l} = (S^{\frown})^{\smile} \cup \{\theta\} = (S^{\smile})^{\frown} \cup \{\theta\},$$

the smallest sublattice of L which contains $S \cup \{\theta\}$.

When $S \subset L$, an *Euler characteristic for* S(relative to L) is a real-valued function χ on S^l which satisfies the following two conditions:

$$\mathfrak{E}' : \chi\theta = 0; \qquad \chi s = 1 \qquad \text{for } \theta \neq s \in S;$$
$$\mathfrak{E}'' : \chi a + \chi b = \chi(a \smile b) + \chi(a \frown b) \qquad \text{for all } a, b \in S^l$$

(i.e., χ is a valuation).

(Recall that in the case of special interest, L is the lattice of all subsets of R^n, with $a \smile b = a \cup b$ and $a \frown b = a \cap b$, and S is the set of all compact convex members of L; S^l is the class $\mathfrak{U}$ mentioned in the Introduction.)

Certain auxiliary functions are useful in discussing Euler characteristics. We define

$$\epsilon\theta = 0; \qquad \epsilon a = 1 \qquad \text{for } \theta \neq a \in L.$$

And for each finite $B \subset L$,

$$\xi B = \sum_{\phi \neq A \subset B} (-1)^{(\text{card } A) - 1} \epsilon(\wedge A).$$

Thus

$$\xi B = \sum_{i=1}^{\text{card } B} (-1)^{i-1}(\mu_i B),$$

where $\mu_i B$ is the number of i-membered subsets A of B for which $\wedge A \neq \theta$. Since B is merely a subset of L and not an indexed family of points, it makes no sense to speak of "repetition" of elements of B in the counting process by which ξB is defined. On the other hand, for each finite set J and indexed family $(a_\iota)_{\iota \in J}$ of points of L, we define

$$\eta(a_\iota)_{\iota \in J} = \sum_{\phi \neq I \subset J} (-1)^{(\text{card } I) - 1} \epsilon\left(\bigwedge_{\iota \in I} a_\iota\right),$$

where $\wedge_{\iota \in I} a_\iota = \wedge\{a_\iota\colon \iota \in I\}$ and it is permitted, of course, that $a_\iota = a_{\iota'}$ for $\iota \neq \iota'$. It is useful to know that η really depends only on the set $\{a_\iota\colon \iota \in J\}$ and not upon how often the various points of this set are repeated in the indexing. This fact, as well as a convenient way of computing η and ξ, follows from the observation that if $\vee_1^n a_i \leqq e \in L$, and $e \neq \theta$ then

$$1 - \eta(a_1, a_2, \cdots, a_n) \doteq (e - a_1)(e - a_2) \cdots (e - a_n)$$

in the sense that $1-\eta(a_1, \cdots, a_n)$ is equal to the result of developing the right-hand expression as an algebraic product, replacing each term by 0 or 1 according as the greatest lower bound of its elements is $=\theta$ or $>\theta$, and then performing the indicated addition and subtraction. In fact, this symbolic device makes it easy to prove

PROPOSITION 1.1. *Suppose A is a finite subset of L and $a_1, \cdots, a_n$ are elements of L such that $A \subset \{a_i\}_1^n$ and each a_i is $\leqq$ some element of A. Then $\xi A = \eta(a_1, \cdots, a_n)$.*

Proof. Since the symbolic product $(e-a_1) \cdots (e-a_n)$ is obviously commutative in its factors, 1.1 results from successive applications of the fact that for $b_i \in L$,

$$b_1 \leqq b_2 \Rightarrow \eta(b_1, b_2, \cdots, b_n) = \eta(b_2, \cdots, b_n).$$

This fact follows from the symbolic equations

$$\begin{aligned}(e - b_1)(e - b_2) \cdots (e - b_n) &\doteq e(e - b_2) \cdots (e - b_n) - b_1(e - b_2) \cdots (e - b_n)\\ &\doteq (e - b_2) \cdots (e - b_n) - b_1e(e - b_3) \cdots (e - b_n) + b_1b_2(e - b_3) \cdots (e - b_n)\\ &\doteq (e - b_2) \cdots (e - b_n),\end{aligned}$$

where $e \geqq \bigvee_1^n b_i$ and the last step results from the equality $b_1 \frown e = b_1 \frown b_2$.

A subset K of L will be called *intersectional* provided $K^\frown = K$, whence of course $K^\iota = K^\smile \subset \{\theta\}$. The following property of the Euler characteristic is the one which is fundamental for combinatorial geometry.

THEOREM 1.2. *If K is an intersectional subset of L and χ is an Euler characteristic for K, then $\chi(\bigvee A) = \xi A$ for each finite $A \subset K$.*

Proof. For card $A \leqq 1$, this follows from condition $\mathfrak{E}'$. Suppose it is known for all sets of cardinality $\leqq n$, and consider a set B of cardinality $n+1$, say $B = A \cup \{b\} \subset K$ where card $A = n$. From the definition of ξ it is evident that

$$\xi B = \xi A + \xi\{b\} - \xi A' \quad \text{where} \quad A' = \{a \frown b : a \in A\}.$$

From $\mathfrak{E}''$ and the inductive hypothesis it follows that

$$\begin{aligned}\chi(\bigvee B) &= \chi(\bigvee A \smile b) = \chi(\bigvee A) + \chi b - \chi((\bigvee A) \frown b)\\ &= \xi A + \xi\{b\} - \xi A',\end{aligned}$$

where the equality $\chi((\bigvee A) \frown b) = \xi A'$ depends on the fact that $(\bigvee A) \frown b = \bigvee A'$ and (since K is intersectional) $A' \subset K$. This completes the proof.

An immediate consequence of 1.2 is

COROLLARY 1.3. *An intersectional subset of L cannot admit more than one Euler characteristic.*

And another is

COROLLARY 1.4. *Suppose K is an intersectional subset of L, K admits an Euler characteristic, and A is a finite subset of K with $\theta \neq \bigvee A \in K$. For $1 \leqq i \leqq \text{card } A$, let μ_i denote the number of sets $B \subset A$ with* card $B = i$ *and $\bigwedge B > \theta$. Then*

$$\sum_{i=1}^{\text{card } A} (-1)^{i-1}\mu_i = 1.$$

The following is of special interest for combinatorial geometry:

COROLLARY 1.5. *With K, A, and μ_i as in* 1.4, *suppose $1 \leqq m < n =$* card A, *and $\mu_i = \binom{n}{i}$ for $1 \leqq i \leqq m$. Then*

$$\sum_{i=m+1}^{n} (-1)^{i-1}\mu_i = (-1)^m \binom{n-1}{m},$$

whence $\mu_{m+1} \neq 0$. Further, $\mu_{m+2} = 0 \Rightarrow \mu_{m+1} = \binom{n-1}{m}$.

Proof. The second and third statements follow from the first, for $\mu_j = 0 \Rightarrow \mu_i = 0$ for $i \geqq j$. The first statement follows from 1.4 in conjunction with the arithmetical identity (†) (in the Introduction).

The following result, though rather technical, is also of interest for combinatorial geometry:

COROLLARY 1.6. *With K and L as in* 1.4, *and W a finite subset of $L \sim \{\theta\}$' suppose the elements $k_w (w \in W)$ satisfy the following conditions*:

(α) *for each $w \in W$, $w \leqq k_w \in K$;*

(β) *for each $w \in W$, there exists $c_w \in K$ such that $\bigvee(W \sim \{w\}) \leqq c_w$ and $c_w \frown k_w = \theta$;*

(γ) *there exists $k \in K$ such that $\bigvee W \leqq k \leqq \bigvee_{w \in W} k_w$.*

Then $\bigwedge_{w \in W} k_w > \theta$.

Proof. The proof is by induction on the cardinality of W, 1.6 being obvious when W has but one element. Suppose it is known for sets of cardinality $n-1$, and consider a situation as described in which W has n elements. For each $w \in W$, let $k_w' = k_w \frown k$, whence all the conditions are satisfied with k_w' in place of k_w. Further, $\bigvee_{w \in W} k_w' = k \in K$, so in order to show that $\bigwedge_{w \in W} k_w' > \theta$ it suffices, in view of 1.5, to show that $\bigwedge_{w \in W \sim \{v\}} k_w' > \theta$ for each $v \in W$. But this follows at once from the inductive hypothesis upon observing that conditions (α), (β), and (γ) are satisfied when W, k_w, c_w, and k are replaced by $W \sim \{v\}$, k_w', c_w and $k \frown c_v$ respectively. (To check (γ), note that

$$\begin{aligned} \bigvee(W \sim \{v\}) &\leqq \bigvee W \frown \bigvee(W \sim \{v\}) \leqq k \frown c_v \\ &\leqq \bigvee_{w \in W} (k_w' \frown c_v) \leqq \Big(\bigvee_{w \in W \sim \{v\}} k_w'\Big) \smile (k_v' \frown c_v), \end{aligned}$$

where $k_v' \frown c_v = \theta$.) The proof of 1.6 is complete.

The next theorem supplements 1.2 by making more explicit the connection between χ and ξ:

THEOREM 1.7. *For an intersectional subset K of L, the following three assertions are equivalent:*

(α) *K admits an Euler characteristic;*

(β) *if A is a finite subset of K and $\bigvee A \in K \sim \{\theta\}$, then $\xi A = 1$;*

(γ) *for finite subsets F of K, ξF depends only on $\bigvee F$; i.e., if A and B are finite subsets of K with $\bigvee A = \bigvee B$, then $\xi A = \xi B$.*

Proof. Clearly $(\gamma) \Rightarrow (\beta)$, and from 1.2 it follows that $(\alpha) \Rightarrow (\gamma)$. We complete the proof by showing that $(\beta) \Rightarrow (\gamma)$ and $(\gamma) \Rightarrow (\alpha)$. It is convenient to work with η rather than ξ, for this obviates concern over the distinctness of certain sets.

Suppose (β) holds and consider elements $a_1, \cdots, a_m$ of K and $b_1, \cdots, b_n$ of K with $\bigvee_1^m a_i = \bigvee_1^n b_j$. From ($\beta$) it follows that if $a \in K$ and $a \leqq \bigvee_1^n b_j$, then $\eta a = \eta(a \frown b_1, \cdots, a \frown b_n)$, whence

$$a \doteq (e - a \frown b_1) \cdots (e - a \frown b_n)$$

in the symbolic sense employed earlier. Applying this for the successive choices $a = a_1$, $a = a_1 \frown a_2, \cdots, a = a_1 \frown a_2 \cdots \frown a_m$, one verifies easily that

$$(e - a_1)(e - a_2) \cdots (e - a_m)$$
$$\doteq (e - a_1 \frown b_1) \cdots (e - a_1 \frown b_n)(e - a_2) \cdots (e - a_m).$$

Similar replacements can be made for the other factors $(e - a_i)$, leading to the conclusion that

$$\eta(a_1, \cdots, a_m)$$
$$= \eta(a_1 \frown b_1, \cdots, a_1 \frown b_n, a_2 \frown b_1, \cdots, a_2 \frown b_n, \cdots, a_n \frown b_1, \cdots, a_n \frown b_n).$$

By the same sort of argument, the right-hand member is also equal to $\eta(b_1, \cdots, b_n)$, and it follows that (β) implies (γ).

Now suppose that (γ) holds, and for each finite $F \subset K$ define $\chi(\bigvee F) = \xi F$. Then χ is a real-valued function on K^l, $\chi\theta = 0$, and $\chi c = 1$ for $\theta \neq c \in K$. To show that χ is an Euler characteristic for K it remains to prove that χ is a valuation, or, equivalently, that if $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ are finite indexed families of points of K, then

$$\eta(a_i)_{i \in I} + \eta(b_j)_{j \in J} = \eta[(a_i)_{i \in I} \cup (b_j)_{j \in J}] + \eta(a_i \frown b_j)_{i \in I, j \in J}.$$

This in turn is equivalent to the symbolic equality

$$(*) \quad \prod_{i \in I} (e - a_i) + \prod_{j \in J} (e - b_j) - \prod_{i \in I, j \in J} (e - a_i \frown b_j) = \prod_{i \in I} (e - a_i) \cdot \prod_{j \in J} (e - b_j).$$

Now it is clear that for each $S \subset I$ and $T \subset J$, the terms $\bigwedge_{s \in S} a_s$ and $\bigwedge_{t \in T} b_t$ appear on both sides of (*) with the coefficients $(-1)^{\text{card } S}$ and $(-1)^{\text{card } T}$ respectively. The term $\bigwedge_{s \in S} a_s \frown \bigwedge_{t \in T} b_t$ appears on the right side with coefficient $(-1)^{\text{card } S + \text{card } T}$, so to complete the proof of 1.7 it will suffice to show that this is also the net coefficient of $\bigwedge_{s \in S} a_s \frown \bigwedge_{t \in T} b_t$ on the left side of (*).

Let $m = \text{card } S$ and $n = \text{card } T$. For each α, let $P_\alpha(m, n)$ denote the number of relations of cardinality α whose domain is $\{1, \cdots, m\}$ and whose range is $\{1, \cdots, n\}$ (where of course a relation is a set of ordered pairs). Then the coefficient in question is equal to $-\sum_{-\infty}^{\infty} (-1)^\alpha P_\alpha(m, n)$, with $P_\alpha(m, n) = 0$ for $\alpha < \max(m, n)$ and for $\alpha > mn$. Thus for 1.7 it suffices to establish

$$(**) \qquad \sum_\alpha (-1)^\alpha P_\alpha(m, n) = (-1)^{m+n+1}.$$

For $m = 1$ and arbitrary n, (**) reduces to $(-1)^n = (-1)^{n+2}$. Now (proceeding to a proof by induction) let us suppose that (**) is known for $m = h-1 \geqq 1$ (and for all n), and let us consider the case $m = h$. Corresponding to each pair of sets (V, U) for which $U \subset V \subset \{1, \cdots, n\}$ and $\{1, \cdots, n\} \sim U \neq \emptyset$, there are $P_{\alpha - \text{card } V}(h-1, n - \text{card } U)$ relations among those which are counted by $P_\alpha(h, n)$; these are exactly those relations ρ which have $\rho(h) = V$ and $\rho\{1, \cdots, h-1\} = \{1, \cdots, n\} \sim U$. Further, each relation counted by $P_\alpha(h, n)$ corresponds in this way to exactly one pair (V, U) as described. Consequently,

$$\begin{aligned} P_\alpha(h, n) &= \sum_{j=1}^{n} \left[\binom{n}{j} \sum_{i=0}^{j} \binom{j}{i} P_{\alpha-j}(h-1, n-i) \right] + \sum_{i=0}^{n-1} \binom{j}{i} P_{\alpha-n}(h-1, n-i) \\ &= \sum_{0 \leqq i \leqq j \leqq n}{}' \binom{n}{j} \binom{j}{i} P_{\alpha-j}(h-1, n-i), \end{aligned}$$

where the $'$ indicates omission of terms corresponding to $i = 0 = j$ and to $i = n = j$. Now (using the inductive hypothesis)

$$\begin{aligned} \sum_\alpha (-1)^\alpha P_\alpha(h, n) &= \sum_{0 \leqq i \leqq j \leqq n}{}' \left[\binom{n}{j} \binom{j}{i} (-1)^j \sum_\alpha (-1)^{\alpha-j} P_{\alpha-j}(h-1, n-i) \right] \\ &= \sum_{0 \leqq i \leqq j \leqq n}{}' \binom{n}{j} \binom{j}{i} (-1)^j (-1)^{h+n-i} \\ &= (-1)^{h+n} \sum_{0 \leqq i \leqq j \leqq n}{}' \binom{n}{j} \binom{j}{i} (-1)^{i+j} \end{aligned}$$

(since $(-1)^{i+j} = (-1)^{j-i}$).
Thus to complete the proof it will suffice to show that

$$\sum_{0 \leqq i \leqq j \leqq n} \binom{n}{j} \binom{j}{i} (-1)^{i+j} = +1,$$

(note omission of $'$). But this is immediate from (†) of the Introduction, according to which

$$\sum_{j=0}^{n}\left[\binom{n}{j}(-1)^j\sum_{i=0}^{j}\binom{j}{i}(-1)^i\right] = \sum_{j=0}^{n}\binom{n}{j}(-1)^j\binom{j-1}{j}(-1)^j$$
$$= \binom{n}{0}(-1)^0\binom{-1}{0}(-1)^0 = 1.$$

The proof of 1.7 is complete.

COROLLARY 1.8. *Suppose that K is an intersectional subset of L, and that for each finite $A \subset K \sim \{\theta\}$ there exists $z \in L$ such that the set $\{k^\frown z: k \in K\}$ admits an Euler characteristic and $a^\frown z > \theta$ for all $a \in A$. Then K admits an Euler characteristic.*

Proof. In view of 1.7, it suffices to show that whenever A_1 and A_2 are finite subsets of K such that $\bigvee A_1 = \bigvee A_2$, then $\xi A_1 = \xi A_2$. Let $A = \{\bigwedge B: B \subset A_1 \cup A_2\} \sim \{\theta\}$, and let z be related to A as in the hypothesis of 1.8. Let $F_i = \{a^\frown z: a \in A_i\}$. Then $\bigvee F_1 = \bigvee F_2$, and since (by hypothesis) the intersectional set $\{k^\frown z: k \in K\}$ admits an Euler characteristic, it follows from 1.7 that $\xi F_1 = \xi F_2$. But the choice of z was such that $\xi F_i = \xi A_i$, so this completes the proof.

2. Applications to combinatorial geometry. A convex set C in a real linear space will be called *closed* ⟨resp. *open*⟩ provided C's intersection with each line is closed ⟨resp. open⟩ in the natural topology of the line.

THEOREM 2.1. *If $\mathfrak{C}$ is the class of all closed ⟨resp. open⟩ convex subsets of a real linear space, then $\mathfrak{C}$ admits an Euler characteristic. That is, there exists a real-valued function χ, defined on the class $\mathfrak{C}^l$ of all finite unions of members of $\mathfrak{C}$, such that $\chi\varnothing = 0$, $\chi C = 1$ for nonempty $C \in \mathfrak{C}$, and*

$$\chi A + \chi B = \chi(A \cup B) + \chi(A \cap B)$$

for all $A, B \in \mathfrak{C}^l$.

Proof. In view of 1.8, it suffices to produce an Euler characteristic for the class of all bounded closed ⟨resp. open⟩ convex subsets of an m-dimensional real linear space R^m (m finite). Hadwiger's proof [6] is by induction on m, the statement being obvious for $m = 0$. Suppose it is known for $m = n-1$, and consider the class $\mathfrak{K}$ of all bounded closed ⟨resp. open⟩ convex subsets of R^n. Let R^{n-1} be a hyperplane through the origin in R^n and Rx a line supplementary to R^{n-1}. For each $r \in R$ (the real field), let H_r denote the hyperplane $R^{n-1} + rx$, K_r the class of all bounded closed ⟨resp. relatively open⟩ convex subsets of H_r, and χ_r the Euler characteristic for $\mathfrak{K}_r$ (existence assured by the inductive hypothesis). Consider an arbitrary $A \in \mathfrak{K}^l$, A being the union of a finite subclass $\mathfrak{F}$ of $\mathfrak{K}$, and let $\mathfrak{D}$ denote the class of all intersections of members of $\mathfrak{F}$. Then for each $D \in \mathfrak{D}$, the set $\{r: D \cap H_r \neq \varnothing\}$ is a closed ⟨resp. open⟩ interval in the real line R, and from 1.2 it follows that $\chi_r(A \cap H_r) = \chi_s(A \cap H_s)$ unless one of these intervals includes exactly one of the numbers r and s. Thus we may define a real number χA by the condition that

$$\chi A = \pm \sum_{r \in R}\left[\chi(A \cap H_r) - \lim_{s \searrow r}\chi_s(A \cap H_s)\right],$$

using $+$ when dealing with closed sets and $-$ with open sets. Then χ is a real-valued function on $\mathcal{K}^l$, and it is evident that $\chi\varnothing=0$ while $\chi K=1$ for each nonempty $K\in\mathcal{K}$. Since the valuation property of χ follows at once from that of the χ_r's, χ must be an Euler characteristic for $\mathcal{K}$ and the proof is complete.

An analogue of 2.1 is valid in a vector space over an arbitrary ordered field, but simple examples show that the sets must be more restricted. (In the rational number space, $[1, \sqrt{2}[$ and $]\sqrt{2}, 2]$ are disjoint closed convex sets whose union is convex.) In any case, there exists an Euler characteristic for the class of finite intersections of closed ⟨resp. open⟩ halfspaces, where these are defined in the natural way in terms of homomorphisms onto the scalar field [**8**].

The following result was recently proved by Ghouila-Houri [**4**] (the second part in sharper form in [**4**]). It is an immediate consequence of 2.1 and 1.5.

Proposition 2.2. *Suppose $C_1, \cdots, C_n$ are closed ⟨resp. open⟩ convex subsets of a real linear space. If $\bigcup_1^n C_i$ is convex and each m of the sets C_i have a common point $(m<n)$, then some $m+1$ have a common point. If each $n-1$ of the sets C_i have a common point but $\bigcap_1^n C_i=\varnothing$, and if C_0 is a closed ⟨resp. open⟩ convex set such that $\bigcap_0^n C_i$ is convex, then C_0 meets each intersection of $n-1$ sets C_i.*

Similarly, 2.1 and 1.6 yield the following special case of a well-known theorem of Sperner [**7**]: If an n-simplex $\operatorname{conv}\{p_0, \cdots, p_n\}$ is covered by $n+1$ closed convex sets $C_0, \cdots, C_n$ in such a way that always $C_j\cap\operatorname{conv}\{p_i: i\neq j\}=\varnothing$, then $\bigcap_0^n C_i\neq\varnothing$.

Hadwiger [**6**] gives (essentially) the above applications and several others as well, in both linear and spherical spaces. Of course, many of the results obtained in this way follow from more general topological theorems, but the elementary method is of interest. (Note that 1.7 and 1.8 are not required for the above applications or for those of Hadwiger.) For an additional application, let us consider a (polyhedral) cell-complex $\mathcal{P}$ in R^n, this being a finite set of (compact) convex polyhedra such that each face of a member of $\mathcal{P}$ is itself a member of $\mathcal{P}$, and such that the intersection of any two members of $\mathcal{P}$ is a face common to both [**1**]. For each such $\mathcal{P}$, let $\delta_i\mathcal{P}$ denote the number of i-dimensional members of $\mathcal{P}$ and $\psi P=\sum_{i=0}^{\infty}(-1)^i\delta_i P$. The most important property of $\mathcal{P}$ is that it is actually determined by the union $|\mathcal{P}|$ of the members of $\mathcal{P}$; that is, if $\mathcal{P}$ and $\mathcal{Q}$ are cell-complexes for which $|\mathcal{P}|=|\mathcal{Q}|$, then $\psi\mathcal{P}=\psi\mathcal{Q}$. This property is a corollary of the following fact (where χ is the Euler characteristic of 2.1).

Theorem 2.3. *For each cell-complex $\mathcal{P}$, $\chi|\mathcal{P}|=\psi\mathcal{P}$.*

Proof. Note first that if $\mathcal{P}$ and $\mathcal{P}'$ are cell-complexes whose union is a cell-complex, then

$$\psi(\mathcal{P}\cup\mathcal{P}')=\psi\mathcal{P}+\psi\mathcal{P}'-\psi(\mathcal{P}\cap\mathcal{P}').$$

For positive integers m and n, let $S_{m,n}$ denote the statement that $\chi|\mathcal{P}|=\psi\mathcal{P}$ for each cell-complex $\mathcal{P}$ in R^n having at most m maximal members. We claim that for $m\geqq 1$, $S_{m,n}\Rightarrow S_{m+1,n}$. Indeed, consider a cell-complex $\mathcal{Q}$ in R^n having $m+1$

maximal members. Let Q be one of these, $\mathcal{Q}_2$ the complex consisting of Q along with its faces, and $\mathcal{Q}_1$ the complex determined by the remaining maximal members of $\mathcal{Q}$. Then $S_{m,n}$ applies not only to $\mathcal{Q}_1$ and $\mathcal{Q}_2$, but also to the complex $\mathcal{Q}_1 \cap \mathcal{Q}_2$, for it is evident that two maximal members of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ cannot lie in the same member of $\mathcal{Q}_1$. Thus from $S_{m,n}$ and the valuation property of χ we see that

$$\begin{aligned} \psi\mathcal{Q} &= \psi\mathcal{Q}_1 + \psi\mathcal{Q}_2 - \psi(\mathcal{Q}_1 \cap \mathcal{Q}_2) \\ &= \chi|\mathcal{Q}_1| + \chi|\mathcal{Q}_2| - \chi|\mathcal{Q}_1 \cap \mathcal{Q}_2| \\ &= \chi(|\mathcal{Q}_1| \cup |\mathcal{Q}_2|) = \chi|\mathcal{Q}|. \end{aligned}$$

To complete the proof of 2.3, it remains to establish $S_{1,n}$ for all n, where of course $S_{1,n}$ is Euler's famous theorem for convex polyhedra. For simplexes this is an immediate consequence of the arithmetic identity (†) of the Introduction, and thus the proof of 2.3 for simplicial complexes is especially easy. For the general case, we employ another induction, showing that $\bigcup_{m=1}^{\infty} S_{m,n} \Rightarrow S_{1,n+1}$. Indeed, consider a complex $\mathcal{P}$ in R^{n+1} having only one maximal member and that an $(n+1)$-dimensional convex polyhedron P. Let A be an n-dimensional face of P, H a hyperplane which supports P and contains A, and z a point of R^{n+1} on the opposite side of H from P, so situated that A is intersected by each segment which joins z to a point of P. Let $\mathcal{Q}$ denote the complex $\mathcal{P} \sim \{P, A\}$, and for each $Q \in \mathcal{Q}$ let Q' denote the intersection with H of the cone from z over Q. Then $\{Q' : Q \in \mathcal{Q}\}$ is a cell-complex $\mathcal{Q}'$ for which $|\mathcal{Q}'| = A$, whence from $\bigcup_{m=1}^{\infty} S_{m,n}$ we conclude that $\psi\mathcal{Q}' = 1$. But then $\psi\mathcal{Q} = 1$, whence $\psi\mathcal{P} = 1$ and the proof is complete. (With a little care, the proof can be phrased so as to apply to cell-complexes in vector spaces over arbitrary ordered fields.)

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MATHEMATICAL INDUCTION AND RECURSIVE DEFINITIONS

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Many students first encounter mathematical induction during a beginning course in algebra, either in secondary school or in college. For some of these students, this can become their first introduction to mathematical ideas, turning their attention away from computational exercises to notions of structure and proof. Their initial distrust is gradually replaced by an appreciation for its power; with reference to dominoes or inductive sets, they will usually become convinced of its reasonableness. Some students will retain a cautious attitude toward certain types of applications of induction; in this paper, I wish to increase the number of these students by discussing some of these problems.

Suppose that we have agreed upon a workable definition of the notion of function. We will deal only with the set $I=\{0, 1, 2, \cdots\}$ of nonnegative integers, or with the set $I\times I$ of pairs of integers, or more generally, with the set $I^k=I\times I\times \cdots \times I$ for some specific k. A function will always be defined on a subset of such a set, and will take its values in I. If we identify a function f with its graph, then a function f on I to I will become a specific set of pairs $\langle n, f(n)\rangle$ for all $n\in I$. More generally, we would say that any nonempty subset S of $I\times I$ that is univalent is a function; its domain will be the projection of S into the first coordinate space.

We now present a student with the following definition of a function f on I to I:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 2 \\ f(n+1) &= f(n)+f(n-1) \qquad \text{all } n = 1, 2, 3, \cdots. \end{aligned} \tag{1}$$

I think that it will be quite convincing to the student that there is such a function f, and that it is uniquely defined by formula (1). He has no difficulty seeing that $f(2)=3$, $f(3)=5$, $f(4)=8$, and so on. In short, he believes in the existence of f. However, there often remains a certain unhappiness in the mind of the student; he may say that he wants a formula for f. If he is pressed to explain, it will be found that he feels that functions ought to be described by means of certain allowed operations such as addition and composition, and ought to be built up from simpler functions. Students are constructivists at heart.

In the present example, of course, a formula can be given and should be given. He will express astonishment that so much complexity is needed for such a simple appearing function.

$$f(n) = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2}$$

Of course, you can convince him that this is correct, first by computing several values to check it, and then by using mathematical induction. We observe that

$$f(0) = \frac{5+3\sqrt{5}}{10} + \frac{5-3\sqrt{5}}{10} = \frac{10}{10} = 1,$$

and that

$$\begin{aligned} f(1) &= \frac{5+3\sqrt{5}}{10}\ \frac{1+\sqrt{5}}{2} + \frac{5-3\sqrt{5}}{10}\ \frac{1-\sqrt{5}}{2} \\ &= \frac{5+8\sqrt{5}+15}{20} + \frac{5-8\sqrt{5}+15}{20} \\ &= 2. \end{aligned}$$

We have verified the correctness of (2) for $n=0$ and $n=1$. Suppose that we have verified it for $n=0, 1, 2, \cdots, k$, can we be sure that it holds for $n=k+1$? For simplicity, set $A = (5+3\sqrt{5})/10$, $B = (5-3\sqrt{5})/10$, $\alpha = (1+\sqrt{5})/2$ and $\beta=(1-\sqrt{5})/2$. Then, formula (2) can be written as

$$f(n) = A\alpha^n + B\beta^n. \tag{3}$$

By (1), we have the right to express $f(k+1)$ as $f(k)+f(k-1)$, so that the inductive hypothesis gives us

$$\begin{aligned} f(k+1) &= [A\alpha^k + B\beta^k] + [A\alpha^{k-1} + B\beta^{k-1}] \\ &= A\alpha^{k-1}[1+\alpha] + B\beta^{k-1}[1+\beta]. \end{aligned}$$

Observing that $\alpha^2=1+\alpha$, and $\beta^2=1+\beta$, we obtain

$$f(k+1) = A\alpha^{k+1} + B\beta^{k+1},$$

verifying (3) and thus (2). Or, if the instructor has been more formal in his presentation, he can point out that this argument has shown that the set $E\subset I$ of those integers n for which (2) holds is an inductive set; since it contains 0, it must, perhaps by an axiom rather than a theorem, be the whole set I.

At this point, the line of development is sure to be interrupted by a clamor to know "where formula (2) came from." This is handled either by suppressive measures, or by embarking upon a brief discussion of difference equations with constant coefficients.

But all of this has, to a certain extent, detoured the real and valid question which was in the student's minds. Is it in fact legitimate to *define* a function by a dodge like that of formula (1)? This certainly does not describe f either as a mapping or as a class of pairs. In this particular example, we were lucky enough to have found a formula. Is this always the case? Moreover, an alert and cautious student may also raise the general problem of how one can tell whether a relation similar to (1) admits a solution. For example, is there a function f which satisfies this relation?

$$
(4) \qquad \begin{aligned} f(1) &= 1 \\ f(2n+1) &= n^2 - n + 1 \\ f(3n+1) &= 2n + f(2n+1) \end{aligned} \qquad n \geqq 1.
$$

[Ans. No; try computing $f(13)$; then, replace n^2-n+1 by $4n+1$ and see what happens.]

Perhaps the following example, which deserves to be better known, will bring the matter more clearly to a head. Suppose we wish to define a function F on the set $I \times I$, which we can for convenience picture as the first quadrant. Suppose we write down the relation:

$$
(5) \qquad F(m+1, n+1) = F(F(m, n+1), n) \qquad m, n = 0, 1, 2, \cdots.
$$

Suppose that we assign the values of F on the edges of the quadrant. Then, a little experimentation leads us to believe that there is such a function, that it is uniquely determined, and that we can calculate any desired value of F.

First, by assumption, we have available a complete knowledge of $F(0, n)$ for each n, and of $F(m, 0)$ for each m; we can calculate their values for any specific choice of m or n. Set $n=0$ in (5), obtaining a simple recursion akin to formula (1):

$$
(6) \qquad F(m+1, 1) = F(F(m, 1), 0) \qquad m = 0, 1, 2, \cdots.
$$

From this, and the knowledge of the boundary values of the supposed F, we can generate the value of $F(x, 1)$ for any desired x; for example, $F(1, 1) = F(F(0, 1), 0)$ which is computable since we know the number $F(0, 1)$, and can compute the specific value of $F(x, 0)$ which results from setting $x = F(0, 1)$. Proceeding, we next have

$$
F(2, 1) = F(F(1, 1), 0)
$$

which is now computable since we know $F(1, 1)$, and so on.

We can thus regard $F(x, 1)$ as known for any specific integer x. Now, put $n=1$ in (4), obtaining

$$
(7) \qquad F(m+1, 2) = F(F(m, 2), 1) \qquad m = 0, 1, 2, \cdots.
$$

Since we know the initial value $F(0, 2)$, we could proceed in the same fashion to compute $F(1, 2)$, $F(2, 2)$, and any later value of $F(x, 2)$. Notice in particular that at any stage, we have needed to know only a finite number of the values of F at *earlier* points; we did not need to know *all* the values of $F(x, 1)$ in order to compute $F(2, 2)$. In Figure 1, we have attempted to make this point clear by shading the region that might be needed in order to compute $F(4, 3)$.

We have thus reached the same spot with this example that we encountered with equations (1). Apparently, we can compute any desired value of F; intuitively, we are therefore convinced that relation (4) admits a solution which

is a function defined on the entire set $I \times I$. However, the behavior illustrated by formula (4) may lead a student to seek assurance that the process outlined above leads to a consistent answer, that the value ascribed to $F(4, 3)$ does not depend upon his mode of procedure; again, he would be much happier if we were to exhibit F as an explicit class of ordered pairs, constructed from the relation (5) by standard set operations, for this would demonstrate existence in a much more satisfying way. We shall in fact do this at the end of this paper, and at the same time show that (5) cannot have two different solutions with the same assigned boundary values; the situation is analogous to the study of the Dirichlet problem in partial differential equations.

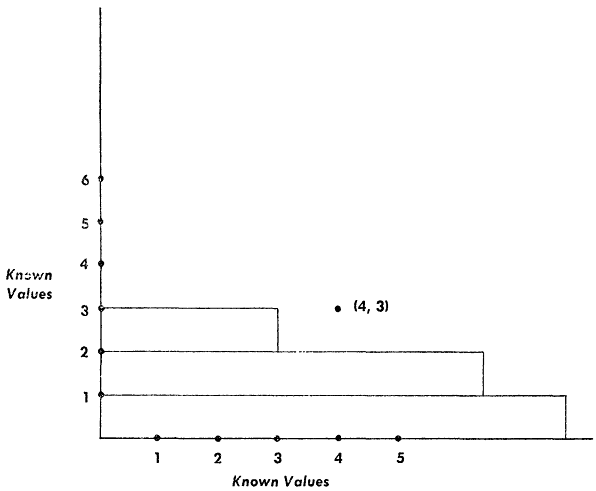

FIG. 1

Before doing this, however, we can gain some appreciation for the latent strength of the scheme (5) by examining the results of a specific choice of boundary values:

$$(8) \qquad \begin{aligned} F(m, 0) &= m + 1 && m = 0, 1, \cdots . \\ F(0, 1) &= 2 \\ F(0, 2) &= 0 \\ F(0, n) &= 1 && n = 3, 4, \cdots . \end{aligned}$$

Formula (6) becomes

$$\begin{aligned} F(m + 1, 1) &= F(F(m, 1), 0) \\ & && m = 0, 1, 2, \cdots . \\ &= F(m, 1) + 1. \end{aligned}$$

With the initial value $F(0, 1)=2$ from (8), this is easily seen to have the solution

(9) $$F(m, 1) = 2 + m \qquad m = 0, 1, \cdots.$$

In the same manner, (7) becomes

$$\begin{aligned} F(m+1, 2) &= F(F(m, 2), 1) \\ &= 2 + F(m, 2) \qquad m = 0, 1, 2, \cdots \end{aligned}$$

with the initial condition $F(0, 2)=0$. From this, we obtain

$$\begin{aligned} F(1, 2) &= 2 + 0 = 2 \\ F(2, 2) &= 2 + 2 = 2(2) \\ F(3, 2) &= 2 + (2)(2) = 2(3) \end{aligned}$$

and in general,

(10) $$F(m, 2) = 2m.$$

Continuing in the same way, we have

$$\begin{aligned} F(0, 3) &= 1 \\ F(m+1, 3) &= 2F(m, 3) \end{aligned}$$

from which we deduce that $F(1, 3)=2$, $F(2, 3)=2^2$, $F(3, 3)=2^3$, and in general

$$F(m, 3) = 2^m.$$

What happens when we go to the next stage? Our simple recursion becomes

$$F(m+1, 4) = 2^{F(m,4)}$$

with $F(0, 4)=1$. We are able to compute the values of F as before:

$$\begin{aligned} F(1, 4) &= 2 \\ F(2, 4) &= 2^2 \\ F(3, 4) &= 2^{2^2} = 16 \\ F(4, 4) &= 2^{2^{2^2}} = 2^{16} = 65536 \\ F(5, 4) &= 2^{2^{2^{2^2}}} = 2^{65536} \end{aligned}$$

and by using dots, we can fake a general formula

$$F(m, 4) = 2^{2^{\cdot^{\cdot^{2}}}} \qquad [\text{with } m \text{ two's}].$$

Let us try the next case; we have the simple recursion

$$\begin{aligned} F(0, 5) &= 1 \\ F(m+1, 5) &= F(F(m, 5), 4) \qquad m = 0, 1, 2, \cdots \end{aligned}$$

and using our value for $F(x, 4)$, we write this as

$$F(m+1, 5) = 2^{2^{2^{\cdot^{\cdot^{2}}}}} \qquad [\text{with } F(m, 5) \text{ two's}].$$

This will suffice to compute some of the early values of F, so that we have for example

$$F(1, 5) = 2$$
$$F(2, 5) = 2^2 = 4$$
$$F(3, 5) = 2^{2^{2}} = 65536$$
$$F(4, 5) = 2^{2^{\cdot^{\cdot^{2}}}} \quad \text{with } 65536 \text{ two's}$$
$$F(5, 5) = 2^{2^{\cdot^{\cdot^{2}}}} \quad [\text{with } 2^{2^{\cdot^{\cdot^{2}}}} \quad [\text{with } 65536 \text{ two's}] \text{ two's}].$$

However, I think that it is quite clear that we do not have a suitable way to write down any nonbogus general formula for $F(m, 5)$ within the notational schemes of the standard terminology.

Still less, then, will this be true for $F(m, 6)$, and manifestly more so for the function ψ of one variable which is now definable by the equation

$$\psi(x) = F(x, x) \qquad \text{for } x = 0, 1, 2, \cdots.$$

However, it is also clear that $\psi(x)$ can be computed for any specific value of x, granting the necessary time and paper—which undoubtedly exceeds both the estimated size of the universe and its duration. Indeed, $\psi(0)=1$, $\psi(1)=3$, $\psi(2)=4$, $\psi(3)=8$, $\psi(4)=65536$, and $\psi(5)=F(5, 5)$, which we have written down just above.

The existence of functions such as ψ yields an unexpected dividend. The following personal illustration may be amusing. I have found that most beginning analysis students seem to accept as plausible the conjecture that, given any increasing sequence of integers $\{c_n\}$, one could find an entire function f such that $f(n) > c_n$ for $n=1, 2, \cdots$. If you suggest $c_n = 2^n$, they counter with $f(z) = \exp(z)$. If you suggest $c_n = n!$, they suggest $f(z) = \exp(\exp(z))$. However, once they have been shown the construction of the special function ψ, and have come to appreciate its stupendous rate of growth, and the obvious possibility of creating functions which grow even more rapidly, their confidence in the conjecture seems to fade; analyticity is too delicate a phenomenon to match such catastrophic growth. Indeed, in one instance, the only student in the class who was able to overcome this feeling and find the simple general proof was one who had been absent the day before, and did not know about the function ψ. (Proof: Put $f(z) = \sum c_n (z/n)^{c_n}$, convergent for all z.)

The power of recursive definitions is now plain to the student; he will not find it hard to modify (5) for a function of three variables, generalizing Peano's recursion, so that:

$$F(x, y, 1) = x + y$$
$$F(x, y, 2) = xy$$
$$F(x, y, 3) = x^y$$

thus obtaining all the usual arithmetic operations at once. (This and the preceding example are slightly modified versions of examples given originally by Ackermann; see [3] or [4].) At this point, the student is also prepared to see the point of general theorems which deal with the more subtle aspects of existence, definability, and computability of functions.

As an illustration, let us re-examine the recursion schemes we have used, and prove that the solution of (5) is unique and can be exhibited as a set of ordered pairs. Let us start from the simple recursion relation:

$$\begin{aligned} f(0) &= a \\ f(m+1) &= g(f(m)) \qquad m = 0, 1, 2, \cdots, \end{aligned} \tag{11}$$

where a is an integer, and g is a previously defined function on I to I. Introduce a special mapping S of $I\times I$ into itself defined by:

$$\text{if } p = \langle u, v\rangle, \quad \text{then} \quad S(p) = \langle u+1, g(v)\rangle.$$

Let us say that a subset $A \subset I\times I$ is *admissible* if it obeys the pair of conditions

$$\begin{aligned} &\langle 0, a\rangle \in A \\ &\text{if } p \in A, \quad \text{then} \quad S(p) \in A. \end{aligned} \tag{12}$$

There are admissible sets, for example $I\times I$. More to the point, if there is a function f that obeys (11), then its graph is an admissible set.

Let A_0 be the *smallest* admissible set, e.g. the intersection of all the admissible sets. We show that A_0 is the graph of a function. Observe first that if A is any admissible set, and q is any point in A other than $\langle 0, a\rangle$, and if q is not of the form $S(p)$ for any $p\in A$, then we can remove q from A and still have an admissible set, since neither condition of (12) will be violated. Consequently, since A_0 is minimal, every point in it, except $\langle 0, a\rangle$, is of the form $S(p)$ for some $p\in A_0$. Let π be the projection of $I\times I$ onto the first factor, sending $\langle u, v\rangle$ into u; it is then immediate that π maps A_0 onto I one-to-one, so that A_0 is a function with domain I, and the desired solution of the recursion (11).

We have therefore produced a new function Φ of two variables, one an integer and the other a function, whose values are functions, and which is described by saying that $\Phi(a, g) = f$, where f is the (unique) solution of (11). If we let $\mathfrak{F}$ denote the class of all functions on I to I, then Φ is a mapping on $I\times\mathfrak{F}$ to $\mathfrak{F}$. Suppose now that α and β are in $\mathfrak{F}$, and let us attempt to define a sequence of functions $F_n\in\mathfrak{F}$ by the format

$$
\begin{aligned}
&F_0 = \alpha \\
&F_{n+1} = \Phi(\beta(n), F_n) \qquad n = 0, 1, 2, \cdots.
\end{aligned}
\tag{13}
$$

The first line means that $F_0(m) = \alpha(m)$ for $m = 0, 1, \cdots$. The second line is harder to interpret; if we set $f = \Phi(\beta(n), F_n)$ then, by (11) which describes Φ,

$$
\begin{aligned}
f(0) &= \beta(n) \\
f(m+1) &= F_n(f(m)) \qquad m = 0, 1, 2, \cdots.
\end{aligned}
$$

Since (13) identifies f with F_{n+1}, these conditions amount to asking that

$$
\begin{aligned}
F_{n+1}(0) &= \beta(n) \\
F_{n+1}(m+1) &= F_n(F_{n+1}(m)) \qquad m = 0, 1, 2, \cdots.
\end{aligned}
$$

If we now write $F(x, y)$ for $F_y(x)$, we see that all together, we have recaptured the form of the double recursion (5) exactly:

$$
\begin{aligned}
F(m, 0) &= \alpha(m) & m &= 0, 1, 2, \cdots \\
F(0, n) &= \beta(n-1) & n &= 1, 2, \cdots \\
F(m+1, n+1) &= F(F(m, n+1), n) & m, n &= 0, 1, 2, \cdots.
\end{aligned}
$$

Thus, we have shown that a multiple recursion of the complicated type which we used to create the function F, and then ψ, can in fact be reduced to a primitive recursion format, provided we allow function valued functions. Does (13) have a solution? If so, it will be a sequence of functions F_n, that is, a function F on I to $\mathfrak{F}$; can we show its existence by exhibiting it as a subset of $I \times \mathfrak{F}$? The pattern used earlier can be repeated exactly. Introduce a special mapping S of $I \times \mathfrak{F}$ into itself by:

$$
\text{if } p = \langle u, \gamma \rangle, \quad \text{then} \quad S(p) = \langle u+1, \Phi(\beta(u), \gamma) \rangle.
$$

Again, say that $A \subset I \times \mathfrak{F}$ is admissible if A contains the point $\langle 0, \alpha \rangle$ and is mapped into itself by S. Then, in exactly the same fashion, the unique minimal admissible set turns out to be the graph of the sought-for function F.

It is clear how this can be continued, basing the study of multiple recursions on that of primitive recursions with more elaborate function valued functions. At this point, we are within range of the concept of a general recursive function and the related notions of computability and constructability. The reader is herewith referred to the bibliography.

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BALANCE SCALE SORTING

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The problems discussed below fall under the general heading of dynamic programming [1], in that they involve the determination of criteria for making a sequence of decisions during a testing procedure, with the motive of minimizing some function of a set of parameters by which the procedure is specified. In particular, we are concerned with *sorting problems*, in which one seeks an optimal program for culling out one or more subsets of a given set. The desired programs are optimal in the sense of minimizing either the maximum, or the expected, number of steps in the process. In any programming problem, the meaning of *optimal* depends on practical considerations motivating the programmer.

A solution of problem $E(n, 1)$, defined below, occupies the bulk of this paper. It appealed to the writer because of the novelty both of certain sequences of numbers to which it led and of the processes for generating them. Despite its "mathematical puzzle" flavor, techniques or principles of more general applicability may conceivably be derived from it.

An interesting paradox revealed by the solution is that the expected length of a sorting process can sometimes be decreased by a deliberate augmentation of the total set from which a subset is to be culled (see Section 6(B)).

1. The problems and some formulations. Let W be a set of n objects, indistinguishable except that the members of some subset H of $h<n$ objects are slightly heavier, in a sense specified below, than the rest. Given a balance scale, consider the problem of finding weighing programs which minimize either of the following:

Problem $M(n, h)$: the maximum number of steps which may be required to cull out H, or

Problem $E(n, h)$: the expected number of such steps.

This paper contains, first, a partial analysis of these general problems; second, a solution (Sections 2–7) of Problem $E(n, 1)$; and third, some comments (Section 8) on Problem $M(n, 2)$.

An interesting treatment of a number of problems closely related to $M(n, h)$ is to be found in a paper by C. A. B. Smith [2].

(A) *Basic assumptions.* (1) The objects in H, to be called *heavy objects*, are of equal weight, and so are those in $W-H$, to be called *light objects.* (2) If λ is the weight of a light object, then the weight of a heavy object is less than $(n+1)\lambda/n$, so that the larger of two numerically unequal subsets of W is always the heavier. (3) The scale reveals which, if either, of two subsets of W is heavier, but not by how much.

Assumption (A2) implies that no information is gained by balancing two numerically unequal sets. Consider, then, a pair of numerically equal disjoint subsets (X, Y) of W. *Step* (X, Y) will mean the balancing of X against Y. It has one of the two following outcomes:

Case B ("*B*" for *balance*). The sets balance, symbolized by $X = Y$.

Case U ("*U*" for *unbalance*). The sets do not balance, symbolized by $X \neq Y$. We reassign the notation, if necessary, so that $X > Y$, where $>$, between two sets, means "is heavier than."

(B) *Notation.* If $S \subset W$, then $S = \{\alpha, \beta\}$ signifies that S consists of α heavy objects and β light ones.

Let $[n/2] = n/2$ or $(n-1)/2$, whichever is an integer, and let $m \in (1, 2, \cdots, [n/2])$ be the number of elements in X, also the number in Y. We symbolize the possibilities under Case U as follows:

$$\text{(1.2)}\quad \textit{Possibility } U_{ji}\text{:}\quad \begin{aligned} X &= \{j, m - j\} \quad (j = 1, \cdots, \min(m, h)) \\ Y &= \{i, m - i\} \quad (i = 0, \cdots, \min(h - j, j - 1)) \\ Z &= W - (X + Y) = \{h - i - j, n - 2m - h + i + j\}. \end{aligned}$$

The probability of this possibility is

$$\text{(1.3)}\quad \begin{aligned} p_{ji} &= \frac{2\binom{h}{j}\binom{h-j}{i}\binom{n-h}{m-j}\binom{n-h-m+j}{m-i}}{\binom{n}{m}\binom{n-m}{m}} \\ &= \frac{2\binom{m}{j}\binom{m}{i}\binom{n-2m}{h-j-i}}{\binom{n}{h}}. \end{aligned}$$

The factor 2 is present because the equally likely case $Y > X$ has been ruled out only by a notational convention.

Case $B(X = Y)$ similarly presents

$$\text{(1.4)}\quad \textit{Possibility } B_j\text{:}\quad \begin{aligned} X &= \{j, m - j\} \\ Y &= \{j, m - j\} \quad (j = 0, 1, \cdots, \min(m, [h/2])) \\ Z &= \{h - 2j, n - 2m - h + 2j\}. \end{aligned}$$

Its probability is

$$\text{(1.5)}\qquad q_j = \frac{\binom{m}{j}^2\binom{n-2m}{h-2j}}{\binom{n}{h}}.$$

Step (X, Y) yields a partition of W into the three sets (X, Y, Z), regarding

which we know, in case U, that X contains more heavy objects than Y; and, in case B, that X and Y contain numerically equal subsets of H. It is clear that a general solution of $E(n, h)$ or $M(n, h)$ must be both complicated and dependent on a number of additional hypotheses; for example, whether or not we know the number, h, of objects we are seeking.

A sequence of steps in a weighing program yields a partition $(W_1, \cdots, W_s)$ of W, and some probabilistic information about the induced partitions of H and $W-H$. Let (X', Y') denote the next step, where $X' = \bigcup_{i=1}^{s} X_i$ and $Y' = \bigcup_{i=1}^{s} Y_i$, with $X_i \subset W_i$, $Y_i \subset W_i$.

(C) Under the conditions just described, one may suppose either (1) that X_i and X_j, also Y_i and Y_j, are distinguishable after Step (X', Y') or (2) that they are so amalgamated on the scale as to be indistinguishable after the weighing.

(D) Suppose that hypotheses have been adopted under which problems $E(n, h)$ and $M(n, h)$ are well-defined. We denote with $\mu_h(n)$ the maximum number of steps in an optimal program for $M(n, h)$ and with $\epsilon_h(n)$ the expected number in an optimal program for $E(n, h)$.

(E) *General principle for* $M(n, h)$. Suppose that an arbitrary first step is taken, balancing m objects against m objects $(1 \leqq m \leqq [n/2])$ and that an optimal program for $M(n, h)$ is followed subsequent to this arbitrary first step. Let μ_U be the maximum number of subsequent steps if case U arises on the first step, and let μ_B be the maximum number of subsequent steps if case B arises. Finally, let $\mu_h(n, m)$ be the maximum total number, first step included, of weighings required to cull out H. Then

$$(1.6) \qquad \begin{aligned} &\text{(a)} \quad \mu_h(n, m) = 1 + \max(\mu_U, \mu_B) \\ &\text{(b)} \quad \mu_h(n) = \min_{(m)} \mu_h(n, m) \qquad (m = 1, \cdots, [n/2]). \end{aligned}$$

(F) *General principle for* $E(n, h)$. Given the conditions of (E) with $E(n, h)$ replacing $M(n, h)$, let ϵ_U and ϵ_B denote the expected numbers of steps, after the first one, under cases U and B respectively. Let $\epsilon_h(n, m)$ be the expected total number of steps. Let Σ' denote a summation over the values of i and j in (1.2) and Σ'' over the values of j in (1.4). Then

$$(1.7) \qquad \begin{aligned} &\text{(a)} \quad \epsilon_h(n, m) = 1 + \Sigma' p_{ij}\epsilon_U + \Sigma'' q_j \epsilon_B, \\ &\text{(b)} \quad \epsilon_h(n) = \min_{(m)} \epsilon_h(n, m) \qquad (m = 1, \cdots, [n/2]). \end{aligned}$$

(G) A *best value* for $M(n, h)$, or $E(n, h)$, is a number m such that an optimal program for $M(n, h)$, or for $E(n, h)$, can commence with the balancing of m objects against m objects.

(H) It follows from (E) and (F) that (1) the set $\{m\}_{n,h}^{\mu}$ of best values for $M(n, h)$ consists of those numbers m for which the minimum in (1.6b) is realized; that is, for which $\mu_h(n) = \mu_h(n, m)$, and (2) the set $\{m\}_{n,h}^{\epsilon}$ of best values for $E(n, h)$ consists of the numbers m for which $\epsilon_h(n) = \epsilon_h(n, m)$.

2. Problem $M(n, 1)$. If $h=1$, we have the familiar puzzle of finding the counterfeit in a set W of coins containing just one counterfeit H heavier than the other coins. Though Problem $M(n, 1)$ is not difficult, we discuss it to illustrate a method.

(A) *Problem* $M^*(n, 1)$ means a modification of $M(n, 1)$, in which a typical step is as follows: a subset of S, known to contain H, is arbitrarily partitioned into (S_1, S_2, S_3), and the step decides which of these three subsets contains H. Such a decision might be made with the aid of a supply of light objects (good coins), some of which could be added to the numerically smaller of two sets S_i, S_j in order to obtain numerically equal sets to balance on the scales. We here use Section 1C(1) as a hypothesis, to permit the separation of the supplementary objects from S_i or S_j after the weighing.

LEMMA $M^*(n, 1)$. *Let j be such that*

$$3^{j-1} < n \leqq 3^j. \tag{2.1}$$

A first step for $M^*(n, 1)$ is consistent with an optimal program if and only if it partitions W into subsets of n_1, n_2, and n_3 members, where

$$3^{j-2} < \max(n_1, n_2, n_3) \leqq 3^{j-1}. \tag{2.2}$$

There is a maximum of j steps in an optimal program.

Proof. If $j=1$, the lemma is trivial. Given $j>1$, assume the lemma for all cases where j is replaced by a smaller positive integer. If (n_1, n_2, n_3) is a partition of n into nonnegative integers, then $\max n_i \geqq n/3 > 3^{j-2}$. Thus the first step may, regardless of the partition, reveal that H belongs to a subset containing more than 3^{j-2} objects. Hence the remaining process may, by inductive hypothesis, require at least $j-1$ steps. Suppose we use a partition subject to (2.2), which is possible since $n \leqq 3^j$. Then, after the first step, at most $j-1$ steps are needed to find H, and the lemma follows.

THEOREM $M(n, 1)$. *Again, let* (2.1) *define j. Then* (a) $\mu_1(n)=j$ [*see Section* 1(D)], *and* (b) $\{m\}^{\mu}_{n,1}$ [*see Section* 1(G)(H)] *is defined by*

$$\{m\}^{\mu}_{n,1}\colon \tfrac{1}{2}(n - 3^{j-1}) \leqq m \leqq \min(3^{j-1}, [n/2]). \tag{2.3}$$

Proof. Problem $M(n, 1)$ is equivalent to $M^*(n, 1)$ with two of the subsets (S_1, S_2, S_3) (see (A) above) required to be equal. An added requirement can lengthen, but not shorten, an optimal program. In this case, it does not lengthen it, because (2.2) is consistent with $n_1=n_2=m$. The theorem follows from the fact that (2.3) is the set of values of m for which (2.2) is fulfilled by $(n_1, n_2, n_3) =(m, m, n-2m)$. To see this, note that the first inequality in (2.3) implies $n-2m \leqq 3^{j-1}$, that $m \leqq 3^{j-1}$ by the second inequality in (2.3), and that the first inequality in (2.2) is valid for each partition (n_1, n_2, n_3) of n.

3. Problem $E'(n, 1)$: Preliminary analysis.

(A) Problem $E'(n, h)$ means Problem $E(n, h)$ modified as follows. Suppose a sequence of weighings has led to a partition $(W_1, \cdots, W_s)$ of W. The next step is then required to be a step (X, Y) with $X \subset W_i$ for some i and $Y \subset W_j$ for some j, possibly equal to i. The symbols $\epsilon_h(n)$, ϵ_U, ϵ_B, $\epsilon_h(n, m)$ are modified by primes in applying them to $E'(n, h)$.

(B) *Introduction to* $E'(n, 1)$. Our solution of $E'(n, h)$ for the case $h=1$ consists in determining the sequence

$$\{\epsilon_1'(n)\} = \epsilon_1'(0), \ \epsilon_1'(1), \ \epsilon_1'(2), \cdots, \tag{3.1}$$

where $\epsilon_1'(0)$ is arbitrarily defined as 0. It is convenient first to determine the series

$$\{w\} = w_0, w_1, w_2, \cdots, \tag{3.2}$$

where $w_n = n\epsilon_1'(n)$ and, as a preliminary to this, to determine the difference sequence

$$\{d\} = d_0, d_1, d_2, \cdots, \tag{3.3}$$

where $d_n = w_{n+1} - w_n$. The sequence $\{d_n\}$ commences as follows:

$$\{d\} = 0 \mid 2 \ 1 \mid 3 \ 2 \ 4 \ 1 \ 3 \ 2 \ \mid 4 \ 3 \ 5 \ 2 \ 4 \ 3 \ 5 \ 2 \cdots. \tag{3.4}$$

The vertical bars are intended to suggest an analysis of $\{d\}$ into subsequences or *blocks*, δ_j $(j=0, 1, 2, \cdots)$, where δ_j consists of the terms d_n $(3^{j-1} < n \leqq 3^j)$ [Compare (2.1)]. Thus, assuming (3.4),

$$\delta_0 = 0, \qquad \delta_1 = 2, 1, \qquad \delta_2 = 3, 2, 4, 1, 3, 2. \tag{3.5}$$

THEOREM 3.1. *Block* δ_j *is given by* (3.5) *for* $j=0, 1, 2$. *For* $j>2$, (a) δ_j *commences with* $j+1, j+2, j-1$, (b) *it is periodic of period* 4 *with a half-period at the end* (*note that* $3^j - 3^{j-1}$, *the number of terms in* δ_j *is congruent to* 2 mod 4).

(C) *Comments.* The δ's are introduced for simplicity of statement. Once they are known, so is $\{d\}$, and hence $\{w\}$. We will think of $\{w\}$ as analyzed, like $\{d\}$, into blocks β_j, where (1) $\beta_0 = 0,0$ and (2) $\beta_j (j>0)$ consists of the terms w_n $(3^{j-1} < n \leqq 3^j)$.

4. Problem $E'(n, 1)$: Computational scheme.

(A) Theorem 3.1 is equivalent to the statement that

$$\begin{aligned} \{w\} &= \underbrace{0 \ 0}_{\beta_0} \mid \underbrace{2 \ 3}_{\beta_1} \mid \ \underbrace{6 \ 8 \ 12 \ 13 \ 16 \ 18}_{\beta_2} \mid \underbrace{22 \ 25 \ 30 \ 32 \ 36 \ 39 \ 44 \ 46 \cdots}_{\beta_3} \end{aligned} \tag{4.1}$$

where the differences of successive terms are the respective d's, as specified in Theorem 3.1.

This statement follows directly from the definitions and is introduced for convenience in expounding our proof.

(B) Given $h=1$, the analysis in (1.2)–(1.5) of the possible outcomes of a step (X, Y) specializes as follows: Possibility U (unbalance) with probability $p_{10}=p=2m/n$, and possibility B (balance) with probability $q_0=q=(n-2m)/n$. Hence, equations (1.7), with primes on ϵ to correspond to $E'(n, 1)$, become [see (3.2)]

$$\text{(4.2)}\qquad \begin{aligned} \text{(a)}\quad \epsilon_1'(n, m) &= 1 + \frac{2m}{n}\epsilon_1'(m) + \frac{n-2m}{n}\epsilon_1'(n-2m) \\ &= 1 + \frac{2w_m + w_{n-2m}}{n}, \\ \text{(b)}\qquad \epsilon_1'(n) &= \min_{(m)} \epsilon_1'(n, m) = 1 + \frac{\min(2w_m + w_{n-2m})}{n}. \end{aligned}$$

Hence, for $n>1$,

$$\text{(4.3)}\qquad w_n = n\epsilon_1'(n) = n + \min_{(m)}(2w_m + w_{n-2m}) \qquad \left(m = 1, 2 \cdots \left[\frac{n}{2}\right]\right).$$

(C) Equation (4.3) suggests a recurrent process for generating $\{w\}$, with $w_0=w_1=0$ as initial step. Suppose that, for some positive integer k, the subsequence $w_0, w_1, \cdots, w_{2k-1}$ is known. Then, from (4.3),

$$\text{(4.4)}\qquad \begin{aligned} &\text{(a)}\quad w_{2k} = 2k + \min(2w_1 + w_{2k-2}, 2w_2 + w_{2k-4}, \cdots, 2w_k + w_0) \\ &\text{(b)}\quad w_{2k+1} = 2k + 1 + \min(2w_1 + w_{2k-1}, 2w_2 + w_{2k-3}, \cdots, 2w_k + w_1). \end{aligned}$$

We next establish an auxiliary result.

LEMMA 4.1. *Let* $\{e\}=w_0, w_2, w_4, \cdots$ *be the sequence of the even-numbered terms of* $\{w\}$, *let* $\{o\}=w_1, w_3, w_5, \cdots$, *and let* $\{2w\}=2w_1, 2w_2, \cdots$. *Let* $d(2w_n)=2w_{n+1}-2w_n=2d_n$, $d(e_n)=w_{2n}-w_{2n-2}$, *and* $d(o_n)=w_{2n+1}-w_{2n-1}$. *These are general terms of difference sequences, specified as follows*: (a) $\{d(2w_m)\}=2d_1, 2d_2, 2d_3, \cdots$, *where the d's are as stated in Theorem* 3.1, (*see also* (3.4) *and note the omission, for convenience, of the initial term* $2d_0=0$), (b) *the jth block of terms in* $d(e_n)$ *consists of* 3^{j-1} *alternating terms* $2j, 2j+2, \cdots, 2j$ $(j=1, 2, \cdots,)$, *and* (c) *the jth block in* $d(o_n)$ *consists of* 3^{j-1} *terms each equal to* $2j+1$.

$$\text{(4.5)}\qquad \begin{aligned} &\text{(a)}\quad \{d(2w_n)\} = 4\ 2\,|\,6\ 4\ 8\ 2\ 6\ 4\,|\,8\ 6\ 10\ 4\ 8\ 6 \cdots \\ &\text{(b)}\quad \{d(e_n)\} = 2\,|\,4\ 6\ 4\,|\,6\ 8\ 6\ 8\ 6\ 8\ 6\ 8\ 6\,|\,8\ 10\ 8 \cdots \\ &\text{(c)}\quad \{d(o_n)\} = 3\,|\,5\ 5\ 5\,|\,7\ 7\ 7\ 7\ 7\ 7\ 7\ 7\ 7\,|\,9\ 9\ 9 \cdots \end{aligned}$$

(D) Before proving Lemma 4.1, we remark that it implies Theorem 3.1 and hence a solution of $E'(n, 1)$. To see this, we need only interpret our entire procedure inductively. As hypothesis for *Step k*, we assume that the first $k-1$ terms have been found in each of the sequences (4.5b) and (4.5c) and are as specified in Lemma 4.1. This implies that the first $2k-1$ terms of $\{w\}$ are known. Step k falls into *Part* 1, in which $d(e_k)$ is determined, and *Part* 2, in which $d(o_k)$ is determined.

We will give details only for Part 2 of Step k. Part 1 can be similarly treated. Since $w_1 = d_0 = 0$, it follows that

$$(4.6) \qquad \text{(a) } 2w_n = \sum_{i=1}^{n-1} d(2w_i), \qquad \text{(b) } w_{2n-1} = \sum_{i=1}^{n-1} d(o_i).$$

From (4.4b),

$$(4.7) \qquad d(o_k) = (2k+1) + \min_{1 \leqq n \leqq k} (2w_n - w_{2k-1} + w_{2k-2n+1}).$$

The last two terms in (4.7) can, by (4.6b), be expressed thus:

$$(4.8) \qquad w_{2k-1} - w_{2k-2n+1} = \begin{cases} 0 & n = 1 \\ \displaystyle\sum_{h=k-n+1}^{k-1} d(o_h) = \sum_{i=1}^{n-1} d(o_{k-i}) & n > 1. \end{cases}$$

Now let

$$(4.9) \qquad \begin{aligned} &(a) \ \omega_i = d(2w_i) - d(o_{k-i}), \\ &\text{(b) } \mu_k = \min_{1 \leqq n \leqq k-1} \left\{0, \sum_{i=1}^{n-1} \omega_i\right\}. \end{aligned}$$

Then, by (4.6a), (4.8), and (4.7),

$$(4.10) \qquad d(o_k) = 2k + 1 + \mu_k.$$

A computational scheme for $d(o_k)$ is as follows:

$$(4.11) \qquad \begin{array}{llll} & d(2w_1) \cdots d(2w_{k-h}) \cdots d(2w_{k-1}) \cdots d(2w_{2k-1}) \\ & d(o_{k-1}) \cdots d(o_h) \quad \cdots d(o_1) \\ \hline \textit{Differences:} & \omega_1 \quad \cdots \omega_{k-h} \quad \cdots \omega_{k-1} \\ \textit{Partial sums:} & \omega_1 \quad \cdots \displaystyle\sum_{i=1}^{k-h} \omega_i \quad \cdots \sum_{i=1}^{k-1} \omega_i \end{array}$$

$$\mu_k = \min \left\{0, \sum_{i=1}^{j} \omega_i \,\middle|\, j = 1, \cdots, k-1\right\}$$

$$d(o_k) = 2k + 1 + \mu_k$$

5. Problem $E'(n, 1)$. **Solution.** The scheme (4.11) suggests the following procedure for recurrently computing all the members of the sequence $\{d(o_n)\}$.

(A) After obtaining a particular term $d(o_k)$ by (4.11), (1) shift the second row one space to the right, (2) insert $d(o_k)$ under $d(2w_1)$, and (3) compute the differences, their partial sums, μ_{k+1}, and finally $d(o_{k+1}) = 2k+3+\mu_{k+1}$.

It is assumed that a similar computational procedure for $\{d(e_n)\}$ is simultaneously carried out. We note, however, that, under our hypothesis [Section 4(C)], $2k-1$ terms of $\{d(2w_n)\}$ are known. Hence, we can apply (A) to the

computation of $d(o_k), \cdots, d(o_{2k})$ before computing any of the intermediate differences $d(e_{k+1}), \cdots, d(e_{2k})$, since there is room in (4.11) for k shifts to the right before $d(o_1)$ appears under $d(2w_{2k-1})$.

(B) *Revised inductive hypothesis.* In the hypothesis of Section 4(C), $k-1 = \sum_{i=0}^{j-1} 3^i$ for some $j>1$.

As a basic step, justifying (B) for $j=2$, let the procedure (4.11) be applied to the successive computation of $d(o_2)$, $d(o_3)$, $d(o_4)$. This leads to the following, based on $\{d(o_1)=1\}$ and $\{d(2w_n)\}=4\ 2\ 6\ \cdots$. If $k=2$, the only term in line 3 of (4.11) is $\omega_1=1$. Hence, $\mu_2=\min(0, 1)$, and $d(o_2)=2k+1+\mu_2=5$. If $k=3$, then $\omega_1=\omega_2=-1$, the partial sums are -1, -2, hence $\mu_3=-2$, and $d(o_3)=7-2=5$. In the next step, where $k=4$, we find that $d(o_4)$ is also 5.

Hypothesis (B) means that (4.11) is, in part:

$$
(5.1)\qquad
\begin{array}{rrcccccccc}
 & 4 & 2 & \mid\ \cdot & \cdot & \cdots\ 2j-2 \mid & 2j+2 & 2j & 2j+4 & 2j-2\ \cdots \\
d(o_{k-1}) = & 2j+1 & 2j+1 & \cdot & \cdot & 2j+1 & 2j+1 \mid & 2j-1 & 2j-1 & 2j-1\ \cdots \\
\hline
\omega_1 = & 3-2j & 1-2j & \cdot & \cdot & -3 & 1 & 1 & 5 & -1\ \cdots
\end{array}
$$

(C) The minimum partial sum μ_k is the sum of the first $3^{j-1}-1$ of the differences ω_i.

Proof of (C). This result is equivalent to the statement that μ_k appears directly under the first $2j-2$ shown in (5.1), if the work is arranged as in (4.11). It is easily verified by inspection of (5.1).

At the stage preceding (5.1), the second row was one space further to the left. It is again easy to verify that μ_{k-1}, like μ_k, appeared under the $(3^{j-1}-1)$th term. Furthermore, since a shift to the left changes none of the first $3^{j-1}-1$ differences in line 3 of (5.1), $\mu_k=\mu_{k-1}$. But, by (4.10) with $k-1$ in place of k, $d(o_{k-1})=2(k-1)+1+\mu_{k-1}$. Hence,

$$
(5.2)\qquad d(o_k) = 2k + 1 + \mu_k = d(o_{k-1}) + 2 = 2j + 3.
$$

Thus, after 3^{j-1} terms equal to $2j+1$, $2j+3$ appears.

To complete an inductive proof based on (B), it remains to show that a total of 3^j successive terms equal to $2j+3$ will now appear. In general, if the minimum partial sum appears in some position, say position h, both before and after a shift, and if the shift changes the first h terms of row 2 only by the replacement of some $2j+1$ by a $2j+3$, then the next computed term of $\{d(o_n)\}$ will be another $2j+3$.

By inspection of (5.1), we see that, for $h=3^{j-1}-1$, these conditions are fulfilled for all shifts through the one which brings $d(o_k)=2j+3$ into the $(3^{j-1}-1)$th position. Hence, $d(o_i)=2j+3$ $(i=k, \cdots, k+3^{j-1}-1)$. But this makes only 3^{j-1} terms equal to $2j+3$, and our object is to show that there are 3^j of them. To accomplish this, let us first revert to the stage shown in (5.1). One shift then brings a $2j+1$ under the $2j$ in (5.1). The corresponding difference is $2j-(2j+1) = -1$. The two ω's [see (4.11)] just to the right of the -3 are thus 1, -1 and contribute 1, 0 respectively to the partial sums. Two shifts later, a $2j+1$ occurs

under each of the terms $(2j+2, 2, 2j+4, 2j-2)$. The ω's under them are therefore $(1, -1, 3, -3)$, and their contributions to the partial sums are $(1, 0, 3, 0)$; hence, the minimum partial sum appears under both the second and the fourth terms of the block starting with $2j+2$. Our proof is completed by easily formulated extensions of these statements.

This completes our inductive proof of Lemma 4.1. As noted in Section 4(D), Theorem 3.1 now follows. The latter yields a solution, briefly discussed below, of problem $E'(n, 1)$.

6. Comments on the solution. In problem $E'(n, 1)$, we are particularly interested in the optimal number of balancings $\epsilon_1'(n)$. The sequence $\{\epsilon_1'(n)\} = \epsilon_1'(1), \epsilon_1'(2), \cdots$ is obtained by discarding the first term of (4.1) [see also (3.2)] and dividing the remaining terms by 1, 2, 3, $\cdots$, respectively. This yields

$$\{\epsilon_1'(n)\} = 0 \,|\, 1 \;\; 1 \,|\, 3/2 \;\; 8/5 \;\; 2 \;\; 13/7 \;\; 2 \;\; 2 \,|\, 11/5 \;\; 25/11 \cdots \tag{6.1}$$

The phenomenon $\epsilon_1'(7) < \epsilon_1'(6)$ means that one can do better, on the average, starting with seven objects than starting with six. A gambler might slowly attain wealth by betting he can detect the heavy object in a set of 7 or 13 more quickly than his opponent can in a set of 6 or 12. This phenomenon implies that, for some values of n, $\epsilon_1'(n) \neq \epsilon_1(n)$. In particular, consider $n=12$. It can be deduced from later arguments that $m=3$ is a best value in problem $E'(12, 1)$. If $m=3$ and case B arises [see Section 1], there is a six-membered set Z [see (1.4)] containing H. The transfer of an object from X to Z, permissible in $E(n, 1)$ but not in $E'(n, 1)$, then improves the expected number of weighings.

To see why $\epsilon_1'(7) < \epsilon_1'(6)$, note that the choice $m=3$ for $n=7$ gives probability 1/7 of case B, which yields H in one weighing. In case U for $n=7$, we get H on the second weighing, agreeing with $\epsilon_1'(7) = 13/7$. In case $n=6$, two weighings are necessary and sufficient under the choices $m=2, 3$; while the choice $m=1$ gives equal probabilities of success in one, two or three weighings, so that $\epsilon_1'(6) = 2$.

The phenomenon $\epsilon_1'(n) < \epsilon_1'(n-1)$ occurs, as one can easily verify, for every

$$n \equiv 3^j \pmod 4 \quad \text{with} \quad 3^j < n \leqq 3^{j+1}. \tag{6.2}$$

These numbers correspond to the minimum differences $d_i = j-1$ in block δ_{j+1}.

We omit, because it is somewhat tedious, a general discussion of the set $\{m\}_{n,1}^{\epsilon'}$ of best values for $E'(n, 1)$. As an example of an interesting phenomenon, we remark that $\{m\}_{20,1}^{\epsilon'} = (5, 7, 8, 9)$, while, by (2.3), $\{m\}_{20,1}^{\mu} = (6, 7, 8, 9)$. Since 5 is in the first set and not in the second, there is an optimal program for $E'(20, 1)$ which is not optimal for $M(20, 1)$. In an optimal program for $E'(20, 1)$ which commences with 5 objects balanced against 5, four weighings may prove necessary, instead of the maximum of three given by an optimal program for $M(20, 1)$.

7. Problem $E^*(n, 1)$ and a solution of $E(n, 1)$. We treated $E'(n, 1)$ in detail,

partly for its own sake and partly as a preliminary to $E(n, 1)$, of which it is a modification. Still another modification is a useful preliminary.

(A) Problem $E^*(n, 1)$ means $E(n, 1)$ modified by adopting the typical step described in Section 2(A); in other words, at each stage in a program for $E^*(n, 1)$, a certain subset $S \subset W$ is known to contain H, and the next step reveals which of three arbitrarily selected subsets, partitioning S, contains H.

The unmodified $E(n, 1)$ is a combination of $E'(n, 1)$ and $E^*(n, 1)$. For, there is no pool of light objects before the first step in $E(n, 1)$, but, as the weighings continue, there is a growing pool of light objects which could be used as described in Section 2(A). Let $\epsilon_1^*(n)$ denote the expected number of weighings in an optimal program for $E^*(n, 1)$, let $v_n = n\epsilon_1^*(n)$ and $d_n = v_n - v_{n-1}$ $(n = 1, 2, \cdots)$.

THEOREM $E^*(n, 1)$. *For each positive integer j, let β_j be the block consisting of the terms $d_n (3^{j-1} < n \leqq 3^j)$ in the sequence $\{d\} = d_1, d_2, \cdots$. Then*

$$\beta_j = j + 1, j, j + 1, j, \cdots, j + 1, j \ (2 \cdot 3^{j-1} \text{ terms}). \tag{7.1}$$

This, with $d_1 = 0$, specifies $\{d\}$; and $\{d\}$, with $v_0 = 0$, determines both $\{v\}$ and $\{\epsilon_1^*(n)\} = \{v_n/n\}$.

We omit the proof of Theorem $E^*(n, 1)$, for the sake of brevity. It is not trivial, but it involves the same kind of reasoning as our solution of $E'(n, 1)$.

A *best partition of n* will mean a partition (n_1, n_2, n_3) such that an optimal program for $E^*(n, 1)$ can commence with a step in which W is partitioned into subsets with n_1, n_2, and n_3 members, respectively. If (n_1, n_2, n_3) is a best partition of n, then $\min(n_1, n_2, n_3)$ will be called a *best n-value.*

THEOREM 7.1. *A best n-value is $[(n+1)/3]$ or $[(n+4)/3]$, whichever is odd.*

This result and the next one are byproducts of the omitted detailed proof of Theorem $E^*(n, 1)$.

(B) The best n-value is unique for each $n < 13$, and so is the best partition of n. The only best 13-value is 3, but 13 has two best partitions: (3, 3, 7) and (3, 5, 5). We have completed an analysis of best partitions which we omit for brevity.

(C) By contrast with problem $E'(n, 1)$, the optimal expected number of weighings $\epsilon_1^*(n)$ is necessarily a monotonically increasing function of n. The value w_n, used in $E'(n, 1)$, will clearly be less than its analog v_n for those values of n such that $\epsilon_1'(n+1) < \epsilon_1'(n)$. [See Section 6(B).] The unique best partition of 6 for $E^*(6, 1)$ is (1, 2, 3), and the best partition of 12 is (3, 4, 5). These are not admissible in $E'(n, 1)$, since $E'(n, 1)$ requires that two numerically equal subsets be balanced. Note that $\epsilon_1^*(6) = 11/6 < \epsilon_1^*(7) = 13/7 = \epsilon_1'(7) < \epsilon_1'(6) = 2$.

(D) A comparison of the difference sequences for $\{w\}$ and for $\{v\}$ leads to the following relations, where $3^{j-1} < n \leqq 3^j$

$$\begin{aligned} &\text{(a) } w_n = v_n + 1, \quad \epsilon_1'(n) = \epsilon_1^* + 1/n \text{ if } n - 3^{j-1} \equiv 3 \pmod 4 \\ &\text{(b) } w_n = v_n, \qquad\quad \epsilon_1'(n) = \epsilon_1^*(n) \text{ otherwise.} \end{aligned} \tag{7.2}$$

This leads to the following result.

THEOREM $E(n, 1)$. *Let j be defined by $3^{i-1} < j \leqq 3^i$. If $n-j \not\equiv 3 \pmod 4$, then $\epsilon_1(n) = \epsilon_1'(n) = \epsilon_1^*(n)$, and there exists at least one program which is simultaneously optimal for $E(n, 1)$, $E'(n, 1)$ and $E^*(n, 1)$. Suppose $n \equiv 3 \pmod 4$, and no light object outside W is available at the start. Then $\epsilon_1(n) = \epsilon_1'(n) = \epsilon_1^*(n) + 1/n$. Suppose $n \equiv 3 \pmod 4$, and a light object outside W is available at the start. Then, under hypothesis* (1) *of Section* 1(C), *the problem reduces to $E^*(n, 1)$; and, under hypothesis* (2) *of* Section 1(C), $\epsilon_1(n) = \epsilon_1'(n+1)$. *In this last case, the supplementary light object is first amalgamated with W, and an optimal program for $E'(n+1, 1)$ is then followed.*

8. Problem $M(n, 2)$. (A) Step $[X, Y]$ results in the following specialization of (1.2)–(1.5) where U_{ji} is replaced by U_j since the range of i reduces to 1 for $j=1$ and $j=2$.

Case U.

$$
\begin{aligned}
&U_1\colon\ X = \{1, m-1\} \qquad Y = \{0, m\} \qquad Z = \{1, n-2m-1\} \\
&U_2\colon\ X = \{2, m-2\} \qquad Y = \{0, m\} \qquad Z = \{0, n-2m\},
\end{aligned} \tag{8.1}
$$

with the following respective probabilities, specializing (1.3):

$$
\begin{aligned}
p_1 &= p(U_1) = \frac{4m(n-2m)}{n(n-1)} \\
p_2 &= p(U_2) = \frac{2m(m-1)}{n(n-1)} \cdot
\end{aligned} \tag{8.2}
$$

Case B.

$$
\begin{aligned}
&B_0\colon\ X = \{0, m\} \qquad Y = \{0, m\} \qquad Z = \{2, n-2m-2\} \\
&B_1\colon\ X = \{1, m-1\} \quad Y = \{1, m-1\} \quad Z = \{0, n-2m\}
\end{aligned} \tag{8.3}
$$

with the respective probabilities

$$
\begin{aligned}
q_0 &= p(B_0) = \frac{(n-2m)(n-2m-1)}{n(n-1)} \\
q_1 &= p(B_1) = \frac{2m^2}{n(n-1)} \cdot
\end{aligned} \tag{8.4}
$$

There are two reasonable procedures immediately after Step 1, which generally leaves us in a state of uncertainty between either U_1 and U_2 or B_0 and B_1, namely: (1) to proceed with weighings specifically designed to resolve the uncertainty, or (2) to proceed with other weighings, without resolving it.

LEMMA 8.1. *The function μ_2 is nondecreasing. That is, $n' > n$ implies $\mu_2(n') \geqq \mu_2(n)$.*

Proof. This can be directly verified, say for $n' \leqq 4$. It can then be proved inductively for all larger values, by showing that if $n' > n$ and $\mu_2(n') < \mu_2(n)$, then there would exist an $m' < n$ and an $m < m'$ such that $\mu_2(m') < \mu_2(m)$. This can be done with the aid of (A) above, applied for n and n', together with the fact that $\mu_1(n)$ is nondecreasing, by Theorem $M(n, 1)$.

LEMMA 8.2. *If it were certain which of the four cases prevailed after step* 1, *the best program for problem* $M(n, 2)$ *would be to proceed independently with the set, or the two sets, known to contain heavy objects, rather than to balance sets chosen from more than one of the subsets* (X, Y, Z).

For example, if X contains one heavy object, the most information the next step can yield, relative to that heavy object, is to reveal which of three subsets of X contains it; and, by the proof of Theorem $M(n, 1)$, nothing is lost by requiring two of these subsets to be equal.

If X contains both of the heavy objects, Lemma 8.2 depends on Lemma 8.1.

LEMMA 8.3. *The following inequalities hold* (*see Section* 1(E) *for notation*):

$$\begin{aligned} \mu_U &\geqq \max\,(\mu_1(m) + \mu_1(n-2m),\, \mu_2(m)) \\ \mu_B &\geqq \max\,(\mu_2(n-2m),\, 2\mu_1(m)). \end{aligned} \tag{8.5}$$

Proof. By Lemma 8.2, the right sides are the maximum numbers of weighings which might be required if there were certainty, instead of uncertainty, as to which subcase of U or of B prevailed. These are lower bounds, since added information cannot increase the required number of weighings.

COROLLARY. *The following inequality holds*:

$$\mu_2(n, m) \geqq 1 + \max\,(2\mu_1(m),\, \mu_1(m) + \mu_1(n-2m),\, \mu_2(m),\, \mu_2(n-2m)). \tag{8.6}$$

Also, if $3^{\nu-1} < n \leqq 3^\nu$,

$$\mu_2(n) \geqq 1 + \max\,(2(\nu - 1),\, \mu_2([n/3])). \tag{8.7}$$

Proof. Inequality (8.6) follows from (8.5) and (1.6). Inequality (8.7) follows from (8.6), Theorem $M(n, 1)$, and Lemma 8.1, which implies (1) that $\max\,(\mu_2(m), \mu_2(n-2m)) = \mu_2(\max\,(m, n-2m))$, and (2) that $m = [n/3]$ is therefore a best choice of m for the purpose of minimizing the maximum.

THEOREM $M(n, 2)$. *The value* $\mu_2(n)$ *satisfies*

$$\mu_2(n) \leqq 2 + \max\,(2(\nu - 1),\, \mu_2([n/3])), \tag{8.8}$$

where

$$3^{\nu-1} < n \leqq 3^\nu. \tag{8.9}$$

Proof. Having found a lower bound in (8.7) on $\mu_2(n)$ by assuming some extra information, we now find an upper bound by computing the maximum number of weighings under a particular program, as follows:

Program. Given $M(n, 2)$, follow an optimum program for problem $M(n, 1)$ until the first weighing on which the scales balance. If no such weighing occurs, one of the heavy objects is obtained in $\mu_1(n) = \nu$ weighings. The remaining heavy object is in the union of all the sets Z which were not put on the scales, and either $\nu - 1$ or ν additional weighings are needed to cull it out. Thus $2\nu - 1$ or 2ν weighings suffice if case B never arises. Now suppose case B arises on some step; the first one for example. Since Z has at least as many elements as Y, we can balance Y against an equal-numbered subset of Z, thus discriminating between B_0 and B_1. We thus find, taking the discriminatory weighing into account,

$$\mu_2(n) \leqq 2 + \max(2(\nu - 1,)\ \mu_2([n/3])) \tag{8.10}$$

(B) *Combining* (8.7) *and* (8.8), *we find*

$$1 + \max(2(\nu - 1), \mu_2([n/3])) \leqq \mu_2(n) \leqq 2 + \max(2(\nu - 1), \mu_2([n/3])) \tag{8.11}$$

This pins $\mu_2(n)$ between a lower and an upper bound which differ by just one unit.

NOTE: The author believes that a slight further analysis would yield a precise formulation of $\mu_2(n)$, and that it has one of the values $2\nu-1$, 2ν, $2\nu+1$, depending on n. He also conjectures that the above program is optimal, provided the optimal program for $M(n, 1)$, on which it depends, is properly selected.

This is a revision of a paper written in 1955 at the Rand Corporation (Rand No. P-736) and presented to the American Mathematical Society, December 1955.

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ON A PROBLEM OF P. ERDÖS

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Introduction. Let $\mathcal{P}$ denote the family of all planar subsets P_n of n points and let $f(n)$ be the minimum number of different distances determined by its n points for all $P_n \in \mathcal{P}$. The determination or even good estimation of $f(n)$ seems to be a difficult problem. The best estimates known for $f(n)$ are:

$$\frac{n^{2/3}}{2\sqrt[3]{9}} - 1 < f(n) < C\frac{n}{(\log n)^{1/2}} \cdot$$

It is the main purpose of the present paper to prove that if the n points are vertices of a convex polygon then $f(n) = [n/2]$. This fact was conjectured by P. Erdös [**1, 2, 3**]. Erdös further conjectured that every convex polygon has a vertex which does not have three vertices equidistant from it. This conjecture was recently disproved by S. Danzer for $n = 9$. The following conjecture of

Erdös is still left open: Every convex polygon of n vertices contains a vertex so that there are at least $[n/2]$ different distances from it. The weaker result, $f(n) \geqq [(n+2)/3]$ was established by L. Moser [4]. A related result to the effect that each convex polygon having $2N+1$ sides and comprising N different distances is regular is also proved.

THEOREM (*proposed by P. Erdös*). *Every plane convex n-sided polygon, (hereafter referred to as "polygon") comprises at least* $[n/2]$ *different distances between corresponding pairs of vertices.*

As first step of its proof, we formulate the following

LEMMA 1. *If a side, appropriately denoted by* A_1A_n, *of a polygon* $A_1A_2 \cdots A_n$ *is of maximum length (i.e. not exceeded in length by any side or diagonal), then for every quadruplet* (p, q, x, y) *satisfying*

$$(A) \qquad 1 \leqq p < y \leqq x < q < n$$

at least one of the diagonals or sides A_pA_x, A_qA_y *is smaller than diagonal* A_pA_q.

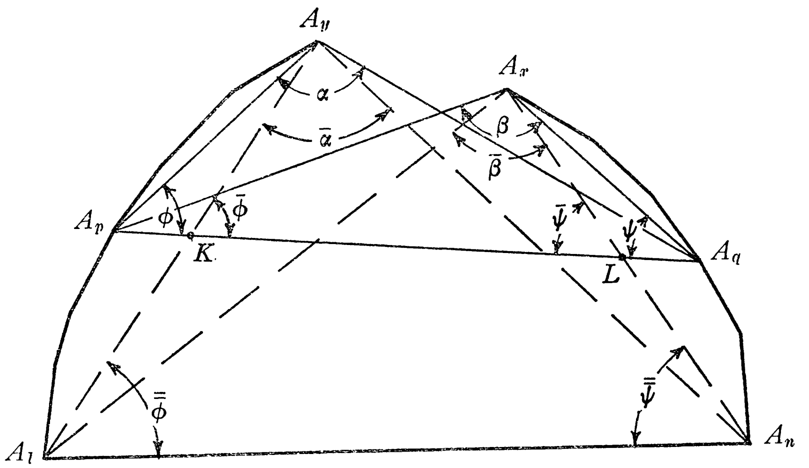

FIG. 1

Proof. Let $A_1A_2 \cdots A_n$ (Fig. 1, in which $x>y$) be an n-sided polygon and A_1A_n a side of maximum length. Furthermore, let A_pA_q, A_pA_x, A_qA_y be diagonals satisfying

$$(\bar{A}) \qquad 1 \leqq p < y < x < q < n.$$

Should neither A_pA_x nor A_qA_y, contrary to Lemma 1, be smaller than A_pA_q, i.e.

$$(\mathrm{B}) \qquad \begin{cases} A_qA_y \geqq A_pA_q \\ A_pA_x \geqq A_pA_q \end{cases}$$

it would follow that

$$\text{(C)} \qquad \begin{aligned} (\phi=)\ \measuredangle A_yA_pA_q &\geqq \measuredangle A_pA_yA_q\ (=\alpha) \\ (\psi=)\ \measuredangle A_xA_qA_p &\geqq \measuredangle A_qA_xA_p\ (=\beta) \end{aligned}$$

and hence

$$\text{(D)} \qquad \phi + \psi \geqq \alpha + \beta.$$

If A_x and A_y are connected with A_1 and A_n, we have by $(\overline{A})$ and by the convexity of the polygon,

$$\text{(1)} \qquad \begin{aligned} (\bar{\alpha}=)\ \measuredangle A_1A_yA_n &< \alpha \\ (\bar{\beta}=)\ \measuredangle A_1A_xA_n &< \beta. \end{aligned}$$

If now K and L denote the intersections of A_1A_y and A_nA_x with A_pA_q respectively, then:

$$\text{(2)} \qquad \begin{aligned} (\bar{\phi}=)\ \measuredangle A_yKA_q &\geqq \phi \\ (\bar{\psi}=)\ \measuredangle A_xLA_p &> \psi, \end{aligned}$$

where $\bar{\phi}=\phi$ if $p=1$. Hence: $\bar{\phi}+\bar{\psi}>\phi+\psi$.

From (D), (1) and (2) we have

$$\text{(E)} \qquad \bar{\phi} + \bar{\psi} > \bar{\alpha} + \bar{\beta}.$$

In triangle $A_1A_yA_n$, angle $\bar{\alpha}$ lies opposite side A_1A_n, hence

$$\bar{\alpha} \geqq \measuredangle A_yA_1A_n\ (=\bar{\bar{\phi}}).$$

Similarly, in triangle $A_1A_xA_n$:

$$\bar{\beta} \geqq \measuredangle A_xA_nA_1\ (=\bar{\bar{\psi}}),$$

hence

$$\text{(F)} \qquad \bar{\alpha} + \bar{\beta} \geqq \bar{\bar{\phi}} + \bar{\bar{\psi}}.$$

Fig. 1 shows that

$$\text{(G)} \qquad \bar{\phi} + \bar{\psi} = \bar{\bar{\phi}} + \bar{\bar{\psi}}$$

and from (F) and (G) it follows that

$$\bar{\alpha} + \bar{\beta} \geqq \bar{\phi} + \bar{\psi}$$

which is contrary to (E). Lemma 1 is hereby proved for $x>y$.

For the case $x=y$ (Fig. 2), $\measuredangle A_pA_yA_q=\alpha>\bar{\alpha}=\measuredangle A_1A_yA_n$ by convexity. Since $\bar{\alpha}$ lies opposite side A_1A_n in triangle $A_1A_yA_n$, $\bar{\alpha}\geqq\pi/3$ and hence $\alpha>\pi/3$. Consequently, at least one of the sides A_pA_y, A_qA_y in triangle $A_pA_yA_q$ must lie

opposite an angle smaller than $\pi/3$, i.e. must be smaller than A_pA_q. Lemma 1 is hereby proved in its entire scope.

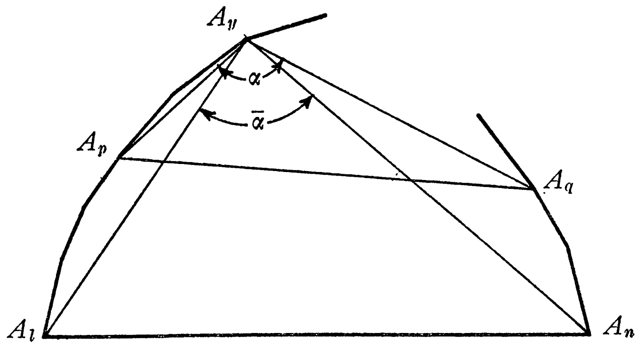

FIG. 2

LEMMA 2. *If a side of an n-sided polygon is of maximum length (i.e. not exceeded by any side or diagonal), the polygon comprises at least* $(n-2)$ *different distances; if however, the side is a maximum in the narrower sense, the polygon contains at least* $(n-1)$ *different distances.*

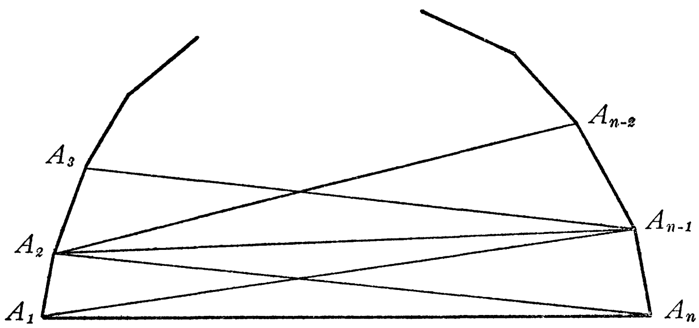

FIG. 3

Proof. Let $A_1A_2 \cdots A_n$ (Fig. 3) be an n-sided polygon and A_1A_n a side of maximum length, denoted by d_1.

Consider the convex quadrilateral $A_1A_2A_{n-1}A_n$. Since A_1A_n is of maximum length, its adjoining angles A_1, A_n are necessarily acute. The quadrilateral must, however, contain at least one obtuse angle, which must thus adjoin side A_2A_{n-1}, i.e. lie opposite one of the diagonals. Side A_2A_{n-1} (denoted by d_2) is then smaller than this diagonal and thus smaller than A_1A_n; $d_2 < d_1$.

By Lemma 1, at least one of the diagonals A_2A_{n-2}, $A_{n-1}A_3$ (denoted by d_3) must be smaller than $A_2A_{n-1}(=d_2)$, i.e. $d_3 < d_2 < d_1$.

Generally, if A_pA_{n-q} is such a diagonal of length d_λ, then by Lemma 1 either $A_{p+1}A_{n-q}$ or A_pA_{n-q-1} must be smaller than A_pA_{n-q}. We have thus introduced a new distance $d_{\lambda+1} < d_\lambda$.

Accordingly, each pair of diagonals, of lengths d_λ, $d_{\lambda+1}$ ($\lambda = 2, 3, \cdots$), have a common vertex, their other end-points being consecutive vertices forming the sequences $A_2, A_3, \cdots$ and $A_{n-1}, A_{n-2}, \cdots$ respectively. Each step in this process towards a new (smaller) distance leads to one more vertex of either of the above sequences. Thus in $(n-3)$ steps the vertices $A_2, A_3, \cdots, A_{n-2}, A_{n-1}$ will be included and $(n-3)$ distinct distances obtained. If the maximum length d_1, that of the side A_1A_n, is also included, the total number of distances thus obtained will be exactly $(n-2)$.

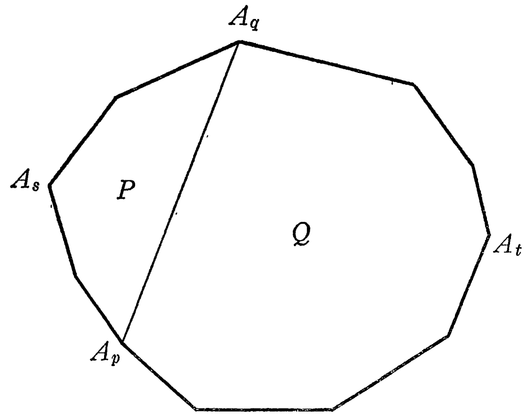

FIG. 4

This number of $(n-2)$ distinct distances: $d_1, d_2 \cdots d_{n-2}$ is obviously minimum, since the described method excludes all other possible distances. The first part of Lemma 2 is hereby proved.

If side A_1A_n is of maximum length in the narrower sense, every diagonal of quadrilateral $A_1A_2A_{n-1}A_n$ (Fig. 3) is smaller than $A_1A_n = d_1$ and at least one of these diagonals is larger than $A_2A_{n-1} = d_2$, with an intermediate length $\bar{d}_1$: $d_1 > \bar{d}_1 > d_2$. Lemma 2 is hereby proved in its entire scope.

Now for the proof of Erdös' theorem: Let $A_1A_2 \cdots A_n$ be a plane convex n-sided polygon (Fig. 4) and A_pA_q a diagonal of maximum length, cutting off a minimum number (denoted by x) of consecutive sides.

We thus obtain two secondary polygons, also convex: $A_p \cdots A_s \cdots A_q$, denoted by P, with a total of $x+1$ sides, $A_p \cdots A_t \cdots A_q$, denoted by Q, with a total of $n-x+1$ sides.

In P, A_pA_q is maximum in a narrower sense and by Lemma 2 it comprises at least x different distances. In Q, A_pA_q is of maximum length and by the same lemma it comprises at least $(n-x-1)$ different distances.

Should the original n-sided polygon, contrary to Erdös' proposition, comprise less than $[n/2]$ different distances, the following two inequalities would hold simultaneously:

$$(x+1)-1<\left[\frac{n}{2}\right], \quad (n-x+1)-2<\left[\frac{n}{2}\right],$$

i.e.

$$x<\left[\frac{n}{2}\right], \quad n-x-1<\left[\frac{n}{2}\right],$$

or

$$x<\left[\frac{n}{2}\right], \quad x>n-\left[\frac{n}{2}\right]-1.$$

For $n=2N$, we have $x<N$, and $x>N-1$, which does not hold for any integer x. Whereas for $n=2N+1$, we have $x<N$, and $x>N$, which is self-contradictory. Erdös' proposition is hereby proved.

In the light of the above, the following theorem can be proved:

THEOREM. *If a plane convex* $(2N+1)$*-sided polygon comprises exactly N different distances, it is regular.*

The following nomenclature is used:

1) All line segments (diagonals or sides) A_kA_{n-k+1}, $K=(2, \cdots [n+1/2])$ of a polygon are said to form a parallel grid $(1, n)$. For odd n, the last parallel is reduced to a point.

2) The diagonals A_kA_{n-k}, $A_{n-k+1}A_{k+1}$, originating in the end-points of a parallel A_kA_{n-k+1} are called transversals.

3) A parallel is said to form ϵ-angles with the sides "above" it and δ-angles with those "below" it. Hence parallel A_kA_{n-k+1}, which is not reduced to a point, forms ϵ-angles with sides A_kA_{k+1}, $A_{n-k+1}A_{n-k}$, and δ-angles with A_kA_{k-1}, $A_{n-k+1}A_{n-k+2}$.

We now prove

LEMMA 3. (a) *If a side* A_1A_n *of a plane convex n-sided polygon* $A_1A_2\cdots A_n$ *is maximum in the narrower sense, and the polygon comprises exactly* $(n-1)$ *different distances* $d_1>d_2>\cdots>d_{n-1}$, *then the parallel* A_kA_{n-k+1}, $k=(1, 2, \cdots [n/2])$ *is of length* d_{2k-1} *and each of the transversals* A_kA_{n-k+2}, $A_{k-1}A_{n-k+1}$, $k=(2, 3, \cdots [n+1/2]$ *of length* d_{2k-2}.

(b) *If a side* A_1A_n *of a plane convex polygon* $A_1, A_2\cdots A_n$ *is of maximum length* d_1 *and the polygon comprises exactly* $(n-2)$ *different distances* $d_1d_2\cdots d_{n-2}$, *then the parallel* A_kA_{n-k+1}, $k=(2, 3, \cdots [n/2])$ *is of length* d_{2k-2} *and each of the transversals* A_kA_{n-k+2}, $A_{k-1}A_{n-k+1}$, $k=(2, 3, \cdots [n+1/2])$ *of length* d_{2k-3}.

Proof. 1. Beginning with the first case, let $A_1A_2 \cdots A_n$ (Fig. 5) be a plane convex n-sided polygon. Further, let A_1A_n be maximum in the narrower sense and the polygon comprise exactly $(n-1)$ different distances.

Consider the quadrilateral $A_1A_2A_{n-1}A_n$. In view of the minimum of $(n-1)$ distances in the polygon by virtue of Lemma 2, only one of the transversals A_1A_{n-1} or A_2A_n need be considered. If the maximum length is d_1, denoting that of the transversal by d_2 and that of the parallel A_2A_{n-1} by d_3, we have

$$d_1 > d_2 > d_3. \tag{K}$$

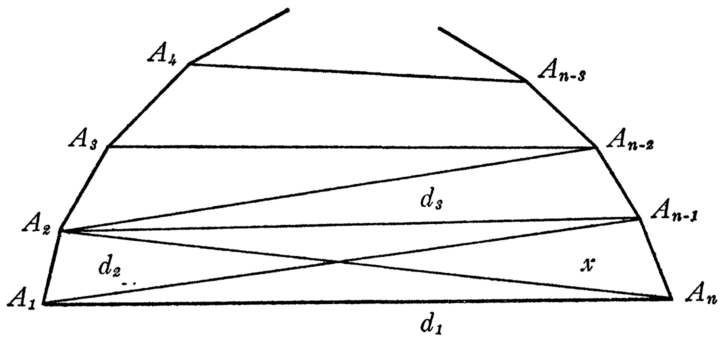

FIG. 5

Let x be the length of the second transversal. By convexity of the quadrilateral in question, the sum of the transversals is larger than that of the including parallels:

$$x + d_2 > d_1 + d_3$$

or

$$x > (d_1 - d_2) + d_3;$$

and by (K), we have $x > d_3$.

Were x to lie between d_2 and d_3, the polygon would comprise n distances contrary to the assumption of the first part of our lemma. Since $x < d_1$, necessarily $x = d_2$, i.e. the transversals of $A_1A_2A_{n-1}A_n$ are equal.

2. It can now be proved that A_2A_{n-1} exceeds all other sides and diagonals of the secondary polygon $A_2A_3 \cdots A_{n-2}A_{n-1}$.

To begin with, a statement regarding the ϵ-angles:

If a side of a convex polygon is of maximum length, the sum of the ϵ-angles at each of the parallels to this side is smaller than π.

In these circumstances the sum of the angles at the maximum side is smaller than π, since every one of them is necessarily acute. In the quadrilateral $A_1A_2A_{n-1}A_n$, the sum of the δ-angles must be greater than π and that of the ϵ-angles at the parallel A_2A_{n-1}, by convexity, smaller than π, etc.

To revert to the secondary polygon $A_2A_3 \cdots A_{n-2}A_{n-1}$, every diagonal or side of maximum length must originate either in A_2 or in A_{n-1}. Were A_pA_q $(2<p<q<n-1)$ maximum, the sum of angles of the quadrilateral $A_2A_pA_qA_{n-1}$ should be smaller than 2π, since there are two acute angles at A_pA_q and ϵ-angles at the parallel A_2A_{n-1}, whose sum is smaller than π.

Were, for example, $A_2A_{n-2} \geqq d_3$, we would have in the convex quadrilateral $A_1A_2A_{n-2}A_{n-1}$:

$$A_2A_{n-1} + A_1A_{n-2} > A_1A_{n-1} + A_2A_{n-2}$$
$$d_3 + A_1A_{n-2} > d_2 + d_3$$
$$A_1A_{n-2} > d_2.$$

It would have followed that $A_1A_{n-2}=d_1$, which is contrary to the first part of Lemma 3. A_2A_{n-1} is thus shown to be maximum in the narrower sense in the secondary polygon $A_2A_3 \cdots A_{n-1}$.

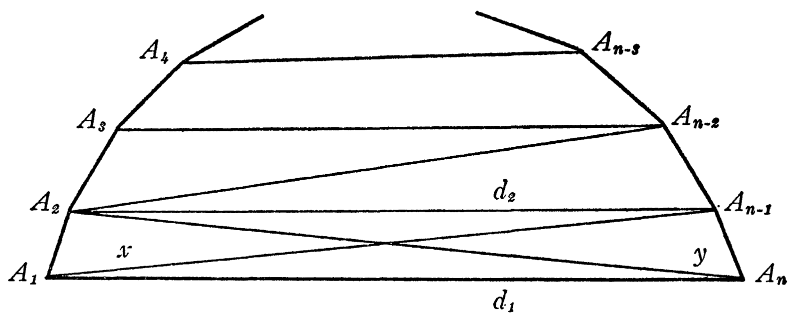

FIG. 6

The number of different distances in this $(n-2)$-sided secondary polygon is exactly $(n-3)$. Applying the considerations of paragraph (1), we find that the transversals A_2A_{n-2}, A_3A_{n-1} should be equal and of length d_4, whence the parallel A_3A_{n-2} should be of length d_5. In accordance with paragraph (1), it is seen that this latter parallel is maximum in the narrower sense in the secondary polygon $A_3A_4 \cdots A_{n-3}A_{n-2}$. Proceeding in this manner, we arrive at the parallel A_kA_{n-k+1} of length d_{2k-1}, which is seen to be maximum in the narrower sense in the secondary polygon $A_kA_{k+1} \cdots A_{n-k}A_{n-k+1}$. This polygon comprises exactly $(n-k+1)$ different distances, and in accordance with paragraph (1) the transversals above this parallel should be equal and of length d_{2k}. The next parallel $A_{k+1}A_{n-k}$ would then be d_{2k+1} in length; and so on. The proposition of the first part of Lemma 3 is hereby proved.

3. Now for the proof of the second part of Lemma 3: If side A_1A_n (Fig. 6) is of maximum length, the lengths of the transversals A_1A_{n-1}, A_2A_n do not figure in stipulating the minimum of $(n-2)$ different distances. If the length of the maximum side is d_1, and that of parallel $A_2A_{n-1}d_2$, then $d_1>d_2$.

If x and y are the respective lengths of the transversals A_1A_{n-1}, A_2A_n, then by convexity of quadrilateral $A_1A_2A_{n-1}A_n$:

$$x + y > d_1 + d_2$$
$$x > (d_1 - y) + d_2$$
$$y > (d_1 - x) + d_2$$

or

$$x > d_2, \quad y > d_2.$$

If the minimum of $(n-2)$ different distances, as stipulated, is not to be exceeded, then necessarily $x = y = d_1$ as in Lemma 3(b).

It is easily seen that, in the secondary polygon $A_2A_3 \cdots A_{n-2}A_{n-1}$, side A_2A_{n-1} is maximum in the narrower sense:

Were for example $A_2A_{n-2} \geqq d_2$, we would have in the convex quadrilateral $A_1A_2A_{n-2}A_{n-1}$

$$A_2A_{n-1} + A_1A_{n-2} > A_1A_{n-1} + A_2A_{n-2}$$
$$d_2 + A_1A_{n-2} > d_1 + d_2$$
$$A_1A_{n-2} > d_1$$

which is contrary to assumption.

Applying the considerations of paragraphs (1) and (2) to the secondary polygon $A_kA_{k+1} \cdots A_{n-k}A_{n-k+1}$ $(k = 2, 3 \cdots [n+1/2])$, the second part of Lemma 3 is proved.

4. The initial theorem can now be proved: Let $A_1A_2 \cdots A_{2N+1}$ (Fig. 7) be a plane convex $(2N+1)$-sided polygon, comprising N different distances. Its diagonal of maximum length must cut off exactly N sides. Were it to cut off, for example, $(N-1)$ sides on one side, we would obtain, on the other side, a secondary polygon of $N+3$ sides with the diagonal in question as a side of maximum length. By Lemma 2, the polygon would then, contrary to assumption, comprise at least $(N+1)$ distances.

Let A_1A_N be a diagonal of maximum length. We obtain the following secondary polygons:

(I) $A_1 \cdots A_p \cdots A_N$ with a total of $(N+1)$ sides, in which A_1A_N is maximum in the narrower sense

(II) $A_n \cdots A_q \cdots A_{2N+1}A_1$ with a total of $(N+2)$ sides, in which A_1A_N is a side of maximum length.

In polygon (II), in accordance with the above, the diagonals A_1A_{N+1} and A_NA_{2N+1} must be of maximum length.

Each of these latter diagonals divides the original polygon $A_1A_2 \cdots A_{2N+1}$ into two secondary polygons of types (I) and (II), so that each secondary polygon of type (II) again comprises a diagonal of maximum length. It is seen that all major diagonals of maximum length, cutting off N sides, are equal.

Introducing in each of the $(2N+1)$ secondary polygons of type (I), above each major diagonal, the parallel grid $(1, N)$, $(2, N+1) \cdots$, the corresponding parallels and transversals must be equal by the proposition of Lemma 3. Hence the $(2N+1)$-sided polygon is regular.

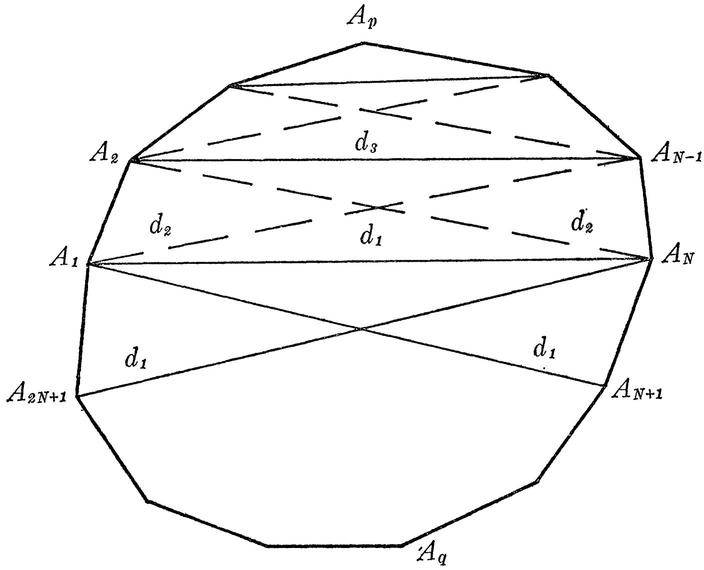

FIG. 7

Remark. A $2N$-sided plane convex polygon comprising exactly N different distances is not necessary regular. For example, by excluding one vertex of a $(2N+1)$-sided regular polygon, we would obtain a $2N$-sided irregular polygon with exactly N distances.

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ON THE CAUCHY-PICARD METHOD

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1. It is well known that the use of the Cauchy-Picard method of successive approximation to prove the existence and uniqueness of the solution $y(x, \mu)$ of the initial value problem $y(x_0, \mu) = y_0$, for the differential equation

$$dy/dx = f(x, y, \mu), \tag{1}$$

with parameter μ, for suitable functions f is a realization of *the principle of the contraction mapping of a complete metric space* (cf., e.g., [2], pp. 46–49). The contraction mapping property is obtained from a Lipschitz condition (usually of constant <1) and the complete metric space sometimes is particularized as a Banach space with distance realized by the norm (cf., e.g., [1], pp. 40–41). Here we point out that for some extra work and some care in statement, the principle of the contraction mapping can be used to prove a more inclusive theorem describing the differentiability properties of the solution as a function of x and μ. This includes, in particular, the behavior of the solution as a function of the initial conditions. To be specific, we will prove the following

THEOREM. *Let x and μ be real variables, y a real variable, and f a real valued function defined on a closed rectangle R_a in the $xy\mu$-space,*

$$R_a: |x - x_0| \leqq a, |y - y_0| \leqq b, |\mu| \leqq c.$$

Let $\partial^{l+m+n}f(x, y, \mu)/\partial x^l \partial y^m \partial \mu^n$ be a continuous function which satisfies a uniform Lipschitz condition with respect to y on R_a. We assert the existence, uniqueness, and continuity of the solution $y(x, \mu)$ and its mixed partial derivatives $\partial^{p+q}y(x, \mu)/\partial x^p \partial \mu^q$ for any $p \leqq \min(l, m) + 1$ and any $q \leqq \min(m, n)$ on R_h for any

$$h < \min \left\{ a, \frac{b}{\max_{R_a} |f|} \right\}.$$

When $l = m = n = 0$, this is the usual theorem (cf., e.g., [1], p. 12). The extension to systems of equations presents no difficulty and is more tedious than informative.

2. The mechanism of our proof is the *principle of the contraction mapping* (cf., e.g., [2], p. 43): Let $\rho(\phi, \psi)$ be the metric of a complete metric space $\mathcal{S}$. Let T be a contraction, i.e., a mapping of the space into itself such that, for all $\phi, \psi \in \mathcal{S}$,

$$\rho(T\phi, T\psi) \leqq \alpha\rho(\phi, \psi), \qquad \alpha < 1. \tag{2}$$

Then T possesses a unique fixed point y; $T^n\phi \to y$, $n \to \infty$ for any ϕ, and the convergence is uniform on any bounded set.

Successive approximation is a method for finding a fixed point of the mapping

$$T\phi = y_0 + \int_{x_0}^{x} f(s, \phi(s, \mu), \mu)ds, \tag{3}$$

defined for functions ϕ such that $(x, \phi, \mu) \in R_a$. We will show that T is a contraction mapping on an appropriate complete metric space; then we will conclude the existence of a unique fixed point, $y(x, \mu)$, in that space, satisfying $y = Ty$ or

$$y(x, \mu) = y_0 + \int_{x_0}^{x} f(x, y(x, \mu), \mu)ds. \tag{4}$$

This function y is the unique desired solution of the initial value problem.

In order to simplify our formulae, we adopt the following notations:

In addition to $f_y = \partial f/\partial y$ and $f_\mu = \partial f/\partial \mu$, we write

$$f_{m,n} = \partial^{m+n} f/\partial y^m \partial \mu^n$$

and

$$\partial^n f = \partial^n f/\partial \mu^n = f_{0,n}.$$

We will denote by G_w the set in $x\mu$-space such that $x_0 \leqq x \leqq w$ and $|\mu| \leqq c$.

We denote by $S_q(h)$ the space of functions $\phi = \phi(x, \mu)$ such that $(x, \phi, \mu) \in R_h$ and $\partial^n \phi (0 \leqq n \leqq q)$ exists and is continuous on G_{x_0+h}. (When h is indexed by q as h_q, this denotes a specific h chosen as a function of q.) $S_q(h)$ is a complete metric space if we set

$$\rho_q(\phi, \psi) = \max_{0 \leqq n \leqq q} \left[\sup_{G_{x_0+h}} |\partial^n \phi - \partial^n \psi| \right],$$

since convergence in this metric is uniform convergence of all partial derivatives $\partial^n \phi$ with $0 \leqq n \leqq q$. The transformation T, defined by (3), maps $S_q(h)$ into itself provided that $h \leqq \min (a, b/\max_{R_a} |f|)$, since our hypotheses permit the differentiations under the integral sign. Unfortunately, T need not be a contraction mapping of $S_q(h)$ into itself. However, we will be able to establish the existence of a closed subset of $S_q(h)$ on which T is a contraction mapping. Since a closed subset of a complete metric space is itself a complete metric space, the principle of the contraction mapping will still apply. We will conclude the existence and uniqueness of $y(x, \mu)$, the solution of (4) (and (1)), on $x_0 \leqq x \leqq x_0 + h_0$ together with the fact that $y(x, \mu)$ is in $S_q(h_q)$ on $G_{x_0+h_q}$. Since $\partial^{l+m+n} f/\partial x^l \partial y^m \partial \mu^n$ is continuous on R_a, differentiation under the integral sign of (4) is permitted; this will imply the existence and continuity of $\partial^{p+q} y(x, \mu)/\partial x^p \partial \mu^q$ on $G_{x_0+h_q}$.

3. The proof of this part of our original assertion proceeds by induction on q; at each stage we need only prove that (2) is satisfied for some $h = h_q > 0$, given that it is satisfied for $0, 1, \cdots, q-1$.

We note first that our hypotheses imply the existence on R_a of a bound M for $f_{m',n'}$ and uniform Lipschitz conditions with respect to y of constant K for $f_{m',n'}$ for all m', n', $0 \leqq m' \leqq m$ and $0 \leqq n' \leqq n$.

Let $q=0$. Then from (3) we see that

$$\sup_{G_x} | T\phi - T\psi | \leqq \int_{x_0}^{x} \sup_{R_a} | f(s, \phi, \mu) - f(s, \psi, \mu) | \, ds.$$

Using the uniform Lipschitz condition with respect to y we obtain

$$\sup_{G_x} | T\phi - T\psi | \leqq (x - x_0) K \sup_{G_x} | \phi - \psi | . \tag{5}$$

We set $h_0 < \min (a, b/M, 1/K)$; then (2) is satisfied, and T is a contraction mapping on the complete metric space $S_0(h_0)$. Hence there exists a unique fixed point y in $S_0(h_0)$ of the mapping T.

In order to make clear the nature of the induction, we will first consider the case $q=1$ and then proceed to arbitrary q.

Let $q=1$. We differentiate (3) with respect to μ and both add and subtract a term $f_y(s, \psi, \mu)\partial\phi$ in the integrand. We see that

$$\sup_{G_x} | \partial T\phi - \partial T\psi | \leqq \int_{x_0}^{x} \sup_{R_a} \{ | f_y(s, \phi, \mu) - f_y(s, \psi, \mu) | \, | \partial\phi | + | f_y(s, \psi, \mu) | \, | \partial\phi - \partial\psi | + | f_\mu(x, \phi, \mu) - f_\mu(s, \psi, \mu) | \} ds.$$

Again we utilize the uniform Lipschitz conditions with respect to y and obtain

$$\sup_{G_x} | \partial T\phi - \partial T\psi | \leqq \int_{x_0}^{x} \left\{ K \sup_{G_x} | \phi - \psi | \sup_{G_x} | \partial\phi | + \sup_{R_a} | f_y | \sup_{G_x} | \partial\phi - \partial\psi | + K \sup_{G_x} | \phi - \psi | \right\} ds.$$

This yields the inequality

$$\sup_{G_x} | \partial T\phi - \partial T\psi | \leqq (x - x_0) \left(K \left(\sup_{G_x} | \partial\phi | + 1 \right) + M \right) \rho_1(\phi, \psi). \tag{6}$$

Since we have no *a priori* bound on $\sup_{G_x} |\partial\phi|$ over the whole space $S_1(h_0)$, (6) does not imply that T is a contraction. If we restrict ϕ and ψ to the closed ball $\rho_1(\phi, 0) \leqq N$, for some positive N, then by (5) and (6), T is a contraction if $h_1 < \min (h_0, (K(N+1)+M)^{-1})$. This is not enough, however, to guarantee that T is a mapping of the closed ball $\rho_1(\phi, 0) \leqq N$ into itself. If we note that the case $q=0$ implies that T maps any closed ball $\rho_0(\phi, y) \leqq N$ into itself, we are led to the following:

LEMMA. *Let $N > |y_0|$. Let $0 < h \leqq h_0$ and let B_h be the common part of the two sets $\rho_0(\phi, y) \leqq N$ and $\rho_1(\phi, 0) \leqq N$. Then B_h is a closed subset of $S_1(h_0)$ and for some positive h, T maps B_h into itself.*

Proof. The closure of B_h is clear. If we set $\psi=y=Ty$ in (5), we obtain

$$\sup_{G_{x_0}+h_0} |T\phi - y| \leqq \rho_0(\phi, y),$$

or $\rho_0(T\phi, y) \leqq \rho_0(\phi, y)$. Next we observe that

$$\sup_{G_x} |T\phi| \leqq |y_0| + M(x - x_0),$$

while

$$\sup_{G_x} |\partial T\phi| \leqq \int_{x_0}^{x} \left(\sup_{R_a} |f_y| \sup_{G_x} |\partial\phi| + \sup_{R_a} |f_\mu|\right) dx \leqq (x - x_0)M(N + 1).$$

Thus for all $h \leqq \min((N-|y_0|)/M, N/M(N+1))$ we have $\rho_1(T\phi, 0) \leqq N$. This shows that T maps B_h into itself.

Finally we see that if $h_1 < \min(h_0, (N - |y_0|)/M, N/M(N + 1), (K(N+1)+M)^{-1})$, then T is a contract of B_{h_1} into itself. (The proof continues as follows: thus T has a unique fixed point in B_{h_1}, which is automatically in $S_0(h_1)$ and hence must coincide with y; then y is in $S_1(h_1)$. This conclusion is not needed for the induction.)

4. Our inductive hypothesis now can be stated: we assume that there is a positive h_{q-1} such that T is a contraction on a closed subset of $S_{q-1}(h_{q-1})$, the common part of the sets $\rho_0(\phi, y) \leqq N$ and $\rho_{q-1}(\phi, 0) \leqq N$. We call this set $B_{h_{q-1},q}$. For any $q > 1$, we may differentiate under the integral sign, so

$$\partial^q T\phi = \int_{x_0}^{x} \partial^q(f(s, \phi(s, \mu), \mu))ds.$$

Now

$$\partial^q(f(x, \phi(x, \mu), \mu) = \sum \alpha_{v_1v_2u_1u_2\cdots u_q} f_{v_1,v_2}(\partial\phi)^{u_1} \cdots (\partial^q\phi)^{u_q}, \tag{7}$$

where the coefficients α are nonnegative integers not greater than $q!$, the summation is over $1 \leqq v_1+v_2 \leqq q$, $1 \leqq u_1 + \cdots + u_q \leqq q$, and thus not more than q^{q+2} terms are present. These coarse estimates are sufficient for our needs. Next we observe that our estimates and Lipschitz conditions imply that

$$\begin{aligned}
&|f_{v_1,v_2}(x, \phi, \mu)(\partial\phi)^{u_1} \cdots (\partial^q\phi)^{u_q} - f_{v_1,v_2}(x, \psi, \mu)(\partial\psi)^{u_1} \cdots (\partial^q\psi)^{u_q}| \\
&\quad \leqq |f_{v_1,v_2}(x, \phi, \mu) - f_{v_1,v_2}(x, \psi, \mu)| \, |\partial\phi|^{u_1} \cdots |\partial^q\phi|^{u_q} \\
&\qquad + |f_{v_1,v_2}(x, \psi, \mu)| \, |(\partial\phi)^{u_1} \cdots (\partial^q\phi)^{u_q} - (\partial\psi)^{u_1} \cdots (\partial^q\psi)^{u_q}| \\
&\quad \leqq K|\phi - \psi| N^q + M|(\partial\phi)^{u_1} \cdots (\partial^q\phi)^{u_q} - (\partial\psi)^{u_1} \cdots (\partial^q\psi)^{u_q}|,
\end{aligned}$$

provided that $\rho_q(\phi, 0) \leqq N$.

Now $|\phi-\psi| \leqq \rho_q(\phi, \psi)$. For any numbers $\alpha_1, \cdots, \alpha_q, \beta_1, \cdots, \beta_q$, we observe that

$$|\alpha_1\alpha_2\cdots\alpha_q - \beta_1\beta_2\cdots\beta_q| \leqq |\alpha_1-\beta_1|\,|\alpha_2\cdots\alpha_q| + |\beta_1|\,|\alpha_2\cdots\alpha_q - \beta_2\cdots\beta_q|.$$

If we apply this inequality successively to the situation where $|\alpha_i| \leqq N$, $|\beta_i| \leqq N$, $|\alpha_i-\beta_i| \leqq \rho_q(\phi, \psi)$, we obtain

$$\begin{aligned} |\alpha_1\alpha_2\cdots\alpha_q - \beta_1\beta_2\cdots\beta_q| &\leqq N^{q-1}\rho_q(\phi,\psi) + N|\alpha_2\cdots\alpha_q - \beta_2\cdots\beta_q| \\ &\leqq qN^{q-1}\rho_q(\phi, \psi). \end{aligned}$$

If we apply this estimate, we see that

$$\begin{aligned} |f_{v_1,v_2}(x, \phi, \mu)(\partial\phi)^{u_1}\cdots(\partial^q\phi)^{u_q} - f_{v_1,v_2}(x, \psi, \mu)(\partial\psi)^{u_1}\cdots(\partial^q\psi)^{u_q}| \\ \leqq (KN^q + MqN^{q-1})\rho_q(\phi, \psi). \end{aligned}$$

Using (7), we see that

$$\sup_{G_x} |\partial^q T\phi - \partial^q T\psi| \leqq (x - x_0)q!\,q^{q+2}N^{q-1}(KN + Mq)\rho_q(\phi, \psi).$$

If $h_q < \min(h_{q-1}, (q!\,q^{q+2}N^{q-1}(KN+Mq))^{-1})$, then T is a contraction on the intersection of $B_{h_{q-1},q}$ and $S_q(h_q)$. It remains to be shown that T maps the closed set $B_{h_q,q}$ into itself. This corresponds to the *lemma* proven above for the case $q=1$. It is easy to see that $B_{h_q,q}$ is a closed subset. That T is a mapping *into* follows from (7), which yields the bound

$$\sup_{G_x} |\partial^q T\phi| \leqq (x - x_0)q!\,q^{q+2}N^qM, \tag{8}$$

and from the inductive hypothesis, since $h_q < h_{q-1}$ already implies $\rho_{q-1}(T\phi, 0) \leqq N$. Since $KN+Mq \geqq M$ for $q \geqq 1$, both sides of (8) are dominated by N if $x_0 \leqq x \leqq x_0 + h_q$.

5. In summary we have shown the existence of an

$$h_q < \min(h_{q-1}, (q!\,q^{q+2}N^{q-1}(KN + Mq))^{-1})$$

such that T is a contraction mapping of a closed subset of $S_q(h_q)$ into itself. The conclusion, as remarked at the close of Section 2, is the existence of a unique solution to the initial value problem on $G_{x_0+h_0}$ which possesses a continuous partial derivative, $\partial^{p+q}y/\partial x^p\partial\mu^q$ on $G_{x_0+h_q}$. Inspection of the proof shows that we could just as well have considered intervals of the form $x_0 - h \leqq x \leqq x_0$, with the same conclusion. At any point of the interval $x_0 - h_0 \leqq x \leqq x_0 + h_0$, we can reapply our proof and obtain the local existence, uniqueness, and differentiability statement. It now follows that the original statement of the theorem is correct.

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SCALAR VALUED MAPPINGS OF SQUARE MATRICES

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In this paper F is a field, n is a positive integer and $L_n(F)$ denotes the algebra of all n by n matrices with entries in F. Let ϕ be a mapping of $L_n(F)$ into F; we shall study conditions which require ϕ to be the determinant on $L_n(F)$.

In [3] Hensel characterized the determinant on $L_n(F)$ as a polynomial of degree n in n^2 variables (that is, $\det(x_{ij})$ is a polynomial in the variables x_{ij} of degree n) which is multiplicative on $L_n(F)$ under the usual definition of the product of two matrices. In [6] Stephanos showed that if F is the complex field, and if ϕ is a multiplicative mapping of $L_n(F)$ into F which is analytic in each entry (when the other n^2-1 entries are fixed), then ϕ is a power of the determinant. In [1] Cater showed that it suffices in Stephanos' Theorem that ϕ is analytic in one of the diagonal entries. In [4] Hosszu showed that if F is the complex field and if ϕ is a multiplicative mapping of $L_n(F)$ into F, then there exists a multiplicative mapping p of F into F, for which $\phi(A)=p[\det A]$ for all A in $L_n(F)$. In [1] Cater let F be the complex field, endowed $L_n(F)$ with the usual topology, and showed that if ϕ is a continuous multiplicative mapping of $L_n(F)$ into F and if $\phi(aI)\geqq 0$ for all scalars $a\geqq 0$, then ϕ is uniquely determined by the scalar $\phi[\exp(1+i)I]$; and in particular if $\phi[\exp(1+i)I]=\exp(n+ni)$ then ϕ is the determinant on $L_n(F)$. Most of these results are established by examining ϕ on elementary matrices. We introduce the following

DEFINITION. *The mapping ϕ is said to be submultiplicative if $\phi(ABC)=\phi(CBA)$ for all matrices A, B, C, in $L_n(F)$.*

Plainly if ϕ is multiplicative on $L_n(F)$, then ϕ is submultiplicative because $\phi(ABC)=\phi(A)\phi(B)\phi(C)=\phi(C)\phi(B)\phi(A)=\phi(CBA)$ for all A, B, C in $L_n(F)$. If p is any mapping of F into F then the mapping $A\rightarrow p(\det A)$ is submultiplicative on $L_n(F)$ because

$$\begin{aligned} p[\det(ABC)] &= p[(\det A)(\det B)(\det C)] \\ &= p[(\det C)(\det B)(\det A)] = p[\det(CBA)]. \end{aligned}$$

In the present paper we will show that any submultiplicative mapping ϕ can be represented in this manner; there must exist a mapping p of F into F for which $\phi(A)=p(\det A)$ for all A in $L_n(F)$. We will prove further that if ϕ is a submultiplicative mapping which is given by a polynomial in the matrix entries, then ϕ is given by a polynomial in the determinant; this generalizes the work of Hensel. Finally we will characterize all the continuous multiplicative mappings of $L_n(F)$ when F is the real field.

THEOREM 1. *Let F be a field not of characteristic 2 and let ϕ be a submultiplicative mapping of $L_n(F)$ into F. Then there exists a unique mapping p of F into F such that $\phi(A)=p(\det A)$, all A in $L_n(F)$. Furthermore ϕ is multiplicative on $L_n(F)$ iff p is multiplicative on F.*

Before proving this Theorem we define elementary matrices of Types I, II and III as Jacobson does in [5], pp. 19–20;

$$T_{pq}(\beta) = \begin{array}{cc} \begin{array}{c} q \end{array} & \\ \begin{bmatrix} 1 & & & & & \\ & \ddots & & \vdots & & \\ & & 1 & \cdots \beta & \cdots & \\ & & & \ddots \vdots & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} & \begin{array}{c} \\ \\ p \\ \\ \\ \\ \\ \end{array} \end{array} \qquad D_p(r) = \begin{array}{c} p \\ \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & r & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix} \end{array}$$

$$P_{pq} = \begin{array}{cc} \begin{array}{cc} p & q \end{array} & \\ \begin{bmatrix} 1 & & & & & & & \\ & \ddots & \vdots & & \vdots & & & \\ & & 1 & & & & & \\ & & & 0 & \cdots & 1 & \cdots & \\ & & & 1 & & & & \\ & & & \vdots & \ddots & \vdots & & \\ & & & & & 1 & & \\ & & & 1 & \cdots & 0 & \cdots & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{bmatrix} & \begin{array}{c} \\ \\ \\ p \\ \\ \\ \\ q \\ \\ \\ \\ \end{array} \end{array}$$

where β and r are in F and all the entries not indicated are 0. Any nonsingular matrix A in $L_n(F)$ is the product of elementary matrices in $L_n(F)$. Furthermore $\det[T_{pq}(\beta)] = 1$, $\det[D_p(r)] = r$ and $\det[P_{pq}] = -1$.

LEMMA 2. *Under the hypothesis of Theorem* 1

$$\phi[AD_p(r)] = \phi[AD_1(r)], \quad \phi[AP_{pq}] = \phi[AD_1(-1)], \quad \phi[AT_{pq}(\beta)] = \phi(A),$$

and in general $\phi(AE) = \phi[AD_1(\det E)]$ *for any elementary matrix* E *and for any matrix* A *in* $L_n(F)$.

Proof. Observe that

$$\phi(BC) = \phi(BIC) = \phi(CIB) = \phi(CB),$$
$$\phi(BCD) = \phi(DCB) = \phi(CBD) = \phi(BDC),$$

all B, C, D in $L_n(F)$. We first give a proof for $n=2$. For any A in $L_2(F)$ we have

$$\phi[AD_2(r)] = \phi\left[A\begin{pmatrix}1 & 0\\ 0 & r\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \phi\left[A\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & r\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]$$
$$= \phi\left[A\begin{pmatrix}r & 0\\ 0 & 1\end{pmatrix}\right] = \phi[AD_1(r)]$$

and

$$\phi[AP_{12}] = \phi\left[A\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\right] = \phi\left[A\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\right]$$
$$= \phi\left[A\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right] = \phi[AD_1(-1)].$$

If β is a nonzero element in F, then

$$\phi(A) = \phi\left[A\begin{pmatrix}\beta^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right]$$
$$= \phi\left[A\begin{pmatrix}\beta^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\right] = \phi\left[A\begin{pmatrix}1 & 0\\ \beta-1 & 1\end{pmatrix}\right].$$

And

$$\phi(A) = \phi\left[A\begin{pmatrix}1 & 0\\ \beta & 1\end{pmatrix}\right] \qquad \text{if } \beta \neq -1.$$

But

$$\phi(A) = \phi\left[A\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right] = \phi\left[A\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\right],$$

and consequently $\phi(A) = \phi[AT_{21}(\beta)]$ for all β in F.

Now suppose that $n>2$. It follows that $\phi[AT_{pq}(\beta)] = \phi(A)$ from the above argument where p replaces 2 and q replaces 1; the ppth, qqth, pqth and qpth entries in the matrices in this argument are given above and we set any other ijth entry equal to δ_{ij} in these matrices (excluding A) where δ denotes the Kronecker delta. Likewise $\phi[AD_p(r)] = \phi[AD_1(r)]$ and $\phi[AP_{pq}] = \phi[AD_1(-1)]$ and the proof is complete.

LEMMA 3. *Under the hypothesis of Theorem* 1 $\phi(A) = \phi[D_1(\det A)]$ *for each nonsingular* A *in* $L_n(F)$.

Proof. Let A be a nonsingular matrix in $L_n(F)$ and put $A=A_1A_2 \cdots A_m$ where each A_i is an elementary matrix. Then

$$\begin{aligned}\phi(A) &= \phi[A_1 \cdots A_m] = \phi[A_1 \cdots A_{m-1}D_1(\det A_m)] \\ &= \phi[D_1(\det A_m)A_1 \cdots A_{m-1}]\end{aligned}$$

by Lemma 2; $\phi(A)=\phi[D_1(\det A_{m-1})D_1(\det A_m)A_1 \cdots A_{m-2}]$ and repeated applications of this principle show that

$$\begin{aligned}\phi(A) &= \phi[D_1(\det A_1) \cdots D_1(\det A_m)] = \phi[D_1(\det A_1 \cdots A_m)] \\ &= \phi[D_1(\det A)].\end{aligned}$$

LEMMA 4. *Under the hypothesis of Theorem* 1 $\phi(A)=\phi[D_1(\det A)]$ *for all* A *in* $L_n(F)$.

Proof. It suffices to show that $\phi(A)=\phi[D_1(0)]$ for singular matrices A in $L_n(F)$. Let V be an n-dimensional vector space over F and select a linear basis of V so that each matrix in $L_n(F)$ can be identified (in the usual manner) with a linear operator on V. Let A be a singular matrix in $L_n(F)$. Then there is an $(n-1)$-dimensional subspace U of V which contains the subspace VA and there is a singular matrix T mapping each vector in U into itself; hence $A=AT$. For each index j, $1 \leqq j \leqq n$, there is a nonsingular S_j in $L_n(F)$ for which $S_jTS_j^{-1}=D_j(0)$. Then

$$\begin{aligned}\phi(A) &= \phi[AT] = \phi[ATS_1^{-1}S_1] = \phi[AS_1TS_1^{-1}] = \phi[S_1TS_1^{-1}A] \\ &= \phi[S_1TS_1^{-1}ATS_2^{-1}S_2] = \phi[S_1TS_1^{-1}AS_2TS_2^{-1}] \\ &= \phi[(S_2TS_2^{-1})(S_1TS_1^{-1})A]\end{aligned}$$

and repeated applications of this principle show that

$$\phi(A) = \phi[(S_nTS_n^{-1}) \cdots (S_1TS_1^{-1})A] = \phi(N),$$

where N denotes the zero matrix. Since $D_1(0)$ is singular we have $\phi(A)=\phi(N)=\phi[D_1(0)]$.

Proof of Theorem 1. We have shown that $\phi(A)=\phi[D_1(\det A)]$ for all A in $L_n(F)$. Define the mapping p of F into F as follows; for each a in F put $p(a)=\phi[D_1(a)]$. Plainly $\phi(A)=p(\det A)$ for each A in $L_n(F)$. Indeed p is uniquely determined by ϕ; for if p' were another such mapping of F into F, then for each matrix A in $L_n(F)$ for which $a=\det A$ we have

$$p'(a) = p'(\det A) = \phi(A) = p(\det A) = p(a).$$

Now suppose p is multiplicative on F; then ϕ must be multiplicative on $L_n(F)$ because

$$\phi(AB) = p(\det AB) = p[(\det A)(\det B)] = p(\det A)p(\det B) = \phi(A)\phi(B).$$

On the other hand if ϕ is multiplicative on $L_n(F)$ then p is multiplicative on F

because

$$\begin{aligned} p(ab) &= p[(\det A)(\det B)] = p(\det AB) = \phi(AB) \\ &= \phi(A)\phi(B) = p(\det A)p(\det B) = p(a)p(b) \end{aligned}$$

for any matrices A, B in $L_n(F)$ for which $a = \det A$, $b = \det B$. This concludes the proof of Theorem 1.

Note that if tr is the trace mapping on $L_n(F)$, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all A, B in $L_n(F)$. However for $n = 2$ we have

$$\operatorname{tr}\left[\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

$$\operatorname{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\right] = \operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1,$$

and tr is not submultiplicative on $L_2(F)$ (in fact, tr is not submultiplicative on $L_n(F)$ for $n \geqq 2$). There is no mapping p of F into F for which $\operatorname{tr}(A) = p(\det A)$ on $L_n(F)$. Consequently in Theorem 1 the hypothesis that ϕ is submultiplicative may not be replaced by $\phi(AB) = \phi(BA)$ for all A, B in $L_n(F)$.

COROLLARY 5. *If in Theorem* 1 *all the elements in F have n-th roots in F and if ψ is another submultiplicative mapping of $L_n(F)$ into F for which $\psi(aI) = \phi(aI)$ for every scalar a in F, then $\psi = \phi$ on $L_n(F)$; in particular if $\psi(aI) = a^n$ for every a in F, then ψ is the determinant on $L_n(F)$.*

Proof. For any a in F there is an element b in F for which $b^n = a$ and by Lemma 4,

$$\phi(bI) = \phi[D_1(b^n)] = \phi[D_1(a)]$$

and likewise $\psi(bI) = \psi[D_1(a)]$. But $\psi(bI) = \phi(bI)$ and $\psi[D_1(a)] = \phi[D_1(a)]$. Again by Lemma 4, $\phi(A) = \phi[D_1(\det A)] = \psi[D_1(\det A)] = \psi(A)$ for all A in $L_n(F)$.

COROLLARY 6. *If in Theorem* 1, *F is a topological field (the set $L_n(F)$ can be regarded as the cartesian product of n^2 copies of F) and $L_n(F)$ is endowed with the cartesian product topology, then ϕ is continuous on $L_n(F)$ iff p is continuous on F.*

Proof. Suppose p is continuous on F. Then $\phi(A) = p(\det A)$ is continuous on $L_n(F)$ because $A \to \det A$ is a continuous mapping of $L_n(F)$ into F. Suppose ϕ is continuous on $L_n(F)$. Then $p(a) = \phi[D_1(a)]$ is a continuous mapping of F into F because $a \to D_1(a)$ is a continuous mapping of F into $L_n(F)$.

THEOREM 7. *Let F, $L_n(F)$ and ϕ be as in Theorem* 1. *Then the following statements are equivalent*:

(1) *There is a polynomial $Q(x_{ij})$ in n^2 variables with coefficients in F for which $Q(a_{ij}) = \phi(a_{ij})$ for all (a_{ij}) in $L_n(F)$.*

(2) *There is a polynomial* $q(x_i)$ *in* n *variables with coefficients in* F *for which* $q(a_{ii}) = \phi(a_{ij})$ *for all diagonal* (a_{ij}) *in* $L_n(F)$.

(3) *There is a polynomial* $s(x)$ *in one variable with coefficients in* F *for which* $s[\det(a_{ij})] = \phi(a_{ij})$ *for all* (a_{ij}) *in* $L_n(F)$.

Proof. The implications (1)$\Rightarrow$(2) is trival. Assume (2). For all A in L_n (F) we have $\phi(A) = \phi[D_1(\det A)]$ by Lemma 4. But

$$\phi[D_1(\det A)] = q(\det A, 1, \cdots, 1)$$

and it suffices to define the polynomial s in (3) to be $s(x) = q(x, 1, \cdots, 1)$. This proves that (2)$\Rightarrow$(3). Now assume (3); there is a polynomial s in one variable for which $\phi(a_{ij}) = s[\det(a_{ij})]$. But $\det(x_{ij})$ is a polynomial in the n^2 variables x_{ij}. It suffices to define the polynomial Q in (1) to be $Q(x_{ij}) = s[\det(x_{ij})]$. Hence (3)$\Rightarrow$(1) and the proof is complete.

COROLLARY 8. *If* F *is an infinite field in Theorem* 7, *if* (1), (2), (3) *hold and if* ϕ *is nonconstant, then*

(i) $n \leqq$ *degree* q *and* $n \leqq$ *degree* Q.

(ii) ϕ *is multiplicative iff* $\phi(a_{ij}) = [\det(a_{ij})]^m$ *for some positive integer* m *and all* (a_{ij}) *in* $L_n(F)$.

Proof. Because F is infinite there is no nonzero polynomial with coefficients in F of which every element in F is a root. Consequently each polynomial Q, q and s given in Theorem 7 is unique. Now $\det(x_{ij})$ is a polynomial in the n^2 variables x_{ij} of degree n. By the proof of (3)$\Rightarrow$(1) in Theorem 7 it follows that $s[\det(x_{ij})] = Q(x_{ij})$ and $n \leqq$ degree Q. By a similar argument $q(a_{ii}) = s[\det(a_{ij})]$ for all diagonal (a_{ij}) in $L_n(F)$ and $n \leqq$ degree q (the details are left to the reader). This proves (i).

To prove (ii) observe that if $\phi(a_{ij}) = [\det(a_{ij})]^m$ then ϕ is obviously multiplicative on $L_n(F)$. Now assume ϕ is multiplicative on $L_n(F)$. Let s be the polynomial given in (3); it suffices only to show that $s(x)$ is of the form x^m for some $m > 0$.

Let m be the least positive integer for which x^m has a nonzero coefficient in $s(x)$. We have $s[(\det A)(\det A)] = [s(\det A)]^2$ for all A in $L_n(F)$, $s(t^2) = [s(t)]^2$ for all t in F and $[s(x)]^2 - s(x^2) = 0$. Consequently the constant term in $s(x)$ is 0; for otherwise the coefficient of x^m in $[s(x)]^2 - s(x^2)$ would be nonzero. And finally $s(x) = x^m$; for if m' were the least integer exceeding m for which $x^{m'}$ has a nonzero coefficient in $s(x)$, then the coefficient of $x^{m'+m}$ in $[s(x)]^2 - s(x^2)$ would be nonzero. This concludes the proof.

Let F be the real field and let n be an even integer. Then the negative elements in F have no nth roots in F. We would not expect Corollary 5 to be valid for this choice of F and n; indeed if ϕ is either det or $|\det|$ on $L_n(F)$ we have $\phi(aI) = a^n$ for all a in F. This leads us to

THEOREM 9. *If F is the real field, if n is a positive even integer and if ϕ is a multiplicative mapping of $L_n(F)$ into F such that $\phi(aI)=a^n$ for all a in F, then either $\phi(A)=\det A$ for all A in $L_n(F)$ or $\phi(A)=|\det A|$ for all A in $L_n(F)$.*

Proof. By Theorem 1 there is a multiplicative mapping p of F into F for which $p(\det A)=\phi(A)$ for all A in $L_n(F)$. In particular $p(a^n)=p(\det aI)=\phi(aI)=a^n$ for all scalars $a\geqq 0$, and $p(a)=a$ for all $a\geqq 0$. On the other hand $p(-1)p(-1)=p(1)=1$ and $p(-1)=\pm 1$. Now plainly if $p(-1)=-1$, then $p(a)=a$ for all scalars a and $\phi=\det$ on $L_n(F)$; if $p(-1)=1$, then $p(a)=|a|$ for all scalars a and $\phi=|\det|$ on $L_n(F)$. This concludes the proof.

THEOREM 10. *If F is the real field, if $L_n(F)$ has the product topology, if n is any positive integer, and if ϕ is a nonconstant, continuous, multiplicative mapping of $L_n(F)$ into F, then there is a positive real number r for which either $\phi(A)=|\det A|^r$, all A in $L_n(F)$ or $\phi(A)=s(A)|\det A|^r$, all A in $L_n(F)$ where $s(A)=1$ if $\det A\geqq 0$ and $s(A)=-1$ if $\det A<0$; in particular if $\phi(2I)=2^n$ then either $\phi=\det$ or $\phi=|\det|$ on $L_n(F)$.*

Proof. There is a nonconstant, continuous, multiplicative mapping p of F into F for which $p(\det A)=\phi(A)$ for all A in $L_n(F)$. Let c be a scalar for which $p(c)\neq 1$. Then $p(0)=p(0)p(c)$ and it follows that $p(0)=0$. On the other hand there is no nonzero scalar c for which $p(c)=0$; for if there were we would have $p(x)=p(c)p(c^{-1}x)=0$ for all scalars x. Furthermore p maps the positive real axis into itself because $p(a^{1/2})p(a^{1/2})=p(a)$ for all scalars $a>0$. Because p is continuous and multiplicative we conclude that there is a real number r for which $p(c)=c^r$ for all $c>0$. To prove this observe that there is a number r for which $p(2)=2^r$; then $p(2^i)=2^{ir}$, $p(2^{i/j})=2^{ir/j}=(2^{i/j})^r$ for integers i and j with $j\neq 0$, and $p(c)=c^r$ for all $c>0$ follows from the continuity of p. Because $p(0)=0$ it follows that $r>0$. Furthermore $p(-1)p(-1)=p(1)=1$ and $p(-1)=\pm 1$.

Now $\phi(A)=p(\det A)=p[s(A)|\det A|]=p[s(A)]|\det A|^r$ and the conclusion is evident.

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RECIPROCAL BASES FOR THE INTEGERS

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1. Introduction.

(1.1) DEFINITION. *A sequence* S: $\{n_1, n_2, \cdots\}$ *of distinct integers is called an R-basis if every positive integer is the sum of reciprocals of finitely many integers of S.*

It is well known that the set of all positive integers is an R-basis. In [**1**] H. S. Wilf suggested the following problems:

(1.2) Are the odd numbers an R-basis?

(1.3) Is every arithmetic progression an R-basis?

(1.4) Does an R-basis necessarily have positive density?

(1.5) If S contains all positive integers and $f(n)$ is the least number required to represent n what is the asymptotic behaviour of $f(n)$?

Problem (1.2) was proposed by E. P. Starke in this MONTHLY 59 (1952). A complete solution was given by R. Breusch [**2**], and by B. M. Stewart [**3**]. In solutions given by E. P. Churchill and B. M. Stewart the proof depended on the theorem: *For every odd A there are infinitely many odd multiples M of A each of which has the property: every integer n such that* $2<n<\sigma(M)-2$, *is a sum of distinct positive divisors of M.* In this paper we give an affirmative answer to (1.3) by generalizing the theorem cited above.

It is easily seen that the answer to (1.4) is negative. Many examples of bases of zero density can be found. Mr. Wilf informed us that this question was also answered by S. Scheinberg and by S. Stein. For the function f of (1.5) we prove

$$\log f(n) = n - \gamma + O\left(\frac{1}{n}\right).$$

Here γ is Euler's constant. Mr. Wilf informed us that $f(n) \sim e^{n-\gamma}$ was also found by P. Erdös.

In the following "integer" always means "positive integer."

2. R-bases of zero density. Obviously, for every integer a, the sequence $S_a = \{a, 2a, 3a, \cdots\}$ is an R-basis. Using this we now construct an R-basis of zero-density. Let $S(1)$ be the set $\{1\}$ and suppose $S(n)$ is a set of integers such that every integer $m \leqq n$ can be expressed as the sum of reciprocals of distinct elements of $S(n)$. Let α_n be the largest element of $S(n)$. We now take an integer $a(n)$ satisfying:

(1) $a(n) > \alpha_n \geqq n$,

(2) the set $S(n)$ has less than $a(n)/n$ elements.

There is a representation of $n+1$ by reciprocals of elements of $S_{a(n)}$. We add these elements to $S(n)$ to form the set $S(n+1)$. By induction we have thus

defined an R-basis $S=\bigcup_{n=1}^{\infty} S(n)$ and clearly the condition (2) implies that S has zero density.

3. Every arithmetic progression is an R-basis. For every integer a, the arithmetic progression S_a is an R-basis. This suggests that every arithmetic progression is an R-basis. To prove this we first prove two lemmas.

Consider the arithmetic progression $a_0, a_1, a_2, \cdots$, where

$$a_i = 1 + iv \qquad (i = 0, 1, 2, \cdots; v \geqq 2).$$

We define $P_n = a_1 a_2 \cdots a_n$. We shall call an integer d^* a *stardivisor* of m if d^* is a divisor of m and d^* is a product of different integers a_i. Let N be an integer satisfying the conditions

$$(3.1) \qquad \begin{aligned} &\text{(a)} \qquad N > v^2 \\ &\text{(b)} \quad 1 + Nv > (1 + 2v)(1 + 3v) \\ &\text{(c)} \quad \sum_{i=1}^{N} a_i^{-1} > 1 \\ &\text{(d)} \quad (1 + v)(1 + 2v) \cdots \{1 + (N - 1)v\} > 1 + Nv. \end{aligned}$$

Condition (d) is trivial. A consequence of conditions (c) and (d) is that the sum of the stardivisors of P_N, which are greater than a_N, is greater than P_N. We take $N' = v^2$. We prove first the

LEMMA (3.2). *Every integer m with $\frac{1}{2}v^3 \leqq m \leqq v^4$ can be written as the sum of $2v$ or less distinct terms a_i $(i \leqq N')$.*

Obviously the sum of k or $v+k$ terms a_j is congruent to k (mod v). So an integer $m \equiv k$ (mod v) can be written as the sum of k terms a_i $(i \leqq N')$ if and only if

$$\sum_{i=0}^{k-1} (1 + iv) \leqq m \leqq \sum_{i=N'-k+1}^{N'} (1 + iv),$$

and m can be written as the sum of $v+k$ terms a_i $(i \leqq N')$ if and only if

$$\sum_{i=0}^{v+k-1} (1 + iv) \leqq m \leqq \sum_{i=N'-v-k+1}^{N'} (1 + iv).$$

Since

$$\sum_{i=0}^{v+k-1} (1 + iv) = \tfrac{1}{2}(v + k)(v^2 + vk - v + 2) \leqq \tfrac{1}{2}k(2v^3 - kv + v + 2)$$
$$= \sum_{i=N'-k+1}^{N'} (1 + iv),$$

the two intervals have a nonempty intersection. Now, taking $k = 1, 2, \cdots, v$

we find a set of integers which can be represented as the sum of $2v$ or less distinct terms a_i $(i \leqq N')$. This set contains the interval $\frac{1}{2}v^3 \leqq m \leqq v^4$ which proves the lemma.

LEMMA (3.3). *If $v^4 < \sigma \leqq P_N$ then σ can be written as the sum of distinct stardivisors of P_N.*

Proof. We wish to show that by successively subtracting stardivisors from σ we can find a number which is the interval of Lemma (3.2) and then (3.3) would follow. In this process no stardivisor may be used twice: Let $d_1^* < d_2^* < \cdots < d_k^*$ be the stardivisors of P_N which are greater than $a_{N'}$. If $\alpha = \sum_{i=1}^k \epsilon_i 2^{i-1}$, $(\epsilon_i = 0$ or $1)$, we define $t(\alpha) = \sum_{i=1}^k \epsilon_i d_i^*$. So now we must show that there is a number α for which

$$\sigma - v^4 \leqq t(\alpha) \leqq \sigma - \tfrac{1}{2}v^3. \tag{1}$$

Now $t(1) = d_1^* = v^3 + v + 1 \leqq \sigma - \frac{1}{2}v^3$ and $t(2^k - 1) = d_1^* + d_2^* + \cdots + d_k^* > P_N > \sigma - \frac{1}{2}v^3$ (by 3.1). Let α be the largest integer for which $t(\alpha) \leqq \sigma - \frac{1}{2}v^3$. If $t(\alpha)$ does not satisfy (1) this would mean $t(\alpha) < \sigma - v^4$ and $t(\alpha+1) > \sigma - \frac{1}{2}v^3$. Hence there is a stardivisor d_j^* for which

$$d_j^* - \sum_{i=1}^{j-1} d_i^* > v^4 - \tfrac{1}{2}v^3 + 1. \tag{2}$$

We now show that (2) is not true and therefore there is a $t(\alpha)$ satisfying (1) which completes the proof of (3.3). As d_1^* and d_2^* are both less than $v^4 - \frac{1}{2}v^3 + 1$ the stardivisor d_j^* cannot be a stardivisor less than a_N because every a_i is less than the sum of a_{i-1} and a_{i-2}. This means that (2) implies $j > 2$ and that $d_j^* > a_N$. So d_j^* is the product of at least 2 different numbers a_i $(i > 0)$. Write $d_j^* = a_{\lambda_1} a_{\lambda_2} \cdots a_{\lambda_l}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_l$ $(l \geqq 2)$. We use:

(a) $a_{\lambda_1 - 1} + a_1 > a_{\lambda_1}$ if $\lambda_1 > 2$,

(b) $a_{\lambda_1} a_{\lambda_2 - 1} + a_1 a_2 > a_{\lambda_1} a_{\lambda_2}$ if $\lambda_1 \leqq 2$, $\lambda_2 > 4$,

(c) if $\lambda_1 \leqq 2$, $\lambda_2 \leqq 3$ then $a_{\lambda_1} a_{\lambda_2} \leqq (1+2v)(1+3v) < a_N$ by (3.1.b) and therefore $a_{\lambda_1} a_{\lambda_2} < (a_{\lambda_1} a_{\lambda_2} - v) + a_1 = a_\nu + a_1$. This shows that we can find two stardivisors, both less than d_j^*, whose sum is greater than d_j^* and this contradicts (2). We now prove the main

THEOREM (3.4). *Every arithmetic progression of positive integers is an R-basis.*

Proof. If $an = \sum_i (1 + \lambda_i v)^{-1}$ then $n = \sum_i (a + a\lambda_i v)^{-1}$ and therefore it is sufficient to prove the theorem for arithmetic progressions with first term $a_0 = 1$. If $a_0, a_1, \cdots$, where $a_i = a + iv$ is the arithmetic progression then $\sum_{i=0}^\infty a_i^{-1}$ is divergent. For every integer m we can find N_0 such that

$$\sum_{i=0}^{N_0} a_i^{-1} \leqq m < \sum_{i=0}^{N_0+1} a_i^{-1}. \tag{3}$$

If equality occurs the proof is complete. If not, then $m=\sum_{i=0}^{N_0} a^{-1}+\nu/P_{N_0}$. We write

$$\frac{\nu}{P_{N_0}}=\frac{\sigma}{P_N},$$

where $\nu^4<\sigma<P_N$ and N satisfies the conditions (3.1). By Lemma (3.3) the numerator σ is the sum of distinct stardivisors of P_N and hence σ/P_N is the sum of reciprocals of distinct terms of the arithmetic progression. Condition (3) guarantees that these terms have index greater than N_0. In this way m has been represented as the sum of reciprocals of distinct terms of the arithmetic progression.

4. Asymptotic behaviour of $f(n)$. We consider the R-basis S consisting of all positive integers. Let $f(n)$ be the least number of elements of S required to represent n as a sum of reciprocals. We prove the

THEOREM (4.1). $\log f(n)=n-\gamma+O(1/n)$.

Proof. Let k be the integer satisfying

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}<n<1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}. \tag{4}$$

Then

$$n=1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{a}{b}$$

in which $1\leqq a<b$ and b is the least common multiple of $1, 2, \cdots, k$. In [4] P. Erdös proved that a/b can be represented as the sum of $N(a, b)$ reciprocals of distinct integers in which

$$N(a,b)\leqq\frac{c\log b}{\log\log b}. \tag{5}$$

As

$$\log b=\sum_{p^\alpha\geqq k}\sum\log p=\psi(k)=k+O\left(\frac{k}{\log k}\right)$$

we find

$$N(a,b)=O\left(\frac{k}{\log k}\right). \tag{6}$$

We use the well-known theorem

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}=\log k+\gamma+O\left(\frac{1}{k}\right). \tag{7}$$

By (4) we have $f(n) > k \geqq n$. By (4) and (6) on the other hand: $f(n) < k + O(k/\log k)$. (4) and (7) lead to $\log k = n - \gamma + O(1/k)$. Combining these we have

$$\log f(n) = n - \gamma + O\left(\frac{1}{n}\right) + O\left(\frac{1}{\log k}\right) = n - \gamma + O\left(\frac{1}{n}\right).$$

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THE EVALUATION OF DETERMINANTS

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1. Introduction. The determinant of a $n \times n$-matrix $[m_{pq}]$ is a sum of $n!$ distinct products, each product containing as its factors n elements, chosen in such a way that only one element out of each row and out of each column is contained in any one product. Each product is given a positive or negative sign according as the total number of inversions occurring in the sequences of first and second suffixes of the elements forming the product is respectively even or odd.

$$D = |m_{pq}| = \sum_{i=0}^{n!-1} d_i,$$

in which

$$d_i = (-1)^k m_{1,a_1} m_{2,a_2} \cdots m_{n,a_n}.$$

Only the second suffixes $a_1, a_2, \cdots, a_n = \{a\}$ will be permuted. First suffixes will remain in their natural order $1, 2, \cdots, n = \{n\}$.

In this paper we show first that the consecutive integers i from 0 to $n!-1$ inclusive can each be uniquely associated with one of $n!$ possible different sequences $\{b\} = b_1, b_2, \cdots, b_n$. The sequence $\{b\}$ associated with the integer i will be referred to as the factorial representation of i (paragraph 2). Next, each of the $n!$ possible permutations $\{a\}$ can be uniquely associated with one of $n!$ possible sequences $\{c\}$, being inversions sequences (paragraph 3). Finally, $\{b\}$ and $\{c\}$ being interchangeable, the contents of the product d_i can be made to depend on the sequential number i of the product (paragraph 4).

2. Factorial representation. Any nonnegative integer i smaller than $n!$ can be uniquely represented as a sum of n products

$$(1) \qquad i = (n-1)!\,b_1 + (n-2)!\,b_2 + \cdots + 1!\,b_{n-1} + 0!\,b_n,$$

in which b_j is a nonnegative integer.

Demanding that each consecutive b_j be given the maximum numerical value compatible with (1) will ensure that

$$b_j \leqq n - j. \tag{2}$$

Hence $b_n \equiv 0$, and if $i < (n-j)!$ then $b_1 = b_2 = \cdots = b_j = 0$. Conditions (1) and (2) limit the number of all possible $\{b\}$ to $n!$. There is a unique relation between the numbers n and i on one side and the factorial representation $\{b\}$ on the other.

Numerical example

Let $n = 7$ and $i = 473$

$$\begin{array}{rlll} & i \div (n-1)! & \text{or} \quad 473 \div 720 & = 0 + 473/720 \\ \text{remainder} & \div (n-2)! & \text{or} \quad 473 \div 120 & = 3 + 113/120 \\ \text{remainder} & \div (n-3)! & \text{or} \quad 113 \div 24 & = 4 + 17/24 \\ & \cdots & \quad\quad\; 17 \div 6 & = 2 + 5/6 \\ & \cdots & \quad\quad\; 5 \div 2 & = 2 + 1/2 \\ & \cdots & \quad\quad\; 1 \div 1 & = 1 + 0 \\ \text{remainder} & \div (n-n)! & \text{or} \quad 0 \div 1 & = \underline{0\quad\;\;} \\ & & & k = 12 \end{array} \tag{3}$$

$\{b\} = 0, 3, 4, 2, 2, 1, 0$ is the factorial representation of $i = 473$. Now let $\{b\} = 7, 4, 5, 4, 1, 1, 0, 0$. Then $n = 8$ and i is uniquely determined:

$$i = 5040 \times 7 + 720 \times 4 + 120 \times 5 + 24 \times 4 + 6 \times 1 + 2 \times 1 + 1 \times 0 + 1 \times 0$$
$$= 38864.$$

3. Conversion of permutation sequence to inversions sequence and vice-versa. The transfer of a permutation sequence $\{a\}$ into its related inversions sequence $\{c\}$, and the reverse process, are best illustrated by numerical examples.

Let $\{a\} = 1, 5, 7, 4, 6, 3, 2$.

The leading integer 1 is compared with all its successors and no inversions (smaller integers) are found. Hence $\{c\} = 0, \cdots$.

The second integer of $\{a\}$ is 5. It is compared with all its successors and 3 inversions viz. 4, 3, and 2, are found; hence

$$\{c\} = 0, 3 \cdots.$$

Continuing we find

$$\{c\} = 0, 3, 4, 2, 2, 1, 0.$$

It is obvious that

$$c_j \leqq n - j \quad \text{and} \quad c_n \equiv 0. \tag{4}$$

Property (4) limits the number of all possible inversions sequences to $n!$. From each permutation sequence it is possible to obtain only one distinct inversions sequence.

The conversion from an inversions sequence to its associated permutation sequence is accomplished by increasing each member of $\{c\} = 0, 3, 4, 2, 2, 1, 0$ by one, thus forming the sequence 1, 4, 5, 3, 3, 2, 1. Consecutive numbers of this sequence are used as the addresses for exhaustive sampling from $\{n\} = 1, 2, 3, 4, 5, 6, 7$. Using the leading address 1, the first member in $\{n\}$ is crossed and placed in $\{a\}$: $\cancel{1}, 2, 3, 4, 5, 6, 7$, whence

$$\{a\} = 1, \cdots.$$

The next address is 4, so the fourth of the remaining numbers in $\{n\}$ is crossed out: $\cancel{1}, 2, 3, 4, \cancel{5}, 6, 7$, whence

$$\{a\} = 1, 5, \cdots.$$

The following address is 5, whence $\{a\} = 1, 5, 7, \cdots$, and so on, until

$$\{a\} = 1, 5, 7, 4, 6, 3, 2.$$

It is to be noted that the conversion from inversions sequence to permutation sequence is unique. Moreover, converting one sequence into another and then reversing the conversion recovers the original sequence. There are in all $n!$ possible $\{a\}$ and $n!$ possible $\{c\}$. This is sufficient to ensure that the relation between $\{a\}$ and $\{c\}$ is unique.

4. Interchangeability of factorial representation and conversions sequence. The properties (2) and (4) make the ensemble of all possible $n!$ sequences $\{b\}$ identical to the ensemble of all $n!$ possible sequences $\{c\}$. That means that each $\{b\}$ can be replaced with a numerically identical $\{c\}$ and vice versa. The only distinguishable property of either $\{b\}$ or $\{c\}$ is their numerical content. Because of the pairwise identical numerical content the notation $\{c\}$ become superfluous, and $\{b\}$ will henceforward be referred to as either factorial representation or inversions sequence.

Techniques described in this paper provide the facility to establish rapidly the contents of any d_i. Let the 474th product of the determinant of the 7×7 matrix $[m_{pq}]$ be required. Here $i=473$ and

$$\{b\} = 0, 3, 4, 2, 2, 1, 0,$$

whence

$$\{a\} = 1, 5, 7, 4, 6, 3, 2$$

and

$$d_{473} = (-1)^{12} m_{1,1} m_{2,5} m_{3,7} m_{4,4} m_{5,6} m_{6,3} m_{7,2}.$$

Here $k = 12$ = total number of inversions as obtained in (3).

Inversely, we would recognize

$$d_i = (-1)^k m_{1,8} m_{2,5} m_{3,7} m_{4,6} m_{5,2} m_{6,3} m_{7,1} m_{8,4}$$

as the 38865th product of an 8th order determinant, because

$$\{a\} = 8, 5, 7, 6, 2, 3, 1, 4,$$
$$\{b\} = 7, 4, 5, 4, 1, 1, 0, 0,$$

whence $k=22$ and $i=38864$.

5. Applications. Either $\{a\}$ or $\{b\}$ can be generated on digital computers. If a matrix is small, or if a large matrix is rich in zeros, its determinant can be calculated by using the principles explained in this paper. The facility which permits permutations to be arranged in a specific numerical order may also be of some use in certain other fields of application.

A COMBINATORIAL PROBLEM

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1. Introduction. A circular wheel is known to have t distinct frames around its circumference. The wheel is housed in such a way that when the wheel is stopped, it is possible to observe two distinct frames which are p positions apart on the circumference. It will be supposed that p and t are relatively prime.

The wheel is caused to rotate and then stopped until n distinct observations have been recorded, $n<t$. Thus the relative positions of n pairs of frames are known.

If some of the pairs can be chained up, however, additional information is available as to the nature of the wheel. For example, if A and B are p positions apart, and B and C are p positions apart, there is a chain A--B--C and A and C are now known to be $2p$ positions apart. Chains of greater length supply additional data as to the relative separations of frames.

The fact that the chain A--B--C supplies a separation number of $2p$ for A and C can be restated by saying that an extra separation number is made available. It is the purpose of this paper to study X, the number of extra separation numbers when n pairs are observed, and the expected value and variance of X.

Since p and t are relatively prime, there is no loss in generality in renumbering the frames so that the separation between the two frames visible when the wheel is stopped is one unit. Furthermore, all separations should be measured in one direction and all arithmetic should be taken modulo t.

It might simplify the description if, instead of talking about the pair of frames visible when the wheel is stopped, one talks about the connecting chord between the two frames. The chord between the frames numbered i and $(i+1)$ could be called the ith chord for convenience.

If $n=t$, then all frames can be chained up into a "circle." If $n<t$ there may be isolated single chords, some chains of two chords, some chains of three chords, etc.

2. Chains. Define an r-chain as a set of r consecutive chords around the circumference of the wheel. Define a true r-chain as one which is exactly r chords long. Define an apparent r-chain as one which is at least r chords long but may be longer. For example, the 4 chain A--B--C--D--E (where it is assumed that the chain cannot be extended at either end) is a true 4 chain, and it contains two 3 chains, namely A--B--C--D and B--C--D--E as well as three 2 chains and four 1 chains.

Let A_r be the actual number of apparent r-chains, and let α_r be the expected value of A_r. Let B_r be the actual number of true r chains. Then, from elementary considerations,

$$A_r = B_r + 2B_{r+1} + 3B_{r+2} + \cdots . \tag{1}$$

The solution of equation (1) for B_r is easily accomplished by the use of the symbolic operator Δ. Indeed,

$$B_r = \Delta^2 A_r. \tag{2}$$

3. Extra separations. A true r-chain contributes $\binom{r}{2}$ extra separation numbers. Therefore, the total number, X, of extra separation numbers is given by

$$\begin{aligned} X &= \sum_{r=2}^{\infty} \binom{r}{2} B_r \\ &= \sum_{r=2}^{\infty} \binom{r}{2} \Delta^2 A_r. \end{aligned} \tag{3}$$

In actual calculation the summations to infinity do not exist, since $n<t$. It will be shown in Section 4 that if m is an even integer then

$$\sum_r \binom{r}{m} \Delta^m A_r = A_m + A_{m+1} + A_{m+2} + \cdots . \tag{4}$$

For the special case $m=2$, equation (4) becomes

$$X = \sum_r \binom{r}{2} \Delta^2 A_r = A_2 + A_3 + A_4 + \cdots . \tag{5}$$

As before the summations terminate when $r=n<t$.

4. Proof of relation (4). Using the symbolic operator E, $E=1+\Delta$,

$$
\begin{aligned}
\sum_r \binom{r}{m} \Delta^m A_r &= \sum_r \binom{r}{m} (E-1)^m A_r \\
&= \sum_r \binom{r}{m} (E-1)^m E^r A_C \\
&= (-1)^m (1-E)^m \cdot E^m \sum_r \binom{r}{m} E^{r-m} A_0 \\
&= (-1)^m (1-E)^m \cdot E^m (1-E)^{-(m+1)} A_0 \\
&= \frac{(-E)^m}{1-E} A_0 = (-E)^m (1 + E + E^2 + \cdots) A_0 \\
&= (-1)^m [A_m + A_{m+1} + \cdots].
\end{aligned} \tag{6}
$$

For $m=2$ this reduces to equation (5).

5. Expected value of X. Given equation (5) and the definitions of α_r, a general probability theorem states that

$$
\begin{aligned}
E(X) &= E(A_2 + A_3 + A_4 + \cdots) \\
&= E(A_2) + E(A_3) + E(A_4) + \cdots \\
&= \alpha_2 + \alpha_3 + \alpha_4 + \cdots .
\end{aligned} \tag{7}
$$

To calculate α_r the following device may be used [1]. Introduce t random variables, $y_1, y_2, \cdots, y_t$ to be determined as follows: $y_i=0$ if any chord in the set $i, i+1, \cdots, i+r-1$ is missing, and $y_i=1$ if all these chords are present. Then

$$A_r = y_1 + y_2 + \cdots + y_t, \tag{8}$$

$$E(A_r) = \sum_{i=1}^{t} E(y_i) = tE(y_1), \tag{9}$$

because of the circular nature of the problem. $E(y_1)$ can be calculated from elementary considerations: n chords are distributed among t positions and one is looking for a sequence of r successive chords beginning with the first position.

$$
\begin{aligned}
E(y_1) &= \frac{n(n-1)\cdots(n-r+1)}{t(t-1)\cdots(t-r+1)} \\
&= \frac{n^{(r)}}{t^{(r)}} \quad \text{with} \quad n < t.
\end{aligned} \tag{10}
$$

Thus,

$$\alpha_r = E(A_r) = tE(y_1) = \frac{n^{(r)}}{(t-1)^{(n-1)}}. \tag{11}$$

The expected value of X then follows from equation (7),

$$E(X) = \frac{n^{(2)}}{(t-1)} + \frac{n^{(3)}}{(t-1)^{(2)}} + \cdots + \frac{n!}{(t-1)^{(n-1)}}. \tag{12}$$

6. Variance of X. Similar but more involved calculations can be used to evaluate the variance of X. For an individual term A_r, one has

$$\operatorname{Var}(A_r) = E(A_r^2) - [E(A_r)]^2. \tag{13}$$

Since

$$\begin{aligned} A_r^2 &= (y_1 + y_2 + \cdots + y_t)^2 \\ &= \sum_{i=1}^{t} y_i^2 + \sum_{i \neq j} y_i y_j \end{aligned} \tag{14}$$

one readily obtains

$$\begin{aligned} E(A_r^2) &= \sum_{i=1}^{t} E(y_i^2) + \sum_{i \neq j} E(y_i y_j) \\ &= tE(y_1^2) + t \sum_{k=2}^{t} E(y_1 y_k) \end{aligned} \tag{15}$$

because of the circular nature of the problem. But

$$y_1^2 = y_1 \tag{16}$$

since y_1 is either 0 or 1 and

$$E(y_1^2) = E(y_1) = \frac{n^{(r)}}{t^{(r)}}. \tag{17}$$

Likewise

$$E(y_1 y_2) = \frac{n^{(r+1)}}{t^{(r+1)}}, \tag{18}$$

since there must be an apparent r chain beginning with chord 1 and another apparent r chain beginning with chord 2, or in effect an $(r+1)$ chain beginning with chord 1 for this situation to be realized. The evaluations

$$E(y_1 y_k), \qquad k = 3, \cdots, t \tag{19}$$

require restrictive conditions on r, k, n, and t, but Var (A_r) can be calculated from (13), (15), (18) and (19).

From (5), and from the general theory of the variance of a linear sum, it is seen that both variance (A_r) and covariance $(A_i A_j)$ are needed to compute variance (X). An extension of the device used in Section 5 can be employed to

calculate the covariances. Again, restrictive conditions on r, i, j, n, and t are required in the tabulation of the forms of the covariance.

The final form for variance (X) does not lend itself to convenient display in equation form and will not be included here.

The device used here was suggested to the author by Professor A. M. Gleason of Harvard University and replaces a mathematically clumsy argument previously used by the author. Previously the same device had been used by W. Feller, *An Introduction to Probability Theory and its Applications*, first edition, 1950, to simplify calculations.

MATHEMATICAL NOTES

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SOME REMARKS ON INVERTIBLE SPACES

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In a recent paper [1], Doyle and Hocking defined invertible space as follows:

DEFINITION. *A topological space (X, τ) is invertible if and only if for every nonempty open subset O, there exists a homeomorphism $h: X \to X$ such that $h(\mathcal{C}O) \subset O$, $\mathcal{C}$ denoting the complement operator. The homeomorphism h will be called an inverting homeomorphism for O.*

In [1], the authors prove several theorems of the following type: Let (X, τ) be invertible and O a nonempty open subset which as a subspace has property Q. Then (X, τ) has property Q.

It is the intent of this note to show that if Q is separability, satisfying the first axiom of countability, satisfying the second axiom of countability, regularity, or normality, then the corresponding theorem is valid.

THEOREM 1. *Let (X, τ) be an invertible space and O a nonempty open subset which as a subspace of X is separable. Then (X, τ) is separable.*

Proof. Let A be a countable dense subset of O and h an inverting homeomorphism for O. Then $A^* = A \cup h^{-1}(A)$ is countable and we now show that A^* is dense in X. Let $O^* \neq \varnothing$ be open in X. *Case* 1: $O \cap O^* \neq \varnothing$. Then $O \cap O^*$ is open in O and thus $O \cap O^* \cap A \neq \varnothing$. Then $O^* \cap A^* \supset O \cap O^* \cap A \neq \varnothing$.

Case 2: $O \cap O^* = \varnothing$. Then $O^* \subset \mathcal{C}O$ and hence $h(O^*) \subset O$ and $h(O^*)$ is open in X. Then $h(O^*)$ is open in O and since A is dense in O, $h(O^*) \cap A \neq \varnothing$. Thus $O^* \cap A^* \supset O^* \cap h^{-1}(A) = h^{-1}h(O^*) \cap h^{-1}(A) = h^{-1}(h(O^*) \cap A) \neq \varnothing$.

THEOREM 2. *Let (X, τ) be an invertible space and O^* a nonempty open subset which as a subspace satisfies the first axiom of countability. Then (X, τ) satisfies the first axiom of countability.*

Proof. Let $x \in X$. *Case* 1: $x \in O^*$. Let $\{O_i^*\}$ be a countable open base for x in O^*. Now let $x \in O$ where O is open in X. Then $x \in O \cap O^*$ which is open in O^* and thus $x \in O_i^* \subset O \cap O^* \subset O$ for some integer i. But $\{O_i^*\}$ is also a family of open sets in X and thus X has a countable open base at x.

Case 2: $x \in \mathcal{C}O^*$. Let h be an inverting homeomorphism for O^*. Then $h(x) \in O^*$. Let $\{O_i^*\}$ be a countable open base for $h(x)$ in O^*. Then $\{O_i^*\}$ is a family of open sets in X and hence so is $\{h^{-1}(O_i^*)\}$. Let $x \in O \in \tau$. Then $h(x) \in h(O) \cap O^*$ which is open in O^* and hence $h(x) \in O_i^* \subset h(O) \cap O^* \subset h(O)$ for some i. Then $x \in h^{-1}(O_i^*) \subset h^{-1}h(O) = O$. Thus $\{h^{-1}(O_i^*)\}$ is a countable open base for x in X.

THEOREM 3. *Let (X, τ) be invertible and O^* a nonempty open subset of X which as a subspace satisfies the second axiom of countability. Then (X, τ) satisfies the second axiom of countability.*

Proof. Let $\{O_i^*\}$ be a countable open base for O^*. Then $\{O_i^*\} \cup \{h^{-1}(O_i^*)\}$ is a countable family of open sets in X, h being an inverting homeomorphism for O^*. Let then $x \in O \in \tau$.

Case 1: $x \in O^*$. Then $x \in O \cap O^*$, an open set in O^* and thus $x \in O_i^* \subset O \cap O^* \subset O$ for some i.

Case 2: $x \in \mathcal{C}O^*$. Then $h(x) \in O^* \cap h(O)$, an open set in O^*. Hence $h(x) \in O_i^* \subset O^* \cap h(O) \subset h(O)$ for some i and thus $x \in h^{-1}(O_i^*) \subset h^{-1}h(O) = O$. Thus in either case $\{O_i^*\} \cup \{h^{-1}(O_i^*)\}$ is a countable open base for x in X.

THEOREM 4. *Let (X, τ) be an invertible space and O^* a nonempty open subset which as a subspace is regular. Then (X, τ) is regular.*

Proof. Let $x \notin F$, F being closed in X. *Case* 1: $x \notin O^*$ and $O^* - F \neq \emptyset$. Then $O^* - F$ is open in X. Let h be an inverting homeomorphism for $O^* - F$. Then $h(x) \in O^* - F$ and $h(F) \subset O^* - F \subset O^*$ and $h(F)$ is closed in O^*. Since O^* is regular, there exist O_1^*, O_2^* open in O^* such that $h(x) \in O_1^*$, $h(F) \subset O_2^*$ and $O_1^* \cap O_2^* = \emptyset$. Then $x \in h^{-1}(O_1^*)$, $F \subset h^{-1}(O_2^*)$ and $h^{-1}(O_1^*)$ and $h^{-1}(O_2^*)$ are disjoint and open in X.

Case 2: $x \notin O^*$ and $O^* - F = \emptyset (O^* \subset F)$. Let h be an inverting homeomorphism for O^*. Then $h(x) \in O^*$ and $h(F - O^*) \subset O^*$ and $h(F - O^*)$ is closed in O^* and $h(x) \notin h(F - O^*)$. Since O^* is regular, there exist O_1^*, O_2^* open in O^* and disjoint such that $h(x) \in O_1^*$, $h(F - O^*) \subset O_2^*$. Then $x \in h^{-1}(O_1^*)$ and $F - O^* \subset h^{-1}(O_2^*)$. But $F = (F - O^*) \cup O^*$ and $x \in h^{-1}(O_1^*) \cap \mathcal{C}F$ and $F \subset h^{-1}(O_2^*) \cup O^*$. Finally, $h^{-1}(O_1^*) \cap \mathcal{C}F \cap \{h^{-1}(O_2^*) \cup O^*\} = \emptyset$.

Case 3: $x \in O^*$. Let h be an inverting homeomorphism for O^* and suppose $h(x) \in \mathcal{C}O^*$. Then $h(x) \in \mathcal{C}O^*$ and $h(F)$ is closed in X and $h(x) \notin h(F)$. By Case 1 and Case 2, there are sets O_1^* and O_2^* open in X such that $h(x) \in O_1^*$ and $h(F) \subset O_2^*$, $O_1^* \cap O_2^*$ being empty. Then $x \in h^{-1}(O_1^*)$ and $F \subset h^{-1}(O_2^*)$.

Case 4: $x \in O^*$ and h is an inverting homeomorphism for O^* for which $h(x) \in O^*$. There exist sets O_1^* and O_2^* open in O^* such that $x \in O_1^*$ and $F \cap O^* \subset O_2^*$ and $O_1^* \cap O_2^* = \emptyset$. But $h(F-O^*) \subset O^*$ and $h(x) \in O^*$ and thus there exist sets O_3^* and O_4^* open in O^* such that $h(x) \in O_3^*$, $h(F-O^*) \subset O_4^*$ and $O_3^* \cap O_4^* = \emptyset$. Then $x \in h^{-1}(O_3^*)$, $F-O^* \subset h^{-1}(O_4^*)$ and thus $x \in O_1^* \cap h^{-1}(O_3^*)$ and $F=(F\cap O^*) \cup (F-O^*) \subset O_2^* \cup h^{-1}(O_4^*)$. It is clear that $\{O_1^* \cap h^{-1}(O_3^*)\} \cap \{O_2^* \cup h^{-1}(O_4^*)\} = \emptyset$.

THEOREM 5. *Let (X, τ) be an invertible space and O^* a nonempty open subset which as a subspace is normal. Then (X, τ) is normal.*

Proof. Let $F_1 \cap F_2 = \emptyset$, F_i being closed in X. *Case* 1: $O^* - (F_1 \cup F_2) \neq \emptyset$. Let h be an inverting homeomorphism for $O^* - (F_1 \cup F_2)$. Then $h(F_i) \subset O^* - (F_1 \cup F_2) \subset O^*$, and are disjoint for $i=1, 2$. Then there exist sets O_1^*, O_2^* open in O^* and hence in X such that $h(F_1) \subset O_1^*$ and $h(F_2) \subset O_2^*$. Then $F_1 \subset h^{-1}(O_1^*)$ and $F_2 \subset h^{-1}(O_2^*)$.

Case 2: Suppose $O^* \subset F_1 \cup F_2$. Let h be an inverting homeomorphism for O^*. Then $O^* = (O^* \cap F_1) \cup (O^* \cap F_2)$ and hence $O^* \cap F_1$ is closed in O^* and so is $O^* \cap F_2$. Since they are disjoint, $O^* \cap F_i$ is also open in O^* and also in X for $i=1, 2$. But $F_1 \cap \mathcal{C}O^*$ and $F_2 \cap \mathcal{C}O^*$ are closed in X and $h(F_1 \cap \mathcal{C}O^*) \cup h(F_2 \cap \mathcal{C}O^*) \subset O^*$. There exist then O_1^* and O_2^* open in O^* and hence in X such that $O_1^* \cap O_2^* = \emptyset$ and $h(F_1 \cap \mathcal{C}O^*) \subset O_1^*$ and $h(F_2 \cap \mathcal{C}O^*) \subset O_2^*$. Then $F_1 \cap \mathcal{C}O^* \subset h^{-1}(O_1^*)$ and $F_2 \cap \mathcal{C}O^* \subset h^{-1}(O_2^*)$. Thus $F_1 = (F_1 \cap O^*) \cup (F_1 \cap \mathcal{C}O^*) \subset \{(F_1 \cap O^*) \cup h^{-1}(O_1^*)\} - F_2$ and $F_2 = (F_2 \cap O^*) \cup (F_2 \cap \mathcal{C}O^*) \subset \{(F_2 \cap O^*) \cup h^{-1}(O_2^*)\} - F_1$. It is clear that $\{(F_1 \cap O^*) \cup h^{-1}(O_1^*)\} - F_2$ and $\{(F_2 \cap O^*) \cup h^{-1}(O_2^*)\} - F_1$ are open and disjoint.

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REMARKS ON A FORMULA FOR PREFERENTIAL ARRANGEMENTS

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In [1, Proposition XXII, p. 87], Whitworth derives the following result: *The number of ways, $Q(n, r)$, in which n different things can be distributed into r different parcels (without blank lots) is given by*

$$Q(n, r) = [D^n(e^x - 1)^r]_{x=0}, \qquad D^n \equiv d^n/dx^n. \tag{1}$$

If f_n denotes the total number of preferential arrangements, then $f_n = \sum_{r=1}^{n} Q(n, r)$.

Recently, Gross [2] has shown that

$$f_n = [D^n(2 - e^x)^{-1}]_{x=0} = \frac{1}{2}\sum_{k=0}^{\infty} k^n 2^{-k} \qquad (n = 0, 1, \cdots), \tag{2}$$

$$f_n \sim n!/(2 \ln^{n+1} 2) \quad \text{as} \quad n \to \infty. \tag{3}$$

The purpose of this note is to give generalizations of (2) and (3), as well as additional results involving the numbers f_n.

The author [3] has shown that

$$(4)\quad (1-x)^{n+1}\sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{n} A_{nk}x^k \qquad (|x|<1,\ n=0,1,\cdots),$$

$$(5)\quad \sum_{k=0}^{\infty} k^n x^k = \left[\frac{d^n}{dr^n}\left(\frac{1}{1-xe^r}\right)\right]_{r=0} \qquad (|x|<1,\ n=0,1,\cdots),$$

$$(6)\quad \begin{aligned}\sum_{k=0}^{\infty} k^n x^k &= -\sum_{k=0}^{\infty}\frac{(k-1)\cdots(k-n)B_k \ln^{k-n-1} x}{k!}\\ &= \frac{n!}{\ln^{n+1}(1/x)} - \sum_{k=n+1}^{\infty}\frac{(k-1)\cdots(k-n)B_k\ln^{k-n-1}x}{k!} \qquad (e^{-2\pi}<x<1).\end{aligned}$$

Here the B_k are Bernoulli numbers, and the A_{nk} are Eulerian numbers (see [4], [5]) defined by

$$(7)\quad A_{nk} = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n \qquad (k=0,1,\cdots,n),$$

where $A_{n0}=0$ for $n\neq 0$, $A_{nn}=1$, $n=0,1,\cdots$, and $\sum_{k=0}^{n} A_{nk}=n!$. We note that for $x=\frac{1}{2}$, (5) yields (2), and since $A_{ns}=A_{n,n-s+1}$ (see [4, p. 250, (2.13)]), (4) yields

$$(8)\quad f_n = \sum_{k=0}^{n} A_{nk}2^{n-k} = \sum_{s=1}^{n} A_{ns}2^{s-1} \qquad (n=1,2,\cdots).$$

The following identity,

$$(9)\quad f_{n+1} = 2(n+1)f_n - \sum_{s=1}^{n} sA_{ns}2^{s-1} \qquad (n=1,2,\cdots),$$

is obtained by differentiating both sides of (4) with respect to x, setting $x=\frac{1}{2}$, and using (8).

Since (see [6, p. 168])

$$(10)\quad [D^n(e^x-1)^r]_{x=0} = \sum_{j=0}^{r}(-1)^{r+j}\binom{r}{j}j^n = r!\mathfrak{S}_n^r,$$

where $\mathfrak{S}_n^r$ is a Stirling number of the second kind, we have

$$(11)\quad f_n = \sum_{r=1}^{n} r!\mathfrak{S}_n^r.$$

In [6, p. 176], it is shown that

$$(12)\quad r!\mathfrak{S}_n^r = n!\sum_{s_1+s_2+\cdots+s_r=n} 1/(s_1!s_2!\cdots s_r!),$$

where the s_i are positive integers, with repetition and permutation allowed. Thus, (11) can also be written as

$$f_n = n! \sum_{r=1}^{n} \sum_{s_1+s_2+\cdots+s_r=n} 1/(s_1!s_2! \cdots s_r!). \tag{13}$$

The following result, which is a generalization of (2) and (3), is motivated by (6).

THEOREM. *If* $p>1$ *then*

$$p^{-1} \sum_{k=0}^{\infty} k^n p^{-k} \sim n!/(p \ln^{n+1} p) \quad as \quad n \to \infty. \tag{14}$$

Proof. Let

$$\phi(z) = (p - e^z)^{-1} - [p\{\ln (p) - z\}]^{-1}, \qquad \phi^{(n)}(0) = a_n - b_n, \tag{15}$$

where

$$a_n = [D^n(p - e^z)^{-1}]_{z=0} = p^{-1} \sum_{k=0}^{\infty} k^n p^{-k}, \qquad \text{(see (5))}, \tag{16}$$

$$b_n = p^{-1}[D^n\{\ln (p) - z\}^{-1}]_{z=0} = p^{-1}(n!)/\ln^{n+1} p. \tag{17}$$

It is readily verified that $\lim_{z\to\ln p} \phi(z) = (2p)^{-1}$ and that $\phi(z)$ has a simple pole at $z=\ln(p) \pm 2\pi i$. Since $\phi(z)$ is regular within any circle about the origin of radius less than $[\ln^2(p)+4\pi^2]^{\frac{1}{2}}$, we choose the circle $|z| = R$, where $\ln(p) < R < [\ln^2(p)+4\pi^2]^{\frac{1}{2}}$, and let $M = \max |\phi(z)|$ for $|z| = R$. Applying Cauchy's inequality, we have $|a_n - b_n| \leqq M(n!)/R^n$; dividing by $|b_n|$, we obtain

$$|(a_n/b_n) - 1| \leqq pM(\ln p)[(\ln p)/R]^n \to 0 \quad \text{as} \quad n \to \infty, \tag{18}$$

the desired conclusion.

Recalling Stirling's formula for $n!$, we see that (14), with $p=e$, yields

$$\sum_{k=0}^{\infty} k^n e^{-k} \sim n! \sim n^n e^{-n}(2\pi n)^{1/2} \quad \text{as} \quad n \to \infty. \tag{19}$$

If we divide both sides of (6) by $n!/\ln^{n+1}(1/x)$ and then apply (14), we obtain

$$\lim_{n\to\infty} \left[\frac{(-1)^n}{n!} \sum_{k=n+1}^{\infty} \frac{(k-1)\cdots(k-n)B_k \ln^k x}{k!} \right] = 0 \qquad (e^{-2\pi} < x < 1). \tag{20}$$

Since $(p-e^x)^{-1}$ may be written as the sum of an even and an odd function of x, we obtain, noting (16),

$$\frac{p - \cosh x}{p^2 + 1 - 2p \cosh x} = \sum_{n=0}^{\infty} a_{2n} x^{2n}/(2n)! \qquad (|x| < \ln p) \tag{21}$$

$$\text{(22)}\qquad \frac{\sinh x}{p^2+1-2p\cosh x}=\sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}/(2n+1)! \qquad (|x|<\ln p).$$

Recalling that $\cosh(ix)=\cos x$, $\sinh(ix)=i\sin x$, we see that (21) and (22) simplify, respectively, to

$$\text{(23)}\qquad \frac{p-\cos x}{p^2+1-2p\cos x}=\sum_{n=0}^{\infty}(-1)^n a_{2n}x^{2n}/(2n)! \qquad (|x|<\ln p)$$

$$\text{(24)}\qquad \frac{\sin x}{p^2+1-2p\cos x}=\sum_{n=0}^{\infty}(-1)^n a_{2n+1}x^{2n+1}/(2n+1)! \quad (|x|<\ln p).$$

If $p=2$, then $a_n\equiv f_n$ (see (2), (16)).

Recursion formulas for the a_n may be obtained from its generating function. Since $(p-e^x)^{-1}=\sum_{n=0}^{\infty}a_nx^n/n!$, we obtain, upon differentiation,

$$\text{(25)}\qquad D(p-e^x)^{-1}=e^x(p-e^x)^{-2}=p(p-e^x)^{-2}-(p-e^x)^{-1}.$$

Equating coefficients of x^n in the series representations in (25) yields

$$\text{(26)}\qquad a_{n+1}=p\sum_{j=1}^{n-1}\binom{n}{j}a_ja_{n-j}-[(p+1)/(p-1)]a_n \qquad (n=2,3,\cdots).$$

It was noted in [2] that the f_n bear a resemblance to the Bernoulli numbers B_n. Noting that

$$\text{(27)}\qquad (p-e^x)^{-1}=\left(\frac{x}{e^x-1}\right)\left[-x+\frac{(p-1)x}{e^x-1}\right]^{-1},$$

where, in symbolic notation,

$$[x/(e^x-1)]=e^{Bx},\ B^n\equiv B_n, \qquad (p-e^x)^{-1}=e^{ax},\ a^n\equiv a_n,$$

we observe that $(p-1)e^{(B+a)x}-xe^{ax}=e^{Bx}$, and thus we obtain the following recursion identity:

$$\text{(28)}\qquad (p-1)(B+a)^n-na_{n-1}=B_n \qquad (n=2,3,\cdots).$$

Interesting relationships may be obtained from (5). As an example, let

$$\text{(29)}\qquad (4-e^x)^{-1}=\sum_{n=0}^{\infty}u_nx^n/n!, \qquad (2+e^x)^{-1}=\sum_{n=0}^{\infty}v_nx^n/n!.$$

Then

$$\text{(30)}\qquad u_n=4^{-1}\sum_{k=0}^{\infty}k^n4^{-k}, \qquad v_n=2^{-1}\sum_{k=0}^{\infty}(-1)^kk^n2^{-k}.$$

Since

$$\text{(31)}\qquad \sum_{n=0}^{\infty}2^nu_nx^n/n!=(4-e^{2x})^{-1}=(2-e^x)^{-1}(2+e^x)^{-1},$$

it follows from (31), upon equating coefficients of x^n, that

$$2^n u_n = \sum_{k=0}^{n} \binom{n}{k} f_k v_{n-k}. \tag{32}$$

From (4), we note that

$$3^{n+1} u_n = \sum_{k=0}^{n} A_{nk} 2^{2n-2k}, \qquad 3^{n+1} v_n = \sum_{k=0}^{n} (-1)^k A_{nk} 2^{n-k}. \tag{33}$$

Remarks. Formulas (34), (35), and (36) cited below are due to Euler (see [3], [4, p. 250, (2.10)]):

$$\frac{1-x}{e^u - x} = \sum_{n=0}^{\infty} H_n(x) u^n/n! \qquad (x \neq 1), \tag{34}$$

$$R_n(x) = (x-1)^n H_n(x), \qquad xR_n(x) = \sum_{s=1}^{n} A_{ns} x^s, \tag{35}$$

$$H_n(x) = \sum_{r=0}^{n} (x-1)^{-r} \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} s^n, \qquad H_0(x) = 1. \tag{36}$$

We note that for $x = 2$,

$$R_n(2) = H_n(2) = \sum_{r=1}^{n} r! \mathfrak{S}_n^r = f_n, \qquad f_n = \sum_{s=1}^{n} A_{ns} 2^{s-1}. \tag{37}$$

Acknowledgment. I want to thank the referee for suggesting the reference [7, pp. 28–29], which gives the generating function of $\{f_n\}$, congruences satisfied by the f_n, and the following symbolic identity:

$$f(f-1) \cdots (f-n+1) = n!, \tag{38}$$

which may be derived from (11). Our (11) appears on p. 29 of [7] as (19).

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ON DIFFERENTIATING ERROR TERMS

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1. A number of well-known formulas in mathematics have associated with them error terms which contain a parameter ξ which is an (unknown) function of

x. When such error terms must be differentiated problems may arise. When the formula is a Taylor series such problems are minimal. But when the parameter arises due to the application of Rolle's theorem the problems are more substantial. This is the case with the Lagrangian interpolation formula [1], [2]. In two important cases in numerical analysis—the derivation of numerical differentiation formulas from the Lagrangian formula [1], [3], and the proof of the simple form of the error term in Newton-Cotes quadrature formulas [4]—it is necessary to differentiate the error term of the Lagrangian formula. Typically in these cases the error term is expressed in terms of divided differences instead of a derivative of the function evaluated at a point ξ. Then theorems on the derivatives of divided differences are applied. It is the purpose of this note to show that, when the error term is expressed in terms of a derivative of the function, it is possible to derive directly the form of the derivative of the error term without resorting to the use of divided differences.

2. The Lagrangian interpolation formula for a function $f(x)$, whose derivatives of the orders required below are assumed to exist and be continuous, is given by

$$f(x) = \sum_{j=1}^{n} \frac{p_n(x)}{(x - a_j)p_n'(a_j)} f(a_j) + p_n(x) f^n(\xi)/n!, \tag{1}$$

where ξ lies between the minimum and maximum of $a_1, \cdots, a_n$ and x and where

$$p_n(x) = (x - a_1) \cdots (x - a_n). \tag{2}$$

We will show directly (i.e. without expressing the error term in divided difference form) that

$$\frac{1}{n!} \frac{d}{dx} [f^{(n)}(\xi)] = \frac{1}{(n+1)!} f^{(n+1)}(\eta), \tag{3}$$

where η lies in the same range as ξ. To do this consider (1) with n replaced by $n+1$. We have then

$$p_{n+1}(x) = p_n(x)(x - a_{n+1}) \tag{4}$$

and

$$p_{n+1}'(a_j) = \begin{cases} p_n'(a_j)(a_j - a_{n+1}) & (j \neq n+1) \\ p_n(a_{n+1}) & (j = n+1). \end{cases} \tag{5}$$

Now dividing (1) through by $p_{n+1}(x)$, using (4) and (5), and rearranging terms, we get

$$\begin{aligned} &[f(x)/p_n(x) - f(a_{n+1})/p_n(a_{n+1})]/(x - a_{n+1}) \\ &\qquad = \sum_{j=1}^{n} \frac{f(a_j)}{(x - a_j)(a_j - a_{n+1})p_n'(a_j)} + f^{n+1}(\tau)/(n+1)!, \end{aligned} \tag{6}$$

with τ in the interval spanned by $a_1, \cdots, a_{n+1}$ and x. Now let $a_{n+1} \to x$ which gives

$$(7) \qquad \frac{d}{dx}[f(x)/p_n(x)] = \sum_{j=1}^{n} \frac{f(a_j)}{-(x-a_j)^2 p_n'(a_j)} + f^{n+1}(\eta)/(n+1)!.$$

But dividing (1) through as it stands by $p_n(x)$ and differentiating we get

$$(8) \qquad \frac{d}{dx}[f(x)/p_n(x)] = \sum_{j=1}^{n} \frac{f(a_j)}{-(x-a_j)^2 p_n'(a_j)} + \frac{d}{dx}[f^n(\xi)]/n!;$$

(7) and (8) together imply (3).

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AFFINE TRANSFORMATIONS AND MIRROR-SYMMETRY

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The set of all images of points of a plane set A under some affine transformation is called an affine image of A. If there is a straight line such that reflection in that line leaves A invariant, we say that A is mirror-symmetric. We propose to prove the following

THEOREM. *A Jordan curve with the property that all its affine images are mirror-symmetric is an ellipse.*

Proof. Call a straight line s a symmetry-axis of the Jordan curve J, if there is a direction d such that for every point P on J there is a point P' on J with the property that the chord PP' is bisected by s and runs parallel to d. We have mirror-symmetry about s, if d is perpendicular to s. The fact that J is bounded and not part of any straight line then implies that the direction d is uniquely determined by s and that different axes determine different directions. One also observes that all symmetry-axes of J are concurrent, because they are at the same time also symmetry-axes for the convex hull $\bar{J}$ of J and as such pass through the center of gravity O of $\bar{J}$. Furthermore, given any two symmetry-axes s and s', the two pairs of directions s, d and s', d' either separate each other or else $s//d'$ and $d//s'$.

Now let J be a Jordan curve with the property as stated in the theorem. It is immediate that J possesses infinitely many symmetry-axes. For, if $s_1, s_2, \cdots, s_n$ are n such axes, $d_1, d_2, \cdots, d_n$ the corresponding symmetry-directions and α_i the angle between s_i and d_i, then by continuity there is an affine transformation T which transforms all angles α_i, $i=1, 2, \cdots, n$, into angles $\neq 90°$. However, $T(J)$ has an axis of mirror-symmetry s, so that the pre-image of s under T

becomes a symmetry-axis of J not represented among the n lines, $s_1, s_2, \cdots, s_n$.

Select now a converging sequence $s_1, s_2, \cdots$ from the set of all symmetry-axes of J and let s be their limit. This can be arranged so that the corresponding symmetry-directions $d_1, d_2, \cdots$ also converge to some direction d. It is easily seen that s and d do not coincide. The curve J being closed and bounded it then readily follows that s is also a symmetry-axis of J with d as the corresponding direction. We may now assume that d is perpendicular to s, as this can always be achieved by a suitable affine transformation. Introduce a rectangular (x, y)-coordinate system with its origin at the point O common to all symmetry-axes of J and let the x-axis coincide with s. For each positive integer n we define an affine transformation T_n by the equations $x'=x$, $y'=k_n y$, where k_n is a real number, such that T_n carries s_n and d_n into a pair of orthogonal directions. This is possible, because the two pairs of directions s, d and s_n, d_n separate each other. Without loss of generality we may also assume that k_n is positive. Thus s and the image s_n' of s_n under T_n become axes of mirror-symmetry for the curve $T_n(J)$. $T_n(J)$ allows rotation by $2\alpha_n$, where α_n is the angle between s and s_n'. Since $\lim s_n = s$ and $\lim d_n = d$ we can find n large enough so as to make α_n as small as we please. Now all curves $T_n(J)$ lie in some infinite rectangular strip containing the y-axis of the coordinate system. From this it follows that as $n \to \infty$ the numbers k_n remain bounded. For otherwise the points $T_n(P)$ for some P on J would form a divergent sequence of points and since $T_n(J)$ allows rotations about $2\alpha_n$, where $\alpha_n \to 0$ as $n \to \infty$, the curves $T_n(J)$ could not possibly be all contained in any vertical rectangular strip. The sequence k_n may now be assumed convergent and, if $\lim k_n = k$, let T be the corresponding transformation $x'=x$, $y'=ky$. It is clear that the x-axis intersects J at some point $A \neq O$. Since A also lies on $T_n(J)$, so do all points obtained from it by a succession of rotations through $2\alpha_n$. Since $\alpha_n \to 0$, a simple argument shows that the limit curve $T(J)$ contains all points of the circle K with center O and radius OA. $T(J)$, being a Jordan curve, must thus be identical with K. Hence J is an ellipse.

ON THE MULTIPLICATION OF SERIES

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Given two series $\sum a_n$ and $\sum b_n$, we have, associated with them in a prescribed manner, a series $\sum c_n$, called the product of the two series $\sum a_n$ and $\sum b_n$. Several types of products—as for example Cauchy, Dirichlet, Laurent and Fourier products—are known and with suitable assumptions on the convergence of the two series $\sum a_n$ and $\sum b_n$, suitable conclusions on the convergence or summability by an appropriate method of the series $\sum c_n$ can be drawn. (See, for instance, [1], Chapter XI.) A basic result in this context is Cesáro's theorem which asserts that the Cauchy product $\sum c_n$ is arithmetic mean summable to AB if the two series $\sum a_n$ and $\sum b_n$ are convergent, with sums A and B respectively. In this note we plan to give a result, which, besides yielding

Cesáro's theorem as a special case, seems to outline the general principle behind results of that type.

Let p denote the matrix $p \equiv (p_{nr})$, assumed to be triangular, and let P denote the matrix $P \equiv (P_{nr})$, where $P_{nr} = p_{nr}/P_n$, $P_n = \sum_{r=0}^{n} p_{nr}$. We shall say that a series $\sum a_n$ is *p-convergent* to A if the sequence $\{A_n\} = \{\sum_{r=0}^{n} p_{nr} a_r\}$ converges (in the ordinary sense) to A and that $\sum a_n$ is (P, p)-*summable* if the sequence $\{A_n\}$ defined above is summable by the matrix method P; i.e. if the sequence $\{\mathcal{A}_n\}$, $\mathcal{A}_n = \sum_{r=0}^{n} P_{nr} A_r$ converges.

We shall say that $\sum c_n$ is the *p-product* of the series $\sum a_n$ and $\sum b_n$ if $c_n = \sum_{r=0}^{n} p_{nr} a_r b_{n-r}$, and there will be no ambiguity about the order in which the two series are taken if we assume, in addition, that $p_{nr} = p_{n,n-r}$. In the sequel this is assumed.

We are now in a position to prove the following theorem.

THEOREM. *Let the series $\sum a_n$ and $\sum b_n$ be p-convergent to A and B respectively. Let the matrix p satisfy the two conditions* (i) $p_{nr} p_{rs} = p_{ns} p_{n-s,r-s}$ *and* (ii) *P is a T-matrix. Then the p-product $\sum c_n$ of the two series $\sum a_n$ and $\sum b_n$ is (P, p)-summable to AB.*

The proof of the theorem is based on the following known lemma (see [2], p. 75).

LEMMA. *If $x_n \to \xi$ and $y_n \to \eta$ and if A is a T-matrix such that $a_{n,n-p} \to 0$ as $n \to \infty$ for each fixed p, then the sequence $\{z_n\}$, $z_n = \sum_{p=0}^{n} a_{n,n-p} x_p y_{n-p}$ converges to $\xi\eta$.*

Proof of the theorem. Let $C_n = \sum p_{nr} c_r$. Then

$$C_n = \sum_{k=0}^{n} p_{nk} \left[\sum_{r=0}^{k} p_{kr} a_r b_{k-r} \right] = \sum_{r} \sum_{k} p_{nk} p_{kr} a_r b_{k-r}$$

$$= \sum_{r=0}^{n} p_{nr} a_r \left\{ \sum_{k=r}^{n} p_{n-r,k-r} b_{k-r} \right\} \qquad \text{(by condition (i))}$$

$$= \sum_{r=0}^{n} p_{nr} a_r B_{n-r}.$$

Therefore, if $\{\mathcal{C}_m\}$ denotes the P-transform of $\{C_n\}$, we have

$$\mathcal{C}_m = \sum_{n=0}^{m} P_{mn} C_n = (1/P_m) \sum_{r=0}^{m} p_{mr} C_r.$$

$$= (1/P_m) \sum_{n=0}^{m} p_{mn} \sum_{r=0}^{n} p_{nr} a_r B_{n-r}$$

$$= (1/P_m) \sum_{r=0}^{m} p_{mr} A_r B_{m-r} = \sum_{r=0}^{m} P_{mr} A_r B_{m-r}.$$

Conditions (i) and (ii) of the theorem together show that the matrix P satisfies the conditions of the lemma and now the result follows from the lemma.

COROLLARY 1. *If* $\sum a_n$ *and* $\sum b_n$ *and their p-product* $\sum c_n$ *are all p-convergent to* A, B *and* C *respectively, it follows that* $C=AB$.

COROLLARY 2. *The choice* $p_{nr}=1(n\geqq r)$ *and* $=0(n<r)$ *reduces p-convergence to ordinary convergence, the p-product to the Cauchy product and p-summability to the arithmetic mean summability. Thus the above choice will yield Cesàro's theorem.*

COROLLARY 3. *The choice* $p_{nr}=\binom{n}{r}$ $(n\geqq r)$ *and* $=0$ $(n<r)$ *satisfies the conditions of the theorem. In this case the p-product is defined by the relation* $c_n=\sum_{r=0}^{n}\binom{n}{r}a_r b_{n-r}$ *and it is easy to verify that if we define* c_n *by the relation*

$$\sum c_n\frac{x^n}{n!}=\sum a_n\frac{x^n}{n!}\sum b_n\frac{x^n}{n!}$$

then we have precisely the previously indicated relation of c_n *to* $\sum a_n$ *and* $\sum b_n$. *This choice of the matrix p reduces the P-summability to the well-known Euler summability* $(E, 1)$ *in the notation of* [1].

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CLASSES OF PAIRS OF COMMUTING MATRICES OVER A FINITE FIELD

L. CARLITZ, Duke University

Let $P(n)$ denote the number of ordered pairs of commuting matrices of order n with coefficients in the finite field $GF(q)$. Feit and Fine [1] have proved that $P(n)$ satisfies the following relation:

$$\sum_{n=0}^{\infty}\frac{P(n)}{q^{n^2}f(n)}x^n=\prod_{i=1}^{\infty}\prod_{j=0}^{\infty}(1-q^{1-j}x^i), \tag{1}$$

where

$$f(n)=\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)\cdots\left(1-\frac{1}{q^n}\right).$$

If $AB=BA$ and U is a nonsingular matrix with elements in $GF(q)$ then of course $U^{-1}AU\cdot U^{-1}BU=U^{-1}BU\cdot U^{-1}AU$. Thus the $P(n)$ ordered pairs of commuting matrices are separated into a certain number of similarity classes. Let $Q(n)$ denote the number of similarity classes. This number can be determined in the following way.

Let A denote a matrix with elements in an arbitrary field F and let $h_1(x)$, $h_2(x),\cdots,h_n(x)$ denote the invariant factors of $xI-A$, so that

$$h_i(x)\mid h_{i+1}(x)\qquad(i=1,\cdots,n-1). \tag{2}$$

Frobenius [2; see also 3, p. 106] has proved that the general solution of the matric equation

$$AX = XA \tag{3}$$

depends upon $r_n+3r_{n-1}+\cdots+(2n-1)r_1$ parameters, where

$$r_n = \deg h_n(x) \qquad (n = 1, 2, \cdots, n).$$

Therefore the number of solutions X of the equation (3) when the elements of the matrices are in $GF(q)$ is equal to

$$q^{r_n+3r_{n-1}+\cdots+(2n-1)r_1}. \tag{4}$$

Clearly if B is similar to A, the number of solutions of $BX=XB$ is equal to the number of solutions of (3) and is given by (4). Thus to find $Q(n)$ it is only necessary to evaluate

$$\sum q^{r_n+3r_{n-1}+\cdots+(2n-1)r_1}, \tag{5}$$

where the summation is over all polynomials $h_i(x)\epsilon GF[q]$ with highest coefficient 1 that satisfy (2) and

$$r_1+r_2+\cdots+r_n = n. \tag{6}$$

Let $r_1, r_2, \cdots, r_n$ be fixed nonnegative integers that satisfy (6). Then $h_1(x)$ can be chosen in q^{r_1} ways. Since $h_1(x)\,|\,h_2(x)$, we can select $h_2(x)$ in $q^{r_2-r_1}$ ways. Similarly $h_3(x)$ can be chosen in $q^{r_3-r_2}$ ways and so on. Thus the total number of choices of the $h_i(x)$ is equal to

$$q^{r_1}q^{r_2-r_1}q^{r_3-r_1}\cdots q^{r_n-r_{n-1}} = q^{r_n}.$$

It follows that

$$Q(n) = \sum q^{r_n}q^{r_n+3r_{n-1}+\cdots+(2n-1)r_1},$$

where the summation is over all r_j such that

$$r_1+r_2+\cdots+r_n = n, \qquad 0 \leqq r_1 \leqq r_2 \cdots \leqq r_n. \tag{7}$$

It is convenient to change the notation slightly and write this in the following way:

$$Q(n) = \sum q^{s_1}q^{s_1+3s_2+\cdots+(2n-1)s_n}, \tag{8}$$

where now the summation is over all s_j such that

$$s_1+s_2+\cdots+s_n = n, \qquad 0 \leqq s_n \leqq \cdots \leqq s_1. \tag{9}$$

This in turn is equivalent to

$$Q(n) = \sum q^{s_1}q^{s_1+3s_2+5s_3+\cdots}, \tag{10}$$

where now

(11) $$s_1 + s_2 + s_3 + \cdots = n, \qquad s_1 \geqq s_2 > \cdots > 0;$$

in other words, the summation in (10) is over all partitions of n into positive parts.

If for fixed k we put

(12) $$Q_k(n) = \sum q^{s_1} q^{s_1+3s_2+\cdots+(2k-1)s_k},$$

where the summation is over all s_j such that

(13) $$s_1 + s_2 + \cdots + s_k = n, \qquad s_1 \geqq \cdots \geqq s_k > 0,$$

we find that

$$\sum_{n=0}^{\infty} Q_1(n)x^n = \frac{q^2x}{1 - q^2x},$$

$$\begin{aligned}\sum_{n=0}^{\infty} Q_2(n)x^n &= \sum_{s_2=1}^{\infty} \sum_{s_1=s_2}^{\infty} q^{2s_1-3s_2}x^{s_1-s_2} \\ &= \sum_{s=1}^{\infty} q^{5s}x^{2s} \sum_{t=0}^{\infty} q^{2t}x^t \\ &= \frac{q^5x^2}{(1 - q^2x)(1 - q^5x^2)},\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty} Q_3(n)x^n &= \sum_{s_3=1}^{\infty} \sum_{s_2=s_3}^{\infty} \sum_{s_1=s_2}^{\infty} q^{2s_1+3s_2+5s_3}x^{s_1+s_2+s_3} \\ &= \sum_{s=1}^{\infty} q^{10s}x^{3s} \sum_{t_2=0}^{\infty} q^{3t_2}x^{2t_2} \sum_{t_3=0}^{\infty} q^{2t_3}x^{t_3} \\ &= \frac{q^{11}x^3}{(1 - q^2x)(1 - q^5x^2)(1 - q^{10}x^3)}.\end{aligned}$$

The general formula is evidently

(14) $$\sum_{n=0}^{\infty} Q_k(n)x^n = \frac{q^{k^2+1}x^k}{(1 - q^2x)(1 - q^5x^2) \cdots (1 - q^{k^2+1}x^k)}.$$

It follows easily from (14) that

(15) $$\sum_{j=0}^{k} \sum_{n=0}^{\infty} Q_j(n)x^n = \prod_{r=1}^{k} (1 - q^{r^2+1}x^r)^{-1}.$$

If we let $k \to \infty$ we get the *formal* generating function

(16) $$\sum_{n=0}^{\infty} Q(n)x^n = \prod_{r=1}^{\infty} (1 - q^{r^2+1}x^r)^{-1}.$$

Since $q>1$, the product converges only when $x=0$. However if we think of q as a complex number of absolute value <1 and define $Q(n)$ by means of (10) then (16) is valid for all x such that $|q^2x|<1$.

Supported in part by National Science Foundation Grant G 16485.

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CLASSROOM NOTES

Edited by John M. H. Olmsted, Southern Illinois University and
A. L. Shields, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, Mathematics Department, University of Michigan, Ann Arbor, Michigan.

ON THE QUOTIENT OF MONOTONE FUNCTIONS

Thomas E. Mott, State University College, Fredonia, N. Y.

A result of some use in analysis is that if $f(t)$ is a nonnegative nondecreasing function in $[0, a]$ then $(1/t)\int_0^t f(u)du$ is nondecreasing in $(0, a]$. An interesting generalization of this result is given below, which admits a simple proof by the use of Fubini's Theorem.

Theorem 1. *Let $f(t)$ and $g(t)$ be respectively nonnegative nondecreasing, and nonnegative nonincreasing, functions on $[a, b]$. If $F(t)=\int_a^t f(u)du$, $G(t)=\int_a^t g(u)du$, $h(t)=F(t)/G(t)$, and $S=\{t: t\in[a, b],\ G(t)\neq 0\}$ then $h(t)$ is defined and nondecreasing on S.*

Proof. Let $a<t_1<t_2\leqq b$, where $t_1, t_2\in S$. Then

$$h(t_2)-h(t_1)=\frac{F(t_2)}{G(t_2)}-\frac{F(t_1)}{G(t_1)}=\frac{1}{G(t_2)}\int_a^{t_2}f(u)du-\frac{1}{G(t_1)}\int_a^{t_1}f(u)du$$

$$=\frac{1}{G(t_2)\cdot G(t_1)}\left[G(t_1)\int_a^{t_2}f(u)du-G(t_2)\int_a^{t_1}f(u)du\right]$$

$$= \frac{1}{G(t_2)\cdot G(t_1)}\left[G(t_1)\int_{t_1}^{t_2} f(u)du + [G(t_1) - G(t_2)]\int_a^{t_1} f(u)du\right]$$

$$= \frac{1}{G(t_2)\cdot G(t_1)}\left[\int_a^{t_1} g(x)dx\cdot\int_{t_1}^{t_2} f(y)dy - \int_{t_1}^{t_2} g(y)dy\cdot\int_a^{t_1} f(x)dx\right]$$

$$= \frac{1}{G(t_2)\cdot G(t_1)}\iint_R [g(x)\cdot f(y) - g(y)f(x)]dA \geqq 0,$$

where R is the closed rectangle $[a_1, t_1]\times[t_1, t_2]$. Since $(x, y)\in R$ implies that $x\leqq y$ it follows that $0\leqq f(x)\leqq f(y)$, $g(x)\geqq g(y)\geqq 0$ and consequently $f(x)\cdot g(y) \leqq g(x)\cdot f(y)$. Therefore the final integrand is nonnegative, $h(t_2)\geqq h(t_1)$, and $h(t)$ is nondecreasing in S.

If $g(b)>0$, then $g(x)>0$ in $[a, b]$ and $S=(a, b]$. If $g(b)=0$ but $g(c)>0$ for some c in (a, b), then $g(x)>0$ in $[a, c]$ and again $S=(a, b]$. Thus $S=(a, b]$ unless $g(x)\equiv 0$ in $(a, b]$, in which case $S=\varnothing$, the empty set.

Note. If $f(t)$ is nonpositive nonincreasing and $g(t)$ is nonpositive nondecreasing the result remains valid. Indeed, we may let $f_0(t)=-f(t)$, $g_0(t)=-g(t)$ and proceed as before.

It is obvious from the proofs given above that in Theorem 1 if the function $g(t)$ is identically equal to a positive (negative) constant on $[a, b]$ then, the nonnegative (nonpositive) hypothesis may be removed from the function $f(t)$. This fact is used in the proof of:

THEOREM 2. *If $f(t)$ is a nondecreasing function on $[a, b]$, then*

$$\frac{1}{t-a}\int_a^t f(u)du \leqq \frac{1}{b-a}\int_a^b f(u)du \leqq \frac{1}{b-t}\int_t^b f(u)du$$

for every t in (a, b).

Proof. Let $F(t)=\int_a^t f(u)du$ and $G(t)=\int_a^t 1du=t-a$. Then it follows from our preceding results that the function $h(t)=F(t)/G(t)$ is defined and nondecreasing on $(a, b]$. Thus the first inequality is verified.

The second inequality is verified as follows:

$$\frac{1}{b-t}\int_t^b f(u)du = \frac{b-a}{b-t}\cdot\frac{1}{b-a}\int_a^b f(u)du - \frac{t-a}{b-t}\cdot\frac{1}{t-a}\int_a^t f(u)du$$

$$\geqq \frac{b-a}{b-t}\cdot\frac{1}{b-a}\int_a^b f(u)du - \frac{t-a}{b-t}\cdot\frac{1}{b-a}\int_a^b f(u)du$$

$$= \frac{1}{b-a}\int_a^b f(u)du.$$

Note. If $f(t)$ is nonincreasing on $[a, b]$ then this theorem will be valid if $\leqq$ is replaced by $\geqq$.

A PROBLEM IN PENETRATION

MILES C HARTLEY, University of Illinois

DEFINITION. *Two polyhedrons are said to be penetrating polyhedrons if their intersection is a third polyhedron and if there exist points of each given polyhedron not belonging to the other.*

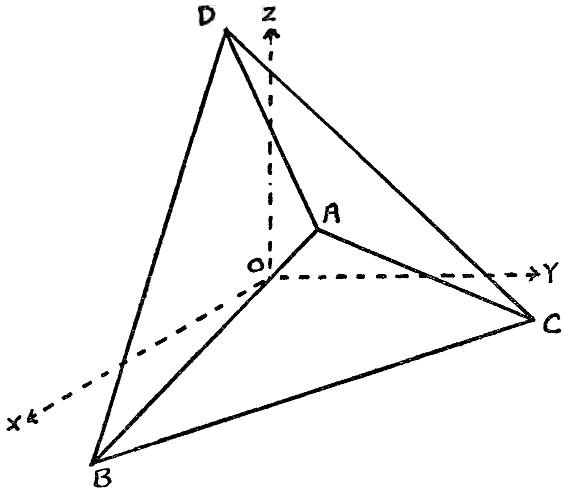

FIG. 1

Consider a regular tetrahedron $ABCD$ with edge $(2\sqrt{2})a$ placed in the coordinate frame with its center at the origin and with its vertices at $A(a, a, a)$, $B(a, -a, -a)$, $C(-a, a, -a)$ and $D(-a, -a, a)$. The centers of the faces (as designated) are:

$$\begin{aligned} E\{\triangle ABC\}&: \quad (a/3,\ a/3,\ -a/3)\\ F\{\triangle ABD\}&: \quad (a/3,\ -a/3,\ a/3)\\ G\{\triangle ACD\}&: \quad (-a/3,\ a/3,\ a/3)\\ H\{\triangle BCD\}&: \quad (-a/3,\ -a/3,\ -a/3). \end{aligned}$$

Since $EF^2 = EG^2 = EH^2 = FG^2 = FH^2 = GH^2 = (8/9)a^2$, there is a regular tetrahedron $EFGH$ with edge $(2\sqrt{2}/3)a$ inscribed in tetrahedron $ABCD$.

The direction numbers of the four lines joining the origin to the vertices E, F, G, and H are: $OE[1, 1, -1]$, $OF[1, -1, 1]$, $OG[-1, 1, 1]$, $OH[-1, -1, -1]$. The equation of plane ABC is

$$x + y - z = a. \tag{1}$$

Now permit the tetrahedron $EFGH$ to increase uniformly so that the new tetrahedron $E'F'G'H'$ is similar to the original tetrahedron $EFGH$. This can be

accomplished by extending each of the lines OE, OF, OG, OH in its specified direction. The parametric equations of the line OE are:

$$(2)\qquad \begin{cases} x = a/3 + (\sqrt{3}/3)t \\ y = a/3 + (\sqrt{3}/3)t \\ z = -a/3 - (\sqrt{3}/3)t, \end{cases}$$

where t is the distance from E to any point on OE. We let $t=EE'$. Hence the coordinates of E' are $(a/3+(\sqrt{3}/3)t,\ a/3+(\sqrt{3}/3)t,\ -a/3-(\sqrt{3}/3)t)$. The line $E'G'$ will have the same direction numbers as EG, namely $[-1, 0, 1]$. Hence the equations of $E'G'$ are:

$$(3)\qquad \begin{cases} x = a/3 + (\sqrt{3}/3)t - (\sqrt{2}/2)T \\ y = a/3 + (\sqrt{3}/3)t \\ z = -a/3 - (\sqrt{3}/3)t + (\sqrt{2}/2)T. \end{cases}$$

To obtain the point of intersection of $E'G'$ with the plane ABC, we substitute the equations (3) in equation (1) and obtain $(\sqrt{3})t-(\sqrt{2})T=0$ or $T=(\sqrt{6}/2)t$. Substituting $T=(\sqrt{6}/2)t$ in (3), we find that $E'G'$ intersects plane ABC in the point I with the coordinates $(a/3-(\sqrt{3}/6)t,\ a/3+(\sqrt{3}/3)t,\ -a/3+(\sqrt{3}/6)t)$. Similarly, $E'F'$ intersects plane ABC in point J with coordinates $(a/3+(\sqrt{3}/3)t,\ a/3-(\sqrt{3}/6)t,\ -a/3+(\sqrt{3}/6)t)$ and $E'H'$ intersects ABC in point K with coordinates $(a/3-(\sqrt{3}/6)t,\ a/3-(\sqrt{3}/6)t,\ -a/3-(\sqrt{3}/3)t)$.

Triangle IJK is equilateral since $IJ^2=IK^2=JK^2=(6/4)t^2$. In other words, tetrahedron $E'F'G'H'$ intersects face ABC in an equilateral triangle with side $(\sqrt{6}/2)t$.

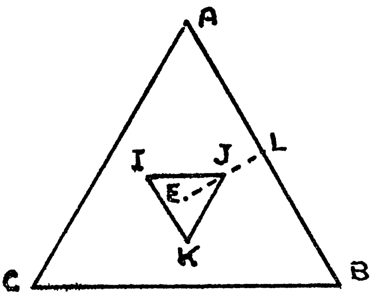

FIG. 2

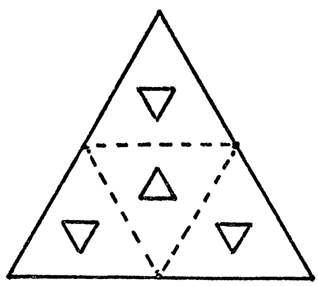

FIG. 3

Let L be the midpoint of line AB (see Fig. 2); the coordinates of L are $(a, 0, 0)$. The direction numbers of EL are $[2, -1, 1]$. The direction numbers for EJ are $[\frac{1}{3}\sqrt{3}, -\frac{1}{6}\sqrt{3}, \frac{1}{6}\sqrt{3}]=[2, -1, 1]$. Hence, L and J lie in the same direction from E and triangle IJK has the position indicated in Fig. 2 relative to triangle ABC. The circumradius R of triangle IJK is $(\sqrt{2}/2)t$, but R cannot

be as large as EL which is one-third the altitude of triangle ABC. Therefore, $(\sqrt{2}/2)t<(\sqrt{6}/3)a$ or $t<(2\sqrt{3}/3)a$.

Since the coordinates of F' are $(a/3+(\sqrt{3}/3)t,\ -a/3-(\sqrt{3}/3)t,\ a/3+(\sqrt{3}/3)t)$, $E'F'^2=2((2/3)a+(2\sqrt{3}/3)t)^2$. Thus, the edge of the tetrahedron $E'F'G'H'$ is $(2\sqrt{2}/3)a+(2\sqrt{6}/3)t$.

THEOREM. *A regular tetrahedron of edge $2\sqrt{2}a$ can be penetrated by a regular tetrahedron of edge $(2\sqrt{2}/3)a+(2\sqrt{6}/3)t$, if $0<t<(2\sqrt{3}/3)a$. The circumradius of each equilateral triangle of intersection is $(\sqrt{2}/2)t$.*

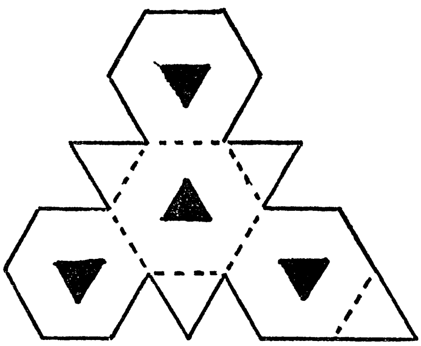

FIG. 4

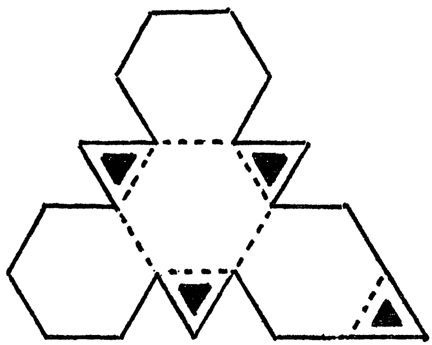

FIG. 5

Figure 3 gives the two-dimensional layout for the exterior tetrahedron.

DEFINITION. *A truncated tetrahedron is a polyhedron bounded by four equal regular hexagons and four equal equilateral triangles and formed by cutting off the vertices of a regular tetrahedron.*

COROLLARY 1. *A truncated tetrahedron of edge $(2\sqrt{2}/3)a$ can be penetrated by a regular tetrahedron of edge $(2\sqrt{2}/3)a+(2\sqrt{6}/3)t$, if $0<t<(2\sqrt{3}/3)a$. The penetration occurs in the hexagonal faces and the circumradius of each equilateral triangle of intersection is $(\sqrt{2}/2)t$.*

Figure 4 gives the two-dimensional layout for Corollary 1.

Since the centers of the triangular faces of a truncated tetrahedron are points of contact for an inscribed tetrahedron, we have

COROLLARY 2. *A truncated tetrahedron of edge $(2\sqrt{2}/3)a$ can be penetrated by a regular tetrahedron of edge $(10\sqrt{2}/9)a+(2\sqrt{6}/3)t$, if $0<t<(4\sqrt{3}/9)a$. The penetration occurs in the triangular faces and the circumradius of each equilateral triangle of intersection is $(\sqrt{2}/2)t$.*

Figure 5 gives the two-dimensional layout for Corollary 2.

The method presented here extends immediately to all regular and semiregular polyhedrons but the computations quickly become very involved.

III. Curriculum for teachers returning for graduate work. Offerings of courses specially designed for teachers, e.g., mathematics courses in which the returning teacher does not have to compete directly with the regular graduate mathematics student.

a. 77% of the respondents offer no courses specially designed for teachers.
b. 10% offer one such course.
c. 3% offer two such courses.
d. 7% offer three or four such courses.
e. 3% offer more than four such courses.

IV. Influence of curriculum study groups in programs for teacher education. Fifty-nine percent of the respondents saw some influence of the curriculum study groups on their own programs of preparation of teachers while forty-one percent saw none.

THE LIMIT CONCEPT IN THE EDUCATION OF TEACHERS

DONALD W. HIGHT, Kansas State College of Pittsburg

In recent years a number of American colleges, the College Entrance Examination Board, the Federal government and numerous teachers' and lay organizations have made special efforts to improve, if not completely rewrite, secondary school mathematics. These efforts are providing stimulus and guidance in reviewing the present program critically, and they are also producing deviation from the traditional texts.

Two basic approaches are noticeable: (1) acceleration, which advocates the teaching of calculus in high school; (2) enrichment, which proposes a stronger, broader set of mathematical experiences. For both of these approaches it is important that students have a good understanding of the concept of a limit prior to entering calculus.

An outgrowth of this need has been the publication of short texts devoted to a study of the real number system. These materials, such as *The Real Number System* by Bates and Kiokemeister and a monograph with the same title by Olmsted, are designed primarily for pre-calculus college students. Their purpose is to provide experience with the real numbers, the limit concept, and the completeness property that will enable the student to study calculus from one of the newer textbooks.

If the need for limits were not confronted prior to the study of calculus, it would be feasible to provide necessary instruction immediately preceding the study. However, this is not the case. The traditional elementary mathematics for high school requires a knowledge of the concept of limits in order to provide an understanding of the real numbers, some geometry theorems, sums of infinite series, asymptotes, and an appreciation of what is called "infinity." It is the introduction of limits that allows one to proceed from elementary mathematics to a study of calculus and all its ramifications.

The teaching of the concept of a limit should begin in secondary schools mainly for two reasons: (1) both the traditional and the newly devised curricula contain topics that cannot be properly handled without the concept of a limit; (2) the students need a gradual introduction to the limit concept in order to grow in mathematical maturity and gain a readiness for calculus.

The limit process has a unique history. Even though problems involving the concept of a limit were considered in the time of Archimedes, the definition accepted today was not widely advocated before the middle of the nineteenth century. Also, the existence of irrational numbers was recognized around the fifth century B.C., but it was not until the limit was defined that the real numbers were algebraically derived from the rational numbers in a sound mathematical manner.

Traditional secondary mathematics, basically Euclidean geometry and algebra, consists almost entirely of the prevalent concepts and points of view held prior to the eighteenth century. It is no wonder then that the concept of a limit as presented today is seldom grasped by the student. The irrational numbers are presented as restricted symbols that sometimes "can't be added" or as "numbers that cannot be developed within the capabilities of the student." Theorems in geometry that involve a limiting process have been accepted with an invalid argument or with no attempted proof. Other topics in secondary mathematics which utilize concepts of a limit are offered to the student with a jargon that is not acceptable mathematics.

A student should not be expected to understand theorems or definitions that are stated in terms of limits when this word has yet to be defined for him. If he is unacquainted with the limit concept, he can hardly be prepared for calculus when the definition of both the derivative and the integral involves limits.

In the 1930's mathematics educators were urging that the limit concept be "skimmed over lightly" by high school teachers leaving vigorous consideration to the colleges. However, the lower-level college mathematics courses of the thirties, forties, and fifties are now appearing in the high schools. This suggestion of our depression forefathers, though still apparent in traditional texts, does not fit today's criteria for a good mathematics program.

The sample textbooks of the School Mathematics Study Group, to which college teachers contributed, are among the most discussed new programs for the secondary schools. Limits are used in topics concerning the geometry of the circle, asymptotes of a graph, and sum of an infinite series. Likewise, they are utilized extensively throughout the book *Elementary Functions*. Even here there is evidence of a reluctance to treat limits as a mathematical topic but rather to use the concept vaguely to obtain other results under the guise of "intuition."

No doubt the writers of the S.M.S.G. sample textbooks, as well as authors of traditional and "modern" textbooks, are aware of the problems of teaching the concept of a limit: How should it be done and with what amount of rigor?

Can the teachers of high school mathematics teach the concept when they may have had little experience with limits? Do high school adolescents have the mental or mathematical maturity to grasp the concept?

Attempts are being made to solve these problems. In the meantime, it is important that college instructors of mathematics, especially those involved in the training of teachers, be cognizant of the state of affairs of the limit concept and make necessary provisions for their charges. The following suggestions are but a few of those possible:

1. Provide pre-calculus instruction in the concepts of the real numbers, limits, and the completeness property until students receive this instruction prior to entering college.

2. Provide opportunities for high school teachers to study the concept of limits and its application to secondary school mathematics and basic definitions of calculus.

3. Become aware of the topics and concepts that have been presented to students in secondary schools. (When a freshman fails calculus we can't always blame his previous teacher and we won't blame ourselves. Therefore, we should compare the mathematics he has taken with the college mathematics confronting him and we might discover that he has little chance for success unless extra provisions are made.)

The greatest gap between the secondary school mathematics program and that of the colleges seems to be due to the present treatment of topics that involve limits. It is hoped that this gap will be narrowed in the near future by objective study and cooperation between teachers in each echelon.

A MATHEMATICS ENRICHMENT PROGRAM FOR HIGH SCHOOL STUDENTS

ROBERT D. LARSSON, Clarkson College of Technology

Clarkson College of Technology is located in an area which includes some 23 high schools within a radius of about 50 miles. Most of these schools have relatively few students taking the standard advanced courses in mathematics. As such, it is not economical, in many cases, for these schools to provide special courses for their high ability students.

For this reason, Clarkson College offered a mathematics enrichment program in the fall of 1960 for junior and senior high school students. This course was not intended to duplicate a regular high school course, nor a standard college freshmen course. It was designed to enrich the mathematical background of those taking it, and endeavor to inspire them to move ahead in mathematics and related fields. It was hoped that it would help solve the financial problems which an individual school would encounter in attempting this, and that it would also offer a college atmosphere, which would be a valuable contribution.

Starting with 59 students from 8 high schools in the fall and spring of 1960–

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Collaborating Editor: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1563. *Proposed by D. L. Silverman, National Security Agency, Fort Meade, Maryland*

If a tetrahedron with a solid right angle has integral edges, prove that the product of the edges is a multiple of 5,702,400.

E 1564. *Proposed by Leon Bankoff, Los Angeles, California*

Show that the sum of the distances from the incenter to the vertices of an acute triangle is not less than twice the sum of the distances from the orthocenter to the sides.

E 1565. *Proposed by R. L. Graham, University of California, and F. D. Parker, University of Alaska*

Show that for any positive integer n there exists a power of 2 with a string of more than n successive zeros.

E 1566. *Proposed by Michael Skalsky, Southern Illinois University*

Show that

$$\sum_{k=0}^{n} (m+k)_m = (m+n+1)_{m+1}/(m+1),$$

where $(x)_m = x(x-1)\cdots(x-m+1)$.

E 1567. *Proposed by Hugh Noland, Chico State College*

If n is a positive integer, determine the number of integral solutions of the equation $x+y+z=n$ satisfying the conditions $x\geqq y\geqq z\geqq 0$.

E 1568. *Proposed by Ralph Greenberg, University of Pennsylvania*

Sum $T=\sum_{n=1}^{\infty}(-1)^s(1/n)$, where s is the number of prime factors of n.

E 1569. *Proposed by R. H. Moorman, Tennessee Polytechnic Institute*

Find the locus of points equidistant from two skew lines.

E 1570. *Proposed by J. S. Brock, David Taylor Model Basin, Washington, D. C.*

Prove that the roots of the cubic equation

$$x^3 - (a + b + c)x^2 + (ab + bc + ca - d^2 - e^2 - f^2)x + (ae^2 + bf^2 + cd^2 - abc - 2def) = 0$$

are all real when a, b, c, d, e, f are real numbers.

SOLUTIONS

A Rational Number

E 1516 [1962, 434]. *Proposed by R. J. Oberg, University of California*

Let $N = 0.1360518 \cdots$, where the kth digit is congruent mod 10 to the kth triangular number. Show that N is rational.

Solution by J. E. Yeager, Temple University. It is easily seen that

$$(n + 20)(n + 21)/2 \equiv n(n + 1)/2 \pmod{10},$$

so that the decimal expansion of N repeats in periods of twenty digits.

Also solved by R. G. Albert, W. T. Bailey, E. D. Bender, Walter Bluger, W. J. Blundon, Alan Bomberger, Robert Bowen, D. A. Breault, Brother R. L. Fitz, Brother Joseph Heisler, Leonard Carlitz, Frederick Carty, S. R. Cavior, P. R. Chernoff, D. I. A. Cohen, R. J. Cormier, J. F. Darling, Monte Dernham, J. W. Ellis, P. G. Engstrom, Stuart Friedman, Anton Glasser, N. D. Glassman, Michael Goldberg, L. D. Goldstone, S. W. Golomb, Ralph Greenberg, Vern Hoggatt, J. E. Homer, Jr., J. A. H. Hunter, A. R. Hyde, R. A. Jacobson, G. J. Janusz, Erwin Just and Norman Schaumberger (jointly), J. D. E. Konhauser, Jack Latimer and O. N. Strand and Lee Thomson (jointly), Jiang Luh, P. B. Manchester, D. C. B. Marsh, Helen M. Marston, G. T. Mitchell, C. S. Ogilvy, F. D. Parker, Walter Penney, J. L. Pietenpol, C. F. Pinzka, Peter Rosenthal, Leo Sauvé, Perry Scheinok, J. E. Scherer, J. P. Scholtz, Donna J. Seaman, J. L. Shores, D. L. Silverman, H. P. Smith, R. L. Syverson, G. C. Thompson, Guy Torchinelli, Stephen Ullom, Simon Vatriquant, John Vinson, Julius Vogel, G. W. Walker, W. C. Waterhouse, Dale Woods, Marvin Wunderlich, L. A. Zalcman, and the proposer.

Editorial Note. In fact, $N = 13605186556815063100/(10^{20} - 1)$, wherein one notes an interesting symmetry in the first 18 digits of the numerator. Parker and Rosenthal pointed out that the result can be generalized for any polygonal numbers. Vinson proved the generalization: If $S_n = \sum_{r=1}^{n} a_r$ and if $\{a_r\}$ is periodic mod m, then so is $\{S_n\}$. Other generalizations were given by Glassman and Wunderlich. Waterhouse called attention to Lemma 1 in W. F. Trench, "On periodicities of certain sequences of residues," this MONTHLY [1960, 652–656].

A Trigonometric Inequality

E 1517 [1962, 434]. *Proposed by R. S. Spira, University of California*

If $u > 0$, show that $(u + 1/u)$ arccot $u > 1$.

I. *Solution by R. G. Albert, Smithsonian Institute, Cambridge, Mass.* Let $w = \operatorname{arccot} u$. Then $0 < w < \pi/2$ and $u = \cot w$. We must show that $w > \sin w \cos w$. This follows from $w > \sin w > 0$ and $1 > \cos w$.

II. *Solution by P. T. Bateman, University of Illinois.* Since $t^2/(1+t^2)$ is an increasing function for positive values of t, we have

$$\operatorname{arccot} u = \int_u^\infty \frac{dt}{1+t^2} = \int_u^\infty \frac{t^2}{1+t^2}\frac{dt}{t^2}$$

$$> \frac{u^2}{1+u^2}\int_u^\infty \frac{dt}{t^2} = \frac{u}{1+u^2},$$

which was to be proved.

III. *Solution by Michael Goldberg, Washington, D. C.* Construct right triangle ABC of legs $AB=u$ and $BC=1$, and denote the angle at A by θ. Then $\cot\theta=u$. Construct perpendicular BE on the hypotenuse and extend it to cut the parallel to AB through C in D. Then $CD=1/u$. Draw an arc through B, centered on A, between the sides AB and AC, and let the length of this arc be p. Similarly, draw an arc through D, centered on C, between CD and CA, and let the length of this arc be q. Then $p=u\theta$ and $q=\theta/u$. Hence

$$p + q = (u + 1/u)\theta = (u + 1/u) \operatorname{arccot} u.$$

The projections of p and q on BC are less than p and q. Furthermore, these projections overlap. Hence $p+q>1$ and the desired inequality is established.

IV. *Solution by R. H. Wilson, Jr., National Aeronautics and Space Administration, Washington, D. C.* The given inequality is equivalent to $\arctan u<u+1/u$, or $u<\tan(u+1/u)$, and expansion of the tangent function by Maclaurin's series gives all positive terms for $u>0$, of which the first term is $u+1/u$.

V. *Solution by David Zeitlin, Minneapolis, Minnesota.* Let $g(u)=u/(u^2+1)-\operatorname{arccot} u$, $u\geqq 0$. Since $g'(u)>0$ for $u\geqq 0$, $g(0)=-\pi/2$, and $g(u)\to 0$ as $u\to\infty$, we conclude that $g(u)<0$ for $u>0$, which is the desired result.

Also solved by Mary C. Andrews, John Avila, Leon Bankoff, T. L. Beatty, Walter Bluger, B. B. Bowen, Robert Bowen, D. A. Breault, J. L. Brown, Jr., Leonard Carlitz, R. L. Carmichael, Frederick Carty, Paul Chernoff, D. I. A. Cohen, R. J. Cormier, T. R. Curry, Molley Cutler and Susan Klein (jointly), G. C. Dodds, D. R. Dunninger, J. W. Ellis, P. G. Engstrom, Sarah Evangelista, Jane Evans, G. P. Farrell, J. A. Faucher, N. D. Glassman, Richard Grassl, Ralph Greenberg, H. W. Guggenheimer, H. W. Hickey, Vern Hoggatt, J. M. Horner, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), Nickolas Konopliv, T. J. Lee, D. L. Linfield, Jiang Luh, E. L. Magnuson, P. B. Manchester, D. C. B. Marsh, Helen M. Marston, G. J. Michaelides, J. B. Muskat, Win Myint, R. J. Oberg, C. S. Ogilvy, J. L. Pietenpol, C. F. Pinzka, B. E. Rhoades, David Rothman, Leo Suavé, Donna J. Seaman, J. L. Shores, H. P. Smith, R. M. Toms, Guy Torchinelli, Simon Vatriquant, Julius Vogel, W. C. Waterhouse, Harry Weingarten, J. S. W. Wong, Dale Woods, J. E. Yeager, and the proposer.

Row Ordered and Column Ordered Matrices

E 1518 [1962, 434]. *Proposed by R. W. Cottle, University of California*

Let $A=(a_{ij})$ be an m by n matrix whose entries are elements of an ordered set $(S, \geqq)$. Suppose A is column ordered—that is, $a_{1j}\geqq a_{2j}\geqq \cdots \geqq a_{mj}$ for each

$j=1, 2, \cdots, n$. Obtain a row ordered matrix A' by arranging the entries of each row of A so that $a'_{i1} \geqq a'_{i2} \geqq \cdots \geqq a'_{in}$ for each $i=1, 2, \cdots, m$. Is A' column ordered?

I. *Solution by J. L. Pietenpol, Columbia University.* Carry out the row ordering as follows: first permute entire columns to order the first row; then permute columns without their first members to order the second row; etc. It is clear that the matrix remains column ordered at each stage of the process.

II. *Solution by Jiang Luh, University of Michigan.* Element $a'_{ij} \geqq$ at least $n-j+1$ entries in the ith row of A' and hence of A. Thus $a'_{ij} \geqq$ at least $n-j+1$ entries in the $(i+1)$th row of A. But there are at most $n-j$ entries $< a'_{i+1,j}$ in the $(i+1)$th row of A, so $a'_{ij} \geqq a'_{i+1,j}$. Therefore A' is column ordered.

Also solved by R. G. Albert, Donald Batman, G. M. Bergman, Marjorie Bicknell, Walter Bluger, Robert Bowen, Brother Joseph Heisler, Frederick Carty, P. R. Chernoff, D. I. A. Cohen, Frank Dapkus, Jane Di Paola, J. W. Ellis, N. D. Glassman, Ralph Greenberg, Vern Hoggatt, R. A. Jacobson, E. L. Magnuson, P. B. Manchester, D. C. B. Marsh, Lois E. Minning, Perry Scheinok, Binyamin Schwarz, Donna J. Seaman, D. L. Silverman, Paul Smith, E. H. Theil, Guy Torchinelli, Dennis Travis, John Vinson, W. C. Waterhouse, J. S. W. Wong, J. E. Yeager, and the proposer.

Bergman pointed out that the problem appeared as Problem 2a in the Winter, 1960, issue of *Particle, a Quarterly by and for Science Students*, and an interesting pictorial solution was given on p. 35 of the Summer, 1962, issue. Schwarz stated that the result of the problem is known for the more general situation of "tableaux," and he gave the reference H. Boerner, *Darstellungen von Gruppen*, Springer, Berlin (1955), p. 137 footnote.

A Minimum Equilateral Triangle

E 1519 [1962, 435]. *Proposed by C. N. Mills, Sioux Falls College*

Find the area of the smallest equilateral triangle inscribed in a right triangle of legs a and b.

I. *Solution by J. L. Pietenpol, Columbia University.* If we place the vertices of the right triangle at (0, 0), (a, 0), (0, b), and let two vertices of an equilateral triangle be (u, 0) and (0, v), the third vertex will be at

$$(u/2 + \sqrt{3}v/2,\ v/2 + \sqrt{3}u/2).$$

This vertex must lie on the hypotenuse $bx+ay=ab$, so that the condition for an inscribed equilateral triangle is

$$(b/2 + \sqrt{3}a/2)u + (a/2 + \sqrt{3}b/2)v = ab.$$

Minimizing u^2+v^2 subject to this condition gives

$$u = ab(b/2 + \sqrt{3}a/2)/(a^2 + \sqrt{3}ab + b^2),$$
$$v = ab(a/2 + \sqrt{3}b/2)/(a^2 + \sqrt{3}ab + b^2).$$

The area of the inscribed equilateral triangle is then

$$\sqrt{3}(u^2 + v^2)/4 = \sqrt{3}a^2b^2/4(a^2 + \sqrt{3}ab + b^2).$$

II. *Solution by J. F. Darling, Woodstown, N. J.* Let P be the Miquel point of any equilateral triangle inscribed in the given triangle. All other Miquel triangles of P are similar and have P as center of similitude, so that the pedal triangle of P must be the smallest. Using the known formula for side x of the pedal triangle in terms of PA and PB and the fact that $\measuredangle APB=150°$, we find

$$x^2 = a^2b^2/(a^2 + \sqrt{3}ab + b^2)$$

and area $=\sqrt{3}x^2/4$.

Also solved by R. G. Albert, J. P. Ballantine, Joel Beer and Perry Scheinok (jointly), Walter Bluger, B. B. Bowen, Robert Bowen, F. Cartuyvels, T. R. Curry, Frank Dapkus, Jane Evans, Michael Goldberg, L. D. Goldstone, S. H. Greene, H. W. Guggenheimer, Ned Harrell, Vern Hoggatt, J. A. H. Hunter, Erwin Just and Norman Schaumberger (jointly), Esther A. Linfield, E. L. Magnuson, D. C. B. Marsh, Helen M. Marston, C. F. Pinzka, Guy Torchinelli, G. W. Walker, and the proposer.

Editorial Note. The more general problem of finding the minimum equilateral triangle inscribed in a given general triangle appears, with solution, in E. W. Hobson, *Treatise on Plane and Advanced Trigonometry*, Dover Publications, Inc. (1957), p. 211, Problem 4. The area of the equilateral triangle is

$$2\sqrt{3}K^2/(a^2 + b^2 + c^2 + 4\sqrt{3}K),$$

where K is the area of the given triangle of sides a, b, c.

The even more general problem of inscribing in a given triangle a minimum triangle of given shape appears as Problem 380 in Julius Petersen, *Methods and Theories for the Solution of Problems of Geometrical Construction*, Stechert (reprint), 1927, p. 83.

The following allied problems were proposed by J. A. H. Hunter and Ned Harrell respectively:

1. A square of area 12 square units is inscribed within an equilateral triangle, at least touching each side of the triangle. Find the sides of the maximum and minimum permissible equilateral triangles.

Answer: $4\sqrt{(2+\sqrt{3})}$ and $2(2+\sqrt{3})$.

2. Find the area of the maximum equilateral triangle which can be circumscribed about a right triangle of legs a and b.

Answer: $\sqrt{3}(a^2+\sqrt{3}ab+b^2)/3$.

The Second Brocard Triangle

E 1520 [1962, 435]. *Proposed by V. E. Hoggatt, Jr. and Charles Phillips, San Jose State College*

Prove that if the circumcenter O of triangle ABC is inverted with respect to the three Apollonian circles of ABC, the images O_a, O_b, O_c are the vertices of the second Brocard triangle of ABC.

Editorial Note. It was discovered too late that the result of this problem is Theorem 494, p. 296, of R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, Inc., 1960.

Solved by J. F. Darling, Sister M. Stephanie, P. D. Thomas, Guy Torchinelli, Roscoe Woods, and the proposers. All solutions were synthetic except that of Sister M. Stephanie, who used complex coordinates.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Collaborating Editors: LEONARD CARLITZ, Duke University, H. S. M. COXETER, University of Toronto, and ALBERT WILANSKY, Lehigh University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, New Jersey. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5072. *Proposed by W. E. Briggs, University of Colorado*

Prove that

$$g(\theta) = \sum_{n=1}^{\infty} \frac{\theta}{n(n + \theta)}$$

is irrational for infinitely many rational θ on $(0, 1)$.

5073. *Proposed by D. J. Newman, Yeshiva University*

Let $f(x)$ be continuous on $[0, n]$ and suppose $f(0) = f(n) = 0$. Show that $f(x) = f(y)$ has at least n different solutions with $x - y$ a positive integer. (The well-known chord theorem asserts that there is at least one solution with $x - y = 1$.)

5074. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Let E_n, $1 \leqq n < \infty$ be an infinite sequence of measurable sets in $(0, 1)$, each E_n having measure $\geqq c$. It is well known that there exists an infinite subsequence E_{n_k}, $1 \leqq k < \infty$ such that $\bigcap_{k=1}^{\infty} E_{n_k}$ is not empty. Prove that the sequence $n_1 < n_2 < \cdots$ can be chosen so as to have upper density c, and that this result is best possible.

5075. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Let $n_1 < n_2 < \cdots$ be a sequence of integers such that $\limsup n_k / n_1 n_2 \cdots n_{k-1} = \infty$ and $\liminf n_k / n_{k-1} > 1$. Prove $\sum 1/n_k$ is irrational. (See also problem 4773 [1958, 782].)

5076. *Proposed by Hans Zassenhaus, University of Notre Dame*

In the correlation theory of multiple events one can define a set of correlation coefficients in the form of cosines of the angles of inclination between two linear subspaces S_1, S_2 of Euclidean n-space. Let q_j be the dimension of S_j, $q_1 \leqq q_2$ and let the row vectors of the $q_j \times n$ matrix A_j form a basis of S_j. Give a

definition of the angles of inclination and show that the characteristic roots of the matrix

$$f(A_1, A_2) = A_1 A_2^T (A_2 A_2^T)^{-1} A_2 A_1^T (A_1 A_1^T)^{-1}$$

are the squares of the cosines of the angles of inclination.

5077. *Proposed by Hewitt Kenyon, George Washington University*

If f is a function and S is a set included in the domain of f, denote by S' the map $f(S)$ of S by f. If f has no fixed points, show that the domain of f can be partitioned into three pairwise disjoint sets A, B, C, none of which intersects its map: $A \cap B = B \cap C = C \cap A = A \cap A' = B \cap B' = C \cap C' = \varnothing$.

5078. *Proposed by Fred Suvorov, Princeton University*

In *Commutative Algebra*, v. I, p. 239, Zariski and Samuel assert that a principal ideal in a noetherian integral domain may have imbedded prime components. Find an example of such an ideal.

5079. *Proposed by Fred Suvorov, Princeton University*

In *Commutative Algebra*, v. I, p. 241, Zariski and Samuel assert that a prime ideal in a noetherian ring need not have finite depth: the depth of a prime ideal is the upper bound (if it exists) of the lengths of strictly ascending chains of prime ideals starting with the given one. Find an example of a prime ideal in a noetherian ring with infinite depth.

5080. *Proposed by A. P. Rollett, Crediton, England*

In the Fibonacci series ($F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$) the first, second and twelfth terms are squares. Are there any others?

5081. *Proposed by D. S. Adorno, California Institute of Technology*

a. Determine the solution to the partial difference equation with indicated boundary conditions: for all $i = 0, 1, \cdots, n+2$; $n = 0, 1, 2, \cdots$; $0 < \alpha < 1$,

$$A_{i,n+1} = A_{i,n} - \alpha^{n+1} A_{i-1,n}, \quad A_{-1,n} = 0, \quad A_{0,n} = 1, \quad A_{n+2,n} = 0.$$

b. Determine $\lim_{n \to \infty} A_{i,n}$.

SOLUTIONS

How to Tie up a Smooth Sphere

4887 [1960, 187; 1960, 1036]. *Proposed by Z. A. Melzak, McGill University*

What is the shortest length of infinitely strong, infinitely thin and perfectly inextensible string, with which a smooth sphere of radius 1 can be tied up so that the resulting parcel can be picked up and carried by the string without any danger of slipping apart?

Note by D. E. Daykin, Reading University, England. The former solution, while the answer is correct, leaves much to be desired in that it is not shown that a more complicated system will not permit a shorter piece of string.

However, the problem has been solved elsewhere. It was a contest problem to Cambridge undergraduates and was solved by R. Schwartzenberger of Trinity College. A less complicated solution by A. S. Besicovitch, *A net to hold a sphere*, can be found in Math. Gazette, 41 (1957), pp. 106–107.

Coefficients of a Power Series

4997 [1961, 1010]. *Proposed by D. J. Newman, Yeshiva University*

Show that the power series of $e^{(z+1)/(z-1)}$ has coefficients $O(n^{-3/4})$, but not $o(n^{-3/4})$.

Solution by Leonard Carlitz, Duke University. Put

$$e^{(z+1)/(z-1)} = e^{-1}e^{-2z/(1-z)} = \sum_{n=0}^{\infty} A_n z^n.$$

Comparing this with the generating function for the Laguerre polynomial:

$$(1-z)^{-\alpha-1}e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)z^n,$$

we get

$$A_n = e^{-1}L_n^{(-1)}(2). \tag{1}$$

Now it is known (see Szegö, *Orthogonal Polynomials*, p. 192) that

$$L_n^{(\alpha)}(x) = \pi^{-\frac{1}{2}}e^{\frac{1}{2}x}x^{-\frac{1}{2}\alpha-\frac{1}{4}}n^{\frac{1}{2}\alpha-\frac{1}{4}} \cdot \cos\left\{2\sqrt{nx} - \tfrac{1}{2}\alpha\pi + \tfrac{1}{4}\pi\right\} + O(n^{\frac{1}{2}\alpha-\frac{3}{4}}) \qquad (x > 0),$$

so that by (1)

$$A_n = (2\pi)^{-\frac{1}{2}}n^{-\frac{3}{4}}\cos(2\sqrt{2n} + \tfrac{1}{4}\pi) + O(n^{-\frac{5}{4}}). \tag{2}$$

It is evident from (2) that $A_n = O(n^{-3/4})$. On the other hand, since for irrational θ, the fractional part of $n\theta$ is everywhere dense in the interval (0, 1), it follows that if we take $n = 2m^2$ in (2) then for a certain subsequence $A_n \neq o(n^{-3/4})$. However, there exist subsequences for which $A_n = o(n^{-3/4})$.

Also solved by I. N. Baker, Robert Breusch, G. T. Cargo, R. G. Kayel, P. G. Rooney, and the proposer.

Editorial Note. J. Koekoek remarks that the solution was given in 1914 by Perron. For an asymptotic expansion of the coefficients see E. M. Wright, Journ. London Math. Soc., v. 7 (1932), 256–262.

Concatenation of Sequences

4998 [1961, 1010]. *Proposed by Meyer Wolf, University of Minnesota*

S. H. Unger has presented a proof concerning the self-overlapping of cyclically equivalent sequences (This MONTHLY, Feb. 1960, pp. 139–143), that depends on the property of being made up of a chain of identical subsequences. Prove that the composition of the sequence formed from the concatenation of two sequences X and Y is independent of the order of concatenation if and only if X and Y are chains of a particular subsequence Z.

Solution by D. L. Silverman, Beverly Hills, California. The sufficiency is obvious. To prove necessity, let $XY=YX$, where XY denotes the ordered concatenation of X and Y. Let $X=Z_1Z_2\cdots Z_m$ and $Y=Z_{m+1}Z_{m+2}\cdots Z_n$, where the Z_i are subsequences each of which has length equal to the greatest common divisor of the lengths of X and Y. Then $Z_i=Z_j$ when $j\equiv i+m \pmod n$ so that $Z_1=Z_{m+1}=Z_{2m+1}=\cdots=Z_{km+1}=\cdots$, where the subscripts are reduced mod n. Since $(m, n-m)=(m, n)=1$, the subscripts run through the complete set of residues mod n, *i.e.*, all the Z_i are identical. Hence X and Y are chains of the subsequence Z_1.

Also solved by the proposer.

Probability Measure for Sets of Positive Integers

4999 [1961, 1010]. *Proposed by R. B. Deal, Oklahoma State University*

Is there a finitely additive probability on the set of all subsets of the positive integers which assigns probability zero to all finite sets?

I. *Solution by W. C. Waterhouse, Harvard University.* Yes. Assigning 0 to finite sets and 1 to their complements defines a finitely additive measure on a subalgebra of the Boolean algebra of all the subsets. By a theorem of Tarski (*cf.* Birkhoff, *Lattice Theory*, p. 185), it can be extended to the entire algebra.

II. *Solution by R. L. Adler and A. G. Konheim, IBM Research, Yorktown Heights, N. Y.* There is. We construct the measure μ in the following manner. Let $\nu(B)$ be the number of elements in the set B. Define the sequence $\{\mu_n(B)\}_{n=1}^{\infty}$ by

$$\mu_n(B) = \frac{1}{n}\,\nu(B \cap \{1, 2, \cdots, n\}).$$

Now $\{\mu_n(B)\}$ is a bounded sequence for any subset B of the positive integers. Thus there exists a Banach limit $\mu(B)=\mathrm{LIM}_{n\to\infty}\,\mu_n(B)$ to the sequence $\{\mu_n(B)\}$. (See Dunford and Schwartz, *Linear Operators*, p. 43.) The set function μ so defined is clearly finitely additive and nonnegative. Moreover, whenever $\{\mu_n(B)\}$ is a convergent sequence then

$$\underset{n\to\infty}{\mathrm{LIM}}\ \mu_n(B) = \underset{n\to\infty}{\mathrm{limit}}\ \mu_n(B).$$

Therefore $\mu\{1, 2, \cdots\}=1$, and if B is any finite subset of $\{1, 2, \cdots\}$ then $\mu(B)=0$.

III. *Solution by Julian H. Blau, Antioch College.* In the ring of all sets of positive integers, let I be a proper prime ideal containing all finite sets. Let $P(E)=0$ for $E\in I$, and $P(E)=1$ for $E\notin I$. The existence of I is easily proved by Zorn's Lemma. The problem was solved by Tarski in Fund. Math., 15 (1930) 42–50. Sierpinski showed that a constructive solution of this problem would lead to a constructible nonmeasurable (Lebesgue) function. (See Fund. Math., 30 (1938) 96–99.)

Also solved by Robert Adams, W. A. Beyer, S. D. Chatterji, Z. Sidak, L. Sucheston, Albert Wilansky, and the proposer.

A Cubic Recurrence

5000 [1961, 1010]. *Proposed by R. C. Lyness, Singleton Lodge, near Blackpool, England*

$x^3-2x^2+2x-1=0$ has roots which are all roots of unity and so $w_{n+3}=2w_{n+2}-2w_{n+1}+w_n$ with $w_0=0$ is a cubic recurrence with an infinite number of zeros, w_{6n}.

$2x^3-2x^2+2x-1=0$ gives the recurrence $w_{n+3}=w_{n+2}-w_{n+1}+\frac{1}{2}w_n$ and with $w_0=w_1=0$ we have $w_m=0$ for $m=0, 1, 4, 6, 13, 52$.

Can a cubic recurrence with only a finite number of zeros have more than six?

Comments by Morgan Ward, California Institute of Technology.

Let $a_0x^3-a_1x^2+a_2x-a_3$ be a cubic polynomial with integral coefficients, $a_0a_3\neq 0$. Let the roots of this polynomial be α, β, γ. Let

$$(w): \qquad w_0, w_1, w_2, \cdots, w_n, \cdots$$

be an associated cubic recurrence defined by $a_0w_{n+3}=a_1w_{n+2}-a_2w_{n+1}+a_3w_n$, $n=0, 1, 2, \cdots$ with w_0, w_1, w_2 rational.

1. If none of the ratios $\alpha/\beta, \beta/\gamma, \gamma/\alpha$ is a root of unity, it follows from a theorem of Kurt Mahler's that $|w_n|\to\infty$ with n. Consequently the recurrence can have only a finite number of zeros in this case.

It is well known, and trivial, that if one of the ratios $\alpha/\beta, \beta/\gamma, \gamma/\alpha$ is a root of unity, the recurrence may have an infinite number of zeros. The proposer's first example shows this; x^3-1 is an even simpler example.

2. Smiley and I proved recently that if α, β, γ are all real, the recurrence can have at most three zeros.

3. Partial results of mine indicate that if α, β, γ are all integers, the recurrence can have at most two zeros if $\alpha+\beta+\gamma\neq 0$. (In the case $\alpha+\beta+\gamma=0$, $w_0=0=w_1$, $w_2=1$; w_3 is also 0.)

4. In extending a result of Carl Siegel's, I proved long ago that if $a_0a_3=\pm 1$, the recurrence (w) can have no more than six zeros.

The example $2x^3-2x^2+2x-1$ is consequently of considerable interest. It should be possible to show that (w) has no more than six zeros by p-adic methods in this particular case.

I think the conjecture of six zeros pretty well founded by virtue of 4., but conjecture counts for little in the Theory of Numbers.

Monotonic Subsequences

5001 [1962, 62]. *Proposed by Melvin Hausner, New York University*

A well-known result of Erdös is that a sequence of n^2+1 terms contains a monotonic subsequence of $n+1$ terms. Prove that if $pq<r$, then a sequence of r terms contains an increasing subsequence of $p+1$ terms or a decreasing subsequence of $q+1$ terms.

Solution by Marlow Sholander, Western Reserve University. If the increasing [resp., decreasing] subsequences which end with a given term have maximal length x [resp., y] we assign the point (x, y) to that term. Thus the first term gives $(1, 1)$, the second term gives $(1, 2)$ or $(2, 1)$ or $(2, 2)$ [the last in the case of equal terms], etc. Of two terms, the relative sizes insure that the latter has either a larger x coordinate or a larger y coordinate or both. Thus the points are distinct and in a sequence of more than pq terms we violate either $1\leqq x\leqq p$ or $1\leqq y\leqq q$.

Also solved by P. T. Bateman, G. M. Bergman, Jane Di Paola, Harley Flanders, Andrzej Makowski, M. M. McWaters, W. C. Waterhouse, A. Ziv, and the proposer.

Editorial Note. Several contributors found proofs in the literature. See Erdös and Szekeres, Compositio Mathematica 2 (1935), 463–470; also A. Seidenberg, Jour. London Math. Soc., 34 (1959), 352.

Composition of a Square Integrable Function

5002 [1962, 62]. *Proposed by I. I. Kolodner, University of New Mexico*

For every $a>0$, let the function $f: R^+\to R$ be square integrable on $[0, a]$, and let the composition $h\circ f$ be defined by $h\circ f(t)=\int_0^t f^2-(\int_0^t f)^2/t$. Show that

1. $h\circ f(t)\leqq A$ if and only if there is a number α such that $(f-\alpha)\in L_2(R^+)$;
2. if $f\in L_2(R^+)$ then as $t\to\infty$, $(\int_0^t f)^2/t\to 0$.

Solution by Brockway McMillan, Air Force Research and Development, Washington, D. C. Define $m(t)$ by $tm(t)=\int_0^t f$. Then

$$h\circ f(t)=\int_0^t [f(r)-m(t)]^2dr\geqq 0. \tag{1}$$

It is then clear that $h\circ(f(t)-\alpha)=h\circ f(t)$ for any constant α. From the inequality in (1) and the original definition of $h\circ f(t)$, one has

$$0\leqq h\circ f(t)\leqq\int_0^t f^2\leqq\int_0^\infty f^2. \tag{2}$$

If $f \in L_2(R^+)$ it follows that $0 \leqq h \circ f(t) \leqq A$, where A can be the right member of (2). If $f-\alpha \in L_2(R^+)$ for some constant α, write $f-\alpha$ for f in the argument above.

Conversely, suppose $h \circ f(t) \leqq A$. Fix a $T \geqq 0$ and consider $t \geqq T$. Since the integrand in (1) is nonnegative,

$$0 \leqq \int_0^T [f(s) - m(t)]^2 ds \leqq A. \tag{3}$$

It follows that $m(t)$, which appears in (3) simply as a constant, is bounded for all $t \geqq T$; therefore there is a sequence $t_n \to \infty$ such that $m(t_n)$ converges to a finite limit, say α. Considering (3) only for $t = t_n$, one has the limiting result

$$\int_0^T (f - \alpha)^2 \leqq A.$$

This being true for any chosen $T \geqq 0$, $f - \alpha \in L_2(R^+)$. This answers part 1.

From the inequality in (1) and the original definition of $h \circ f(t)$

$$0 \leqq tm^2(t) \leqq \int_0^t f^2.$$

If $f \in L_2(R^+)$ it follows that $m(t) \to 0$ as $t \to \infty$, and that

$$0 \leqq \limsup tm^2(t) = \beta < \infty. \tag{4}$$

Then, for any given $T \geqq 0$,

$$\int_0^\infty f^2 - \beta = \liminf h \circ f(t) \geqq \liminf \int_0^T (f - m(t))^2$$
$$= \lim \int_0^T (f - m(t))^2 = \int_0^T f^2.$$

This being true for any $T \geqq 0$, we have $\beta \leqq 0$ and from (4) then that $\beta = 0$. This establishes part 2.

Also solved by D. M. Friedlen, W. A. J. Luxemburg, and the proposer.

Cyclic Partitions

5003 [1962, 62]. *Proposed by D. R. Morrison, Sandia Corporation, Albuquerque, N. M.*

A crew of m men is to be partioned into teams, each team to include at least one man, and the teams to be arranged in cyclic order. Show that for $m > 1$, the number of ways in which this can be done using an even number of teams is equal to the number of ways it can be done using an odd number of teams. (Example: 3 men, say A, B, C. Even numbers of teams: (A, BC), (B, CA), (C, AB); odd number: (ABC), (A, B, C), (A, C, B).)

Solution by B. J. M. Morselt, Technical University, Eindhoven, Netherlands. Let $a(m, k)$ be the number of ways of putting m men into k teams, arranged in a cyclic order, with no team empty. We have to prove $\sum_{k=1}^{m}(-1)^k a(m, k)=0$. With $S(m, k)$ the Stirling number of the second kind, defined by

$$t_{(0)} = 1, \qquad t_{(k)} = t(t-1)\cdots(t-k+1) \quad \text{for} \quad k \geqq 1,$$

$$t^m = \sum_{k=0}^{m} S(m, k)t_{(k)} \quad \text{for} \quad m \geqq 0,$$

we have

(1) the number of ways of putting m men into k teams, with no team empty, is $S(m, k)$;

(2) $S(m, k)=S(m-1, k-1)+kS(m-1, k)$ for $1\leqq k\leqq m-1$.

(See Riordan, *An Introduction to Combinatory Analysis* (1958), p. 33, 99.) From this we have

$$\sum_{k=1}^{m}(-1)^k a(m, k) = \sum_{k=1}^{m}(-1)^k(k-1)!S(m, k)$$

$$= \sum_{k=1}^{m-1}(-1)^k(k-1)!\{S(m-1, k-1) + kS(m-1, k)\} + (-1)^m(m-1)!S(m, m)$$

$$= \sum_{k=0}^{m-2}(-1)^{k+1}k!S(m-1, k) + \sum_{k=0}^{m-1}(-1)^k k!S(m-1, k) + (-1)^m(m-1)!$$

$$= -S(m-1, 0) + (-1)^{m-1}(m-1)! + (-1)^m(m-1)!$$

$$= 0 \quad \text{for} \quad m \geqq 2.$$

Also solved by M. T. L. Bizley, Harley Flanders, Jim Shilleto, Michael Skalsky, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Introduction to Probability and Mathematical Statistics. By Z. W. Birnbaum. Harper, New York, 1962. 325 pp. $6.50.

Here is a text in probability and mathematical statistics "for students who have a firm grasp of calculus and some knowledge of the theory of matrices and determinants." The required level of calculus should include change of variables

R. L. Vaught, Denumerable models of complete theories. 18 pp.

Group V. Recursive Functions and Related Problems.

R. Fraïssé, Une notion de récursivité relative. 5 pp.
R. Péter, Über die Verallgemeinerung der Rekursionsbegriffe für abstrakte Mengen als Definitionsbereiche. 6 pp.
Gr. C. Moisil, Les logiques à plusieurs valeurs et l'automatique. 8 pp.
L. Kálmar, A practical infinitistic computer. 15 pp.

HENRY E. KYBURG, JR.
University of Denver

Inequalities. By Edwin F. Beckenbach and Richard Bellman. Ergebnisse der Mathematik und ihrer Grenzgebiete, New Series, volume 30, Springer-Verlag, Berlin, 1961. xi+198 pp. DM 48.60.

This is a monograph on topics that "have particularly delighted and intrigued" the authors, and "to the study of which" they "have contributed." Particular emphasis is given to inequalities related to positive definite matrices, characteristic roots, positive matrices, moment spaces, resonance theorems and the positivity of operators. The contents are presented in over two hundred short sections (many less than half a page in length). A skeletal outline of many fields is given, the reader being asked to consult original sources for details. A great many references are provided.

Particularly good is the discussion of the inequality between arithmetic and geometric means; twelve proofs are given that illustrate several important concepts. One of these proofs is attributed to Euler, but to the reviewer, it is at least suggested by remarks of Hardy, Littlewood and Pólya on page 20 of their treatise, *Inequalities*, Cambridge University Press, 1934.

The authors are to be commended for presenting Čaplygin's theorem on comparison of solutions of two differential equations (p. 139). This result has been exploited almost exclusively by Soviet mathematicians and has yet to find its way into U.S. texts on differential equations, although it appears in Lusin's *Calculus*.

The two chapters on matrices, characteristic roots, and moment spaces constitute the strong point of the book. Their pages are full of fascinating theorems.

The final chapter is on inequalities for differential operators. While many portions of other chapters, for example, the section on Gärding's Inequality, are relevant to the theory of partial differential equations, to head a section of one third of a page in this chapter "Partial Differential Equations" is misleading. The gap between the authors' recognition in the Preface of the role of inequalities in the modern theory of partial differential equations and their treatment of it in the text is disappointing.

This book is not particularly suited for the student. It is rather for the research mathematician. He will often find it an invaluable source.

NICHOLAS D. KAZARINOFF
The University of Michigan

that is not necessary, but it is also not the one that was claimed to be necessary. The treatment of empathic understanding and scientific induction is also conventional and in the reviewer's opinion superficial, as also is the failure to follow up on the suggestion of Dieudonnée and Wittgenstein that $n<n+1$ cannot be regarded as intuitive for very large n. But these are minor complaints about a very good book.

MICHAEL SCRIVEN
Center for Advanced Study
in the Behavioral Sciences
Stanford, California

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Brandeis University: Assistant Professors J. J. Kohn and R. S. Palais have been promoted to Associate Professors.

University of British Columbia: Professor G. de B. Robinson, University of Toronto, and Assistant Professor Bomshik Chang, Ewha Women's University, Korea, have been appointed Visiting Professors; Assistant Professors R. R. Christian and C. A. Swanson have been promoted to Associate Professors; Messrs. R. C. Thompson and A. H. Cayford have been promoted to Assistant Professors; Assistant Professor Maurice Sion is on leave of absence for 1962–63 and will visit Princeton University and the University of California, Berkeley; Assistant Professor A. H. Cayford is on leave of absence for 1962–63 at the National Aeronautics and Space Administration, Pasadena, California.

Carleton College: Assistant Professor S. F. Dice, Bucknell University, has been appointed Assistant Professor; Mr. P. S. Jorgensen has been promoted to Assistant Professor.

Central Michigan University: Associate Professor Joseph Csendes, Michigan State University, has been appointed Associate Professor; Associate Professor Wilbur Waggoner has been promoted to Professor.

University of Cincinnati: Dr. Roger Chalkley, Knox College, has been appointed Assistant Professor; Messrs. C. F. Pinzka and Cecil Craig, Jr. have been promoted to Assistant Professors.

Clarkson College: Messrs. E. M. Stone, University of Wisconsin, and E. A. Unwin, University of Arizona, have been appointed Assistant Professors; Associate Professor R. D. Larsson has been promoted to Professor; Associate Professor J. M. Perry has been awarded the 3rd Annual Clarkson College Excellent Teacher Award.

Fresno State College: Assistant Professor Burke Zane, Central Washington State College, has been appointed Assistant Professor; Assistant Professor T. C. Kipps has been promoted to Associate Professor.

University of Georgia: Assistant Professors S. E. Puckette, University of the South· and E. E. Grace, Emory University, have been appointed Visiting Associate Professors·

Institute for Advanced Study: Professors R. H. Bing, University of Wisconsin, S. S· Cairns, University of Illinois, Casper Goffman, Purdue University, R. C. James, Harvey Mudd College, A. H. Taub, University of Illinois, Associate Professors Robert Ellis, University of Pennsylvania, Anil Nerode, Cornell University, Marian B. Pour-El, Pennsylvania State University, Assistant Professor Louis Solomon, Haverford College, and Mr. R. P. Langlands, Princeton University, have been appointed Members in Mathematics for the academic year 1962–63.

University of Michigan: Dr. P. L. Duren, Stanford University, has been appointed Assistant Professor; Assistant Professor B. A. Galler has been promoted to Associate Professor; Messrs. R. D. Low and G. W. Hedstrom have been promoted to Assistant Professors.

University of Minnesota: Assistant Professors Alfred Aeppli and D. G. Aronson have been promoted to Associate Professors.

Montclair State College: Mr. W. R. Westphal, University of Illinois, has been appointed Assistant Professor; Associate Professor M. A. Sobel has been promoted to Professor; Assistant Professor A. J. Pettofrezzo has been promoted to Associate Professor.

Mount Holyoke College: Associate Professor W. H. Durfee has been promoted to Professor; Professor F. L. Kiokemeister is on sabbatical leave for 1962–63.

Ohio State University: Mr. J. A. Zilber, Associate Editor of "Mathematical Reviews," American Mathematical Society, has been appointed Assistant Professor; Assistant Professors Angelo Margaris and T. W. Hildebrandt have been promoted to Associate Professors; Associate Professor Margaret Jones retired with the title of Associate Professor Emeritus.

Tennessee Agricultural and Industrial State University: Dr. R. O. Abernathy, University of California, Berkeley, has been appointed Professor; Associate Professor Sadie Gasaway has been promoted to Professor.

University of Utah: Professor Emeritus T. H. Hildebrandt, University of Michigan, and Professor Alexander Peyerimhoff, University of Marburg, Germany, have been appointed Visiting Professors; Associate Professor E. O. Nelson, University of North Dakota, has been appointed Assistant Professor; Professor E. A. Davis has returned from leave of absence at the National Science Foundation.

Dr. N. S. Andrews, Clemson College, has been appointed Assistant Professor at Southeastern Louisiana College.

Dr. C. E. Antle, Oklahoma State University, has been appointed Associate Professor at Missouri School of Mines.

Assistant Professor George Bachman, Polytechnic Institute of Brooklyn, has been promoted to Associate Professor.

Mr. A. F. Beninati, Nyack High School, Nyack, New York, has been appointed Associate Professor at the State University of New York College at New Paltz.

Assistant Professor D. C. Benson, University of California, Davis, has been promoted to Associate Professor.

Dr. L. J. Berenson, Hofstra College, has been appointed Associate Professor at Nassau Community College.

Associate Professor T. A. Botts, University of Virginia, has been appointed Visiting Professor at the University of Puerto Rico.

Dr. Harold Chatland, Sylvania Corporation of America, Mountain View, California, has been appointed Professor at Western Washington State College.

Assistant Professor G. L. Curme, Illinois Institute of Technology, has been appointed Assistant Professor at St. Mary's College.

Mr. J. W. Daniels, University of Illinois, has been appointed Assistant Professor.

Dr. Raymond Doby, Westinghouse Electric Corporation, Philadelphia, Pennsylvania, has accepted a position as Senior Staff Scientist with the AVCO Corporation, Wilmington, Massachusetts.

Professor Meyer Dwass, University of Minnesota, has been appointed Professor at Northwestern University.

Mr. E. LaV. Eagle, Martin Company, Denver, Colorado, has accepted the position as Reliability Staff Engineer with the Missile Division of Chrysler Corporation, Warren, Michigan.

Dr. J. A. Fickes, University of Southern Florida, has been appointed Assistant Professor at Quinnipiac College.

Dr. C. H. Frick, U. S. Naval Weapons Laboratory, Dahlgren, Virginia, has been appointed Professor at the University of South Dakota.

Professor B. H. Gere, Hamilton College, is on leave during the academic year 1962-63 at the U. S. Naval Postgraduate School.

Dr. Leon Gleiberman, Brooklyn College, has accepted a position as Program Mathematician with Service Bureau Corporation, New York, New York.

Mr. A. R. Goddard, Tarleton State College, has been promoted to Assistant Professor.

Mr. M. L. Goemaat, Chico State College, has been promoted to Assistant Professor.

Mr. V. W. Goldsberry, University of Nebraska, has accepted a position as Mathematician with the Phillips Petroleum Company, Idaho Falls, Idaho.

Associate Professor Lillian Gough, Wisconsin State College, River Falls, has been promoted to Professor.

Mr. P. M. Grabarkewitz, Bemidji State College, has been promoted to Assistant Professor.

Mr. G. B. Grunwald, Minot State Teachers College, has been appointed Assistant Professor at Ball State Teachers College.

Associate Professor M. C. Hartley, University of Illinois, has been appointed Professor and Chairman of the Department of Mathematics at the University of Tampa.

Dr. K. L. Hillam, University of Colorado, has been appointed Assistant Professor at Brigham Young University.

Dr. R. E. Horton, Assistant Dean of Educational Services, Los Angeles City College, has been appointed Dean.

Dr. H. F. Hunter, Rensselaer Polytechnic Institute, has been promoted to Assistant Professor.

Assistant Professor A. A. Johnson, Ohio Wesleyan University, has been appointed Associate Professor at the University of Toledo.

Associate Professor Constantine Kassimatis, North Carolina State University, has been appointed Associate Professor at Assumption University.

Associate Professor Ali Kyrala, Arizona State University, has been appointed Professor at the U. S. Naval Postgraduate School.

Associate Professor W. D. Lindstrom, Kenyon College, has received a NSF Faculty Fellowship for 1962–63 and will study at the University of California, Berkeley.

Assistant Professor G. F. Lowerre, Washington and Jefferson College, has been appointed Assistant Professor at Lake Erie College.

Dr. D. O. McKay, University of Western Ontario, has been promoted to Assistant Professor.

Rev. J. J. MacDonnell, S.J., College of the Holy Cross, has been promoted to Assistant Professor.

Dr. A. M. Peiser, M. W. Kellogg Company, New York, New York, has accepted a

position as Senior Associate in the Engineering Department of the Socony Mobil Oil Company, New York, New York,

Professor W. A. Pierce, West Virginia University, has been appointed Visiting Associate Professor at Oklahoma State University.

Mr. A. W. Ransom, Purdue University, has accepted a position as Senior Mathmatician with the Paul Moore Research Center of the Republic Aviation Corporation, Farmingdale, Long Island.

Mr. E. W. Robinson, University of Wichita, has been appointed Assistant Professor at Nebraska Wesleyan University.

Mr. H. S. Schloss, Burroughs Corporation, Pasadena, California, has accepted a position as Senior Research Engineer with North American Aviation, Downey, California.

Associate Professor R. J. Silverman, Illinois Institute of Technology, has been appointed Professor at the University of New Hampshire.

Mr. L. J. Simonoff, University of Massachusetts, has been appointed Assistant Professor at Idaho State College.

Mr. R. D. Sinkhorn, University of Wisconsin, has been appointed Assistant Professor at the University of Houston.

Sister Margaret Ann Raphael, Immaculate Heart College, has been promoted to Assistant Professor.

Sister Marion Beiter, Rosary Hill College, has been promoted to Associate Professor.

Sister Patricia Anne, College of Notre Dame, has been promoted to Assistant Professor.

Mr. M. J. Smith, Norden Laboratories, Stamford, Connecticut, has accepted a position as Reliability Statistician with the Grumman Aircraft Engineering Corporation, Bethpage, New York.

Mr. M. E. Sperline, Colorado State University, has been promoted to Assistant Professor.

Associate Professor O. E. Stanaitis, St. Olaf College, has been promoted to Professor.

Mr. E. F. Stiel, University of California, Los Angeles, has been appointed Assistant Professor at Orange State College.

Associate Professor Konrad Suprunowicz, Utah State University, has been appointed Visiting Associate Professor at the University of Nebraska.

Associate Professor L. R. Tappan, Nicholls State College, has been appointed Associate Professor at Northern Michigan College.

Mr. J. A. Voytuk, Carnegie Institute of Technology, has been appointed Assistant Professor at Western Reserve University.

Mr. R. P. Walker, Kent State University, has accepted a position as Mathematician with the Research Center of the Babcock and Wilcox Company, Alliance, Ohio.

Dr. D. W. Willett, California Institute of Technology, has been appointed Assistant Professor at the University of Alberta.

Associate Professor Louise Wolf, University of Wisconsin, retired January 1962 with the title of Associate Professor Emeritus.

Mr. Gregory Wulczyn, Bucknell University, has been promoted to Assistant Professor.

Associate Professor A. H. Zemanian, New York University, has been appointed Professor at the State University of New York at Stony Brook.

Mr. J. J. Aeberly, Bureau of Heating Ventilation and Industrial Sanitation, Chicago, Illinois, died June 18, 1962. He was a member of the Association for 4 years.

Mr. J. B. Geiser, Brookline, Massachusetts, died September 26, 1962. He was a member of the Association for 4 years.

Professor Emeritus B. M. Turner, West Virginia University, died September 5, 1962. She was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE 1962 HIGH SCHOOL MATHEMATICS CONTEST

There was a modest increase in the number of students participating in the Annual High School Mathematics Contest sponsored by the Mathematical Association of America and the Society of Actuaries, from approximately 160,000 in 1961 to 170,000 in 1962, with the number of schools remaining constant at about 5300. Registrations of APO (Army) and FPO (Navy) schools increased from fourteen (1961) to nineteen.

In rounded figures the overall median team score rose from 75 in 1961 to 91, surpassing even the 1960 median by 2 points, and the overall median highest individual score rose from 31 in 1961 to 38, and again surpassing the 1960 median by 4 points. These increases are reflected in the corresponding changes in each of the percentiles listed in the Summary (available upon request), namely, 75, 50, and 25, for each of the ten regions. Assuming that the body of participants over the years is a pedagogical constant, we conclude that we have, this year, partly succeeded in our consciously-set goal of easing the examination with respect to Part I for the avowed purpose of attracting and holding the smaller schools. In pursuing our objective of appealing to a very wide spectrum of student abilities we must necessarily have the problems in the examination cover the full gamut from very easy to strongly challenging.

For the first time in the history of this activity there were two perfect scores (150) by Michael J. Razar, Forest Hills High School, Queens, New York and Barry M. Simon, James Madison High School, Brooklyn, New York. In team performance Forest Hills High School (Queens) ranked first with a score of 391.25 out of a possible 450, and Bronx High School of Science (Bronx) second with a score of 386.25.

With nothing more on our part than a willingness to co-operate, international participation last year comprised 1325 students in 36 schools in the Netherlands, 300 students in England, and, for the first time, 45 students in Luxembourg.

The 1963 examination will again pose a "Modern Mathematics" problem in dual wording, one classic and one modern. As the curricula in the secondary schools indicate acceptance of the newer content and the newer points of view, increased emphasis will be given to them in the examination.

The demand for career brochures by the schools and by some of the contest chairmen continues very strong. The distribution last year was a cooperative venture with the National Council of Teachers of Mathematics and the National Academy of Sciences-National Research Council.

CHARLES T. SALKIND, *Chairman*
Committee on High School Contests

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York 53, N. Y.

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WHAT DOES THE SPECTRAL THEOREM SAY?

P. R. HALMOS, University of Michigan

Most students of mathematics learn quite early and most mathematicians remember till quite late that every Hermitian matrix (and, in particular, every real symmetric matrix) may be put into diagonal form. A more precise statement of the result is that every Hermitian matrix is unitarily equivalent to a diagonal one. The spectral theorem is widely and correctly regarded as the generalization of this assertion to operators on Hilbert space. It is unfortunate therefore that even the bare statement of the spectral theorem is widely regarded as somewhat mysterious and deep, and probably inaccessible to the nonspecialist. The purpose of this paper is to try to dispel some of the mystery.

Probably the main reason the general operator theorem frightens most people is that it does not obviously include the special matrix theorem. To see the relation between the two, the description of the finite-dimensional situation has to be distorted almost beyond recognition. The result is not intuitive in any language; neither Stieltjes integrals with unorthodox multiplicative properties, nor bounded operator representations of function algebras, are in the daily toolkit of every working mathematician. In contrast, the formulation of the spectral theorem given below uses only the relatively elementary concepts of measure theory. This formulation has been part of the oral tradition of Hilbert space for quite some time (for an explicit treatment see [6]), but it has not been called the spectral theorem; it usually occurs in the much deeper "multiplicity theory." Since the statement uses simple concepts only, this aspect of the present formulation is an advantage, not a drawback; its effect is to make the spirit of one of the harder parts of the subject accessible to the student of the easier parts.

Another reason the spectral theorem is thought to be hard is that its proof is hard. An assessment of difficulty is, of course, a subjective matter, but, in any case, there is no magic new technique in the pages that follow. It is the statement of the spectral theorem that is the main concern of the exposition, not the proof. The proof is essentially the same as it always was; most of the standard methods used to establish the spectral theorem can be adapted to the present formulation.

Let ϕ be a complex-valued bounded measurable function on a measure space X with measure μ. (All measure-theoretic statements, equations, and relations, e.g., "ϕ is bounded," are to be interpreted in the "almost everywhere" sense.) An operator A is defined on the Hilbert space $\mathfrak{L}^2(\mu)$ by

$$(Af)(x) = \phi(x)f(x), \qquad x \in X;$$

the operator A is called the *multiplication* induced by ϕ. The study of the relation between A and ϕ is an instructive exercise. It turns out, for instance, that

the adjoint A^* of A is the multiplication induced by the complex conjugate $\bar{\phi}$ of ϕ. If ψ also is a bounded measurable function on X, with induced multiplication B, then the multiplication induced by the product function $\phi\psi$ is the product operator AB. It follows that a multiplication is always normal; it is Hermitian if and only if the function that induces it is real. (For the elementary concepts of operator theory, such as Hermitian operators, normal operators, projections, and spectra, see [**3**]. For present purposes a concept is called elementary if it is discussed in [**3**] before the spectral theorem, i.e., before p. 56.)

As a special case let X be a finite set (with n points, say), and let μ be the "counting measure" in X (so that $\mu(\{x\})=1$ for each x in X). In this case $\mathcal{L}^2(\mu)$ is n-dimensional complex Euclidean space; it is customary and convenient to indicate the values of a function in $\mathcal{L}^2(\mu)$ by indices instead of parenthetical arguments. With this notation the action on f of the multiplication A induced by ϕ can be described by

$$A\langle f_1, \cdots, f_n\rangle = \langle \phi_1 f_1, \cdots, \phi_n f_n\rangle.$$

To say this with matrices, note that the characteristic functions of the singletons in X form an orthonormal basis in $\mathcal{L}^2(\mu)$; the assertion is that the matrix of A with respect to that basis is diag $\langle\phi_1, \cdots, \phi_n\rangle$.

The general notation is now established and the special role of the finite-dimensional situation within it is clear; everything is ready for the principal statement.

SPECTRAL THEOREM. *Every Hermitian operator is unitarily equivalent to a multiplication.*

In complete detail the theorem says that if A is a Hermitian operator on a Hilbert space $\mathcal{H}$, then there exists a (real-valued) bounded measurable function ϕ on some measure space X with measure μ, and there exists an isometry U from $\mathcal{L}^2(\mu)$ onto $\mathcal{H}$, such that

$$(U^{-1}AUf)(x) = \phi(x)f(x), \qquad x \in X,$$

for each f in $\mathcal{L}^2(\mu)$. What follows is an outline of a proof of the spectral theorem, a brief discussion of its relation to the version involving spectral measures, and an illustration of its application.

Three tools are needed for the proof of the spectral theorem.

(1) *The equality of norm and spectral radius.* If the spectrum of A is $\Lambda(A)$, then the *spectral radius* $r(A)$ is defined by

$$r(A) = \sup\{|\lambda| : \lambda \in \Lambda(A)\}.$$

It is always true that $r(A) \leq \|A\|$ ([**3**, Theorem 2, p. 52]); the useful fact here is that if A is Hermitian, then $r(A)=\|A\|$ ([**3**, Theorem 2, p. 55]).

(2) *The Riesz representation theorem for compact sets in the line.* If L is a positive linear functional defined for all real-valued continuous functions on a com-

pact subset X of the real line, then there exists a unique finite measure μ on the Borel sets of X such that

$$L(f) = \int f d\mu$$

for all f in the domain of L. (To say that L is linear means of course that

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$

whenever f and g are in the domain of L and α and β are real scalars; to say that L is positive means that $L(f) \geqq 0$ whenever f is in the domain of L and $f \geqq 0$.) For a proof, see [**4**, Theorem D, p. 247].

(3) *The Weierstrass approximation theorem for compact sets in the line.* Each real-valued continuous function on a compact subset of the real line is the uniform limit of polynomials. For a pleasant elementary discussion and proof see [**1**, p. 102].

Consider now a Hermitian operator A on a Hilbert space $\mathcal{H}$. A vector ξ in $\mathcal{H}$ is a *cyclic vector* for A if the set of all vectors of the form $q(A)\xi$, where q runs over polynomials with *complex* coefficients, is dense in $\mathcal{H}$. Cyclic vectors may not exist, but an easy transfinite argument shows that $\mathcal{H}$ is always the direct sum of a family of subspaces, each of which reduces A, such that the restriction of A to each of them does have a cyclic vector. Once the spectral theorem is known for each such restriction, it follows easily for A itself; the measure spaces that serve for the direct summands of $\mathcal{H}$ have a natural direct sum, which serves for $\mathcal{H}$ itself. Conclusion: there is no loss of generality in assuming that A has a cyclic vector, say ξ.

For each real polynomial p write

$$L(p) = (p(A)\xi, \xi).$$

Clearly L is a linear functional; since

$$\begin{aligned} |L(p)| &\leqq \|p(A)\| \cdot \|\xi\|^2 = r(p(A)) \cdot \|\xi\|^2 \\ &= \sup \{ |\lambda| : \lambda \in \Lambda(p(A)) \} \cdot \|\xi\|^2 \\ &= \sup \{ |p(\lambda)| : \lambda \in \Lambda(A) \} \cdot \|\xi\|^2, \end{aligned}$$

the functional L is bounded for polynomials. (The last step uses the spectral mapping theorem; cf. [**3**, Theorem 3, p. 55].) It follows (by the Weierstrass theorem) that L has a bounded extension to all real-valued continuous functions on $\Lambda(A)$. To prove that L is positive, observe first that if p is a real polynomial, then

$$((p(A))^2\xi, \xi) = \|p(A)\xi\|^2 \geqq 0.$$

If f is an arbitrary positive continuous function on $\Lambda(A)$, then approximate $\sqrt{f}$ uniformly by real polynomials; the inequality just proved implies that $L(f) \geqq 0$

(since f is then uniformly approximated by squares of real polynomials). The Riesz theorem now yields the existence of a finite measure μ such that

$$(p(A)\xi, \xi) = \int p d\mu$$

for every real polynomial p.

For each (possibly complex) polynomial q write

$$Uq = q(A)\xi.$$

Since A is Hermitian, $(q(A))^*(=\bar{q}(A))$ is a polynomial in A, and so is $(q(A))^*q(A)$ $(=|q|^2(A))$; it follows that

$$\int |q|^2 d\mu = (\bar{q}(A)q(A)\xi, \xi) = ((q(A))^*q(A)\xi, \xi) = \|q(A)\xi\|^2 = \|Uq\|^2.$$

This means that the linear transformation U from a dense subset of $\mathcal{L}^2(\mu)$ into $\mathcal{H}$ is an isometry, and hence that it has a unique isometric extension that maps $\mathcal{L}^2(\mu)$ into $\mathcal{H}$. The assumption that ξ is a cyclic vector implies that the range of U is in fact dense in, and hence equal to, the entire space $\mathcal{H}$.

It remains only to prove that the transform of A by U is a multiplication. Write $\phi(\lambda)=\lambda$ for all λ in $\Lambda(A)$. Given a complex polynomial q, write $\tilde{q}(\lambda)=\lambda q(\lambda)=\phi(\lambda)q(\lambda)$; then

$$U^{-1}AUq = U^{-1}Aq(A)\xi = U^{-1}\tilde{q}(A)\xi = U^{-1}U\tilde{q} = \tilde{q}.$$

In other words $U^{-1}AU$ agrees, on polynomials, with the multiplication induced by ϕ, and that is enough to conclude that $U^{-1}AU$ is equal to that multiplication. This completes the outline of the proof of the spectral theorem for Hermitian operators.

The formulation of the spectral theorem given above yields fairly easily all the information contained in the more common versions. Thus if A is the multiplication on $\mathcal{L}^2(\mu)$ induced by the real function ϕ on X, and if F is a (complex) Borel measurable function that is bounded on $\Lambda(A)$, then $F(A)$ can be defined as the multiplication induced by the composite function $F\circ\phi$. The mapping $F\rightarrow F(A)$ is the homomorphism that is frequently known by the impressive name of "the functional calculus." If, in particular, $F=F_M$ is the characteristic function of a Borel set M in the real line, and if $E(M)$ is the multiplication induced by $F_M\circ\phi$, then E is the spectral measure of A. The verification that E is indeed a spectral measure is easy. To prove that it belongs to A (i.e., that $A=\int\lambda dE(\lambda)$), proceed as follows. Fix f and g in $\mathcal{L}^2(\mu)$ and write

$$\nu(M) = (E(M)f, g)$$

for each Borel set M; it is to be proved that

$$(Af, g) = \int \lambda d\nu(\lambda).$$

Since $(E(M)f, g) = \int (F_M \circ \phi) f\bar{g} d\mu$ and $\nu(M) = \int F_M d\nu$, it follows that

$$\int (F_M \circ \phi) f\bar{g} d\mu = \int F_M d\nu$$

for all Borel sets M. This implies that

$$\int (F \circ \phi) f\bar{g} d\mu = \int F d\nu$$

whenever F is a simple function, and hence, by approximation, whenever F is a bounded Borel measurable function. This conclusion (for $F(\lambda) \equiv \lambda$) is just what was wanted.

The multiplication version of the spectral theorem implies the spectral measure version, but the latter is canonical (E is uniquely determined by A) whereas the former is not. Consider, for instance, the identity operator on a separable infinite-dimensional Hilbert space in the role of A. It is unitarily equivalent to multiplication by the constant function 1 on, say, the unit interval (with Lebesgue measure); it is also unitarily equivalent to multiplication by the constant function 1 on the set of positive integers (with the counting measure).

There is a spectral theorem for normal operators also; its statement can be obtained from the one given above by substituting "normal" for "Hermitian." It is a well-known technical nuisance that the proof of the spectral theorem for normal operators involves some difficulties that do not arise in the Hermitian case. The source of the trouble is that it is not enough just to replace polynomials in a real variable by polynomials in a complex variable; the Weierstrass theorem demands the consideration of polynomials in two real variables. There is a consequent difficulty in extending the spectral mapping theorem to the kind of functions (polynomials in the real and imaginary parts of a complex variable) that arise in the imitation of the proof above. Even the equality of norm and spectral radius, while true for normal operators, requires a proof quite a bit deeper than in the real case. One way around all this is not to imitate the proof but to use the result. In [**3**, p. 72], for instance, the spectral theorem for normal operators (spectral measure version) is derived from the Hermitian theorem (spectral measure version); the only additional tool needed is an essentially classical extension theorem for measures in the plane.

In any case, all this talk about proof is somewhat beside the point in this paper. The reason a proof is outlined above is not so much to induce belief in the result as to clarify it. The emphasis here is not on *how* but on *what*, not on *proof* but on *statement*, not on *How does the spectral theorem come about*? but on *What does the spectral theorem say*?

To see how the multiplication point of view can be used, consider the Fuglede commutativity theorem [**2**]. A possible statement is this: if A is normal and if B is an operator that commutes with A, then B commutes with $F(A)$ for each Borel measurable function bounded on $\Lambda(A)$. (An alternative state-

ment, only apparently weaker, is that if B commutes with A, then B commutes with A^*; for a recent elegant proof see [5].) The spectral theorem shows that there is no loss of generality in assuming that A is the multiplication induced by ϕ, say, on a measure space X with measure μ. If F_M is, for each Borel set M in the complex plane, the characteristic function of M, and if $E(M)$ is the multiplication induced by $F_M \circ \phi$, then it is sufficient to prove that B commutes with each $E(M)$. (Approximate the general F by simple functions, as before.) If $\mathcal{E}(M)$ is the range of the projection $E(M)$, then the desired result is that $\mathcal{E}(M)$ reduces B, but it is, in fact, sufficient to prove that $\mathcal{E}(M)$ is invariant under B. Reason: apply the invariance conclusion, once obtained, to the complement of M, and infer that both $\mathcal{E}(M)$ and $(\mathcal{E}(M))^\perp$ are invariant under B.

Observe now that $\mathcal{E}(M)$ is the set of all those functions in $\mathcal{L}^2(\mu)$ that vanish outside $\phi^{-1}(M)$, and consider first the case of the closed unit disc,

$$M = \{\lambda: |\lambda| \leqq 1\};$$

then

$$\phi^{-1}(M) = \{x: |\phi(x)| \leqq 1\}.$$

Assertion: $\mathcal{E}(M)$ consists of all f in $\mathcal{L}^2(\mu)$ for which the sequence $\{\|Af\|, \|A^2f\|, \|A^3f\|, \cdots\}$ is bounded. Indeed, if f vanishes outside $\phi^{-1}(M)$, then

$$\|A^nf\|^2 = \int |\phi^n f|^2 d\mu = \int_{\phi^{-1}(M)} |\phi^n|^2 \cdot |f|^2 d\mu \leqq \int |f|^2 d\mu.$$

If, on the other hand, there is a set S of positive measure on which $f \neq 0$ and $|\phi| > 1$, then

$$\|A^nf\|^2 = \int |\phi^n f|^2 d\mu \geqq \int_S |\phi^2|^n |f|^2 d\mu \to \infty.$$

The assertion is proved, and the invariance of $\mathcal{E}(M)$ under B follows: if $\|A^nf\| \leqq c$ for all n, then $\|A^nBf\| = \|BA^nf\| \leqq \|B\| \cdot \|A^nf\| \leqq \|B\| \cdot c$ for all n.

If M is any closed disc, $M = \{\lambda: |\lambda - \lambda_0| \leqq r\}$, then

$$\phi^{-1}(M) = \{x: |\phi(x) - \lambda_0| \leqq r\} = \left\{x: \left|\left(\frac{\phi - \lambda_0}{r}\right)(x)\right| \leqq 1\right\}.$$

Since B commutes with multiplication by ϕ, it commutes with multiplication by $(\phi - \lambda_0)/r$ also, and it follows from the preceding paragraph that $\mathcal{E}(M)$ is invariant under B.

The rest of the proof is easy measure theory; from this point of view spectral measures behave even better than numerical measures. Since $\mathcal{E}(M)$ is invariant under B whenever M is a disc, the same is true whenever M is the union of countably many discs. This implies that $\mathcal{E}(M)$ is invariant under B whenever M is open, and hence (regularity) for arbitrary Borel sets M.

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AN EXTENSION OF THE FERMAT THEOREM

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1. If p is a prime, $e \geqq 1$ and $(a, p)=1$ then

$$a^w \equiv 1 \pmod{p^e} \tag{1}$$

provided $p^{e-1}(p-1) \mid w$. It follows from (1) that

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a^{n+sw} = a^n(a^w - 1)^r \equiv 0 \pmod{p^{re}} \tag{2}$$

for all $r \geqq 0$. This congruence is useful for example in deriving Kummer's congruence for the Euler and Bernoulli numbers [1, Ch. 14].

It may be of interest to examine the sum

$$\Delta^r a^{n^k} = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a^{(n+sw)^k}, \tag{3}$$

where k is a fixed integer $\geqq 1$. We remark that (3) is suggested by some recent work [2] on colored graphs. We shall prove

THEOREM 1. *Let $p^{e-1}(p-1) \mid w$ and let $\lambda \geqq e$ be the largest integer such that*

$$a^w \equiv 1 \pmod{p^\lambda}. \tag{4}$$

Then

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} a^{(n+sw)^k} \equiv 0 \pmod{p^{\lambda r_k}}, \tag{5}$$

where

$$r_k = [(r + k - 1)/k], \tag{6}$$

the greatest integer $\leqq (r+k-1)/k$.

2. The proof of the theorem depends upon the following

LEMMA. *Let*

$$f(x) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} x^{c_1 s + \cdots + c_k s^k}, \tag{7}$$

where the c_j are arbitrary nonnegative integers and $k \geqq 1$. Then

$$f(x) = (x-1)^{r_k} g(x), \tag{8}$$

where $g(x)$ is a polynomial with integral coefficients. Moreover, if $r = kt$, then

$$g(1) = r! c_k^t / t! \tag{9}$$

Proof. Clearly for $r \geqq 1$ we have

$$f(1) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} = 0.$$

Let $1 \leqq j < r_k$. Then for the jth derivative we have

$$D^j f(1) = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \prod_{i=0}^{j-1} (c_1 s + \cdots + c_k s^k - i). \tag{10}$$

We may put

$$\begin{aligned}\prod_{i=0}^{j-1} (c_1 s + \cdots + c_k s^k - i) = A_0^{(j)} + A_1^{(j)} s + A_2^{(j)} s(s-1) + \cdots \\ + A_m^{(j)} s(s-1) \cdots (s-m+1),\end{aligned} \tag{11}$$

where

$$m = jk \leqq (r_k - 1)k < r \tag{12}$$

and the $A_i^{(j)}$ are integers. Then (10) becomes

$$\begin{aligned} D^j f(1) &= \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} \sum_{i=0}^{m} A_i^{(j)} s(s-1) \cdots (s-i+1) \\ &= \sum_{i=0}^{m} A_i^{(j)} r(r-1) \cdots (r-i+1) \sum_{s=i}^{r} (-1)^{r-s} \binom{r-i}{s-i}. \end{aligned} \tag{13}$$

Hence by (12)

$$D^j f(1) = 0 \qquad (0 \leqq j < r_k)$$

and (8) follows at once.

In the next place if $r = kt$, so that $t = r_k$, we take (10) with $j = t$. Then (13) becomes

$$D^t f(1) = \sum_{i=0}^{r} A_i^{(t)} r(r-1)\cdots(r-i+1) \sum_{s=i}^{r} (-1)^{r-s} \binom{r-i}{s-i},$$

which reduces to

$$D^t f(1) = r! A_r^{(t)}. \tag{14}$$

But since (11) is an identity in s, it is clear that

$$A_r^{(t)} = c_k^t.$$

Therefore (14) becomes

$$D^t f(1) = r! c_k^t. \tag{15}$$

By (8) we have $D^t f(1) = t! g(1)$, so that

$$t! g(1) = r! c_k^t. \tag{16}$$

This evidently proves (9).

3. It is now easy to prove Theorem 1. Indeed by (3) we have

$$\Delta^r a^{n^k} = a^{n^k} \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (a^w)^{e_s},$$

where

$$e_s = \frac{(n+sw)^k - n^k}{w} = \sum_{j=1}^{k} \binom{k}{j} n^{k-j} s^j w^{j-1}. \tag{17}$$

Applying (8) we get

$$\Delta^r a^{n^k} = a^{n^k} (a^w - 1)^{rk} g(a^w).$$

Therefore (4) implies

$$\Delta^r a^{n^k} \equiv 0 \pmod{p^{\lambda rk}}, \tag{18}$$

so that we have proved (5).

4. Theorem 1 can be improved by making use of (9). It is clear from (4) that

$$g(a^w) \equiv g(1) \pmod{p^\lambda}. \tag{19}$$

Then if $r = kt$, it follows from (9), (17) and (19) that

$$g(a^w) \equiv \frac{r!}{t!} w^{(k-1)t} \pmod{p^\lambda}. \tag{20}$$

Thus (18) becomes

$$\Delta^r a^{n^k} \equiv 0 \pmod{p^{\lambda t + \min(\lambda,\mu)}}, \tag{21}$$

where p^μ is the highest power of p dividing

$$\frac{r!}{t!}\, p^{(e-1)(r-t)}. \tag{22}$$

Since by (3)

$$\Delta^{r+1} a^{n^k} = \Delta^r a^{(n+w)^k} - \Delta^r a^{n^k}, \tag{23}$$

we can remove the restriction $k|r$ in (21). We state

THEOREM 2. *If the hypotheses of Theorem* 1 *are satisfied then*

$$\Delta^r a^{n^k} \equiv 0 \pmod{p^{\lambda rk+\min(\lambda,\mu)}}, \tag{24}$$

where p^μ is the highest power of p dividing (22). *In particular if $k>1$ then*

$$\Delta^r a^{n^k} \equiv 0 \pmod{p^{\lambda rk+e-1}}. \tag{25}$$

5. It follows from (20) that for $k>1$, $g(a^w)$ is prime to p if and only if $e=1$ and $p\nmid r!/t!$. Thus when these conditions are satisfied $\Delta^{kt}a^{n^k}$ is divisible by exactly $p^{\lambda t}$.

For arbitrary r we make use of (23). If we assume that

$$\Delta^r a^{n^k} \equiv 0 \pmod{p^{\lambda rk+1}} \qquad (n = 0, 1, 2, \cdots), \tag{26}$$

where $p\nmid r$, it follows from (6) that

$$\Delta^{r+1} a^{n^k} \equiv 0 \pmod{p^{\lambda rk+1}}.$$

This yields

THEOREM 3. *Let $k>1$ and $r\geqq 1$. Then the congruence*

$$\Delta^r a^{n^k} \equiv 0 \pmod{p^{\lambda rk}} \tag{27}$$

is best possible if and only if $e=1$ and $(r!/r_k!)\not\equiv 0$, $(\bmod\ p)$.

By the statement that (27) is best possible is meant that (26) does not hold for all $n\geqq 0$. When $k|n$ we can assert that (26) holds for no n.

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ON AN INEQUALITY FOR THE GENERALIZED ARITHMETIC AND GEOMETRIC MEANS

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0. Introduction. The Hölder inequality suggests the set function

$$H(E) \equiv \left\{\int_E |\phi_1|^p dx\right\}^{1/p} \left\{\int_E |\phi_2|^q dx\right\}^{1/q} - \left|\int_E \phi_1\phi_2 dx\right|. \tag{0.0}$$

It was shown in [2], under suitable conditions, that if $E' \subset E$ then

$$H(E) - H(E') \geqq H(E - E') \geqq 0. \tag{0.1}$$

This result may be written in the equivalent form

$$H(E_1 \cup E_2) \geqq H(E_1) + H(E_2) \tag{0.2}$$

provided that the sets E_1 and E_2 are disjoint.

It is natural to ask if there are similar properties for other well-known inequalities. It is the purpose of this note to indicate such properties for the generalized arithmetic and geometric means.

If $k>1$ and $\{a_n;\ 1 \leqq n \leqq k\}$ is a set of nonnegative numbers, it may be possible to obtain a result analogous to (0.1) by considering $\rho^*(p)$, $1 \leqq p \leqq k$, where

$$\rho^*(p) \equiv p^{-1}\sum_{n=1}^{p} a_n - \left\{\prod_{n=1}^{p} a_n\right\}^{1/p}.$$

It is clear that $\rho^*(p) \geqq 0$ but it may be seen from examples that ρ^* is not, in general, monotonic in p. However if $\rho(p)$ is defined by $\rho(p) \equiv p\rho^*(p)$ then not only do we have an analogy with (0.1), inasmuch as $\rho(p)$ is monotonic increasing in p, but in analogy with (0.2) the following inequality holds for $1 \leqq p < k$:

$$\begin{aligned}\sum_{n=1}^{k} a_n - k\left\{\prod_{n=1}^{k} a_n\right\}^{1/k} \geqq \sum_{n=1}^{p} a_n - p\left\{\prod_{n=1}^{p} a_n\right\}^{1/p} &+ \sum_{n=p+1}^{k} a_n \\ &- (k-p)\left\{\prod_{n=p+1}^{k} a_n\right\}^{1/(k-p)}\end{aligned} \tag{0.3}$$

That $\rho(p)$ above is monotonic increasing is due to R. Rado (see [1] p. 61, example 60); the extension of this result to (0.3) and the further generalizations which follow seem to be new.

1. Notations. I and J will denote nonempty *finite* sets of positive integers, $N(I)$ the number of integers in the set I, $\square$ the empty set, and I_p the set $\{1, 2, \cdots, p\}$ where $p \geqq 1$.

In all that follows $a \equiv \{a_n;\ n \geqq 1\}$ will denote a sequence of nonnegative real numbers.

Following [1] Section 2.1 we define, for $0 \leqq r < \infty$,

$$M_r(a; I) \equiv N(I)^{-1/r} \left\{ \sum_{n \in I} a_n^r \right\}^{1/r} \qquad (0 < r < \infty)$$

(1.1)

$$M_0(a; I) \equiv \left\{ \prod_{n \in I} a_n \right\}^{N(I)^{-1}}$$

so that $M_r(a; I)$ is the rth mean of those members of the sequence a such that $n \in I$.

For $0 \leqq s \leqq r < \infty$ we define

$$\rho_{r,s}(a; I) \equiv N(I)\{M_r(a; I) - M_s(a; I)\}; \tag{1.2}$$

it is well known (see [1] Section 2.9) that the rth mean $M_r(a; I)$, with I fixed, increases with r and this implies that

$$\rho_{r,s}(a; I) \geqq 0 \tag{1.3}$$

with equality if and only if either $r = s$ or $s < r$ and all the a_n for $n \in I$ are equal.

Note that $\rho_{1,0}(a; I_p)$ is identical with $\rho(p)$ defined in the introduction.

2. Statements of the theorems.

THEOREM 1. *Let* $I \cap J = \square$; (i) *if* $1 < r < \infty$ *then*

$$N(I \cup J) M_r(a; I \cup J) \geqq N(I) M_r(a; I) + N(J) M_r(a; J) \tag{2.1}$$

with equality if and only if

$$M_r(a; I) = M_r(a; J), \tag{2.2}$$

(ii) *if* $r = 1$ *there is equality in* (2.1) *for all disjoint* I *and* J,

(iii) *if* $0 \leqq r < 1$ *the inequality* (2.1) *is reversed with equality holding if and only if* (2.2) *is satisfied.*

THEOREM 2. *If* $0 \leqq s \leqq 1 \leqq r < \infty$ *and* $I \cap J = \square$, *then*

$$\rho_{r,s}(a; I \cup J) \geqq \rho_{r,s}(a; I) + \rho_{r,s}(a; J), \tag{2.3}$$

where:

(i) *if* $s = r = 1$ *there is equality in* (2.3) *for all disjoint* I *and* J, *both sides being zero,*

(ii) *if* $0 \leqq s < 1 < r < \infty$ *there is equality if and only if*

$$M_r(a; I) = M_r(a; J) \tag{2.4}$$

$$M_s(a; I) = M_s(a; J), \tag{2.5}$$

(iii) *if* $0 \leqq s < 1 = r$ *there is equality if and only if* (2.5) *is satisfied,*

(iv) *if* $s = 1 < r < \infty$ *there is equality if and only if* (2.4) *is satisfied.*

THEOREM 3. *If* $0 \leqq s \leqq 1 \leqq r < \infty$ *then for all* $p \geqq 1$

$$\rho_{r,s}(a;\, I_{p+1}) \geqq \rho_{r,s}(a;\, I_p). \tag{2.6}$$

Necessary and sufficient conditions for equality in (2.6) *are as follows*:

(i) $s = r = 1$; *there is equality in* (2.6) *for all* $p \geqq 1$, *both sides being zero*;

(ii) $0 \leqq s < 1 < r < \infty$; *all* a_n, $1 \leqq n \leqq p+1$, *are equal, both sides then being zero*;

(iii) $0 \leqq s < 1 = r$; $a_{p+1} = M_s(a;\, I_p)$;

(iv) $s = 1 < r < \infty$; $a_{p+1} = M_r(a;\, I_p)$.

Note that (2.6) and condition (iii) of Theorem 3 with $s = 0$, give the inequality of Rado mentioned in the introduction.

If the condition $I \cap J = \square$ is not satisfied then the various set functions of the above inequalities are not comparable in the sense of [1] Section 1.6; furthermore, the inequalities (2.3) and (2.6) are not comparable if the condition $s \leqq 1 \leqq r$ fails to hold.

3. Certain lemmas. To prove the above theorems the following lemmas are required.

LEMMA 1. *If* $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, $A \geqq 0$, $B \geqq 0$, *then*

$$A^\alpha B^\beta \leqq \alpha A + \beta B \tag{3.1}$$

with equality if and only if $A = B$.

See [1] Section 2.14.

LEMMA 2. *Let* $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$, $A \geqq 0$, $B \geqq 0$;

(i) *if* $1 < \tau < \infty$ *then*

$$(\alpha A + \beta B)^\tau \leqq \alpha A^\tau + \beta B^\tau \tag{3.2}$$

with equality if and only if $A = B$,

(ii) *if* $\tau = 1$ *there is equality in* (3.2) *for all* A *and* B,

(iii) *if* $0 < \tau < 1$ *the inequality* (3.2) *is reversed with equality if and only if* $A = B$.

Excluding the obvious case when either of A or B is zero we can reduce the proof of (3.2), by using homogeneity, to showing that $(\alpha p + \beta)^\tau \leqq \alpha p^\tau + \beta$ for $\tau > 1$, $p \geqq 1$ with equality if and only if $p = 1$. Consider then

$$\phi(p) \equiv \alpha p^\tau + \beta - (\alpha p + \beta)^\tau \qquad (p \geqq 1);$$

clearly $\phi(1) = 0$ and

$$\phi'(p) = \alpha\tau(p^{\tau-1} - (\alpha p + \beta)^{\tau-1}).$$

Now $p = p(\alpha + \beta) \geqq p\alpha + \beta$ and, since $\tau - 1 > 0$, $p^{\tau-1} \geqq (\alpha p + \beta)^{\tau-1}$ with equality if and only if $p = 1$. Thus $\phi'(p) > 0$ if $p > 1$ and $\phi(p) > 0$ if $p > 1$. The proof of (iii) is similar.

4. Proofs of the theorems.

Proof of Theorem 1. (i) In (iii) of Lemma 2 we set

$$\tau = r^{-1}, \qquad \alpha = N(I)\{N(I \cup J)\}^{-1}, \qquad \beta = N(J)\{N(I \cup J)\}^{-1},$$
$$A = \{M_r(a; I)\}^r, \qquad B = \{M_r(a; J)\}^r,$$

and obtain the result together with the condition for equality. (Note that $\alpha+\beta=1$ only when $I\cap J=\Box$ and that $\alpha>0$ only when $I\neq\Box$.)

(ii) Clear.

(iii) If $0<r<1$ the inequality follows as in the proof of (i) above except that (i) of Lemma 2 is required. If $r=0$ in the statement of Lemma 1 we set:

$$\alpha = N(I)\{N(I \cup J)\}^{-1} \qquad \beta = N(J)\{N(I \cup J)\}^{-1}$$
$$A = M_0(a; I) \qquad B = M_0(a; J)$$

which then gives the required result.

Proof of Theorem 2. From the definition (1.2)

$$\rho_{r,s}(a; I \cup J) = N(I \cup J)M_r(a; I \cup J) - N(I \cup J)M_s(a; I \cup J).$$

Since $0\leqq s\leqq 1$ and $1\leqq r<\infty$ the inequalities of Theorem 1 can be applied to the two terms on the right hand side above and a rearrangement of the resulting expression gives (2.3).

The conditions for equality follow from an application of the condition (2.2) for equality in Theorem 1.

Proof of Theorem 3. Write $I=I_p$, $J=I_{p+1}-I_p$; then $I\cap J=\Box$ and $I\cup J=I_{p+1}$. It follows from definition (1.2) that $\rho_{r,s}(a;\, I_{p+1}-I_p)=0$ for all r and s. An application of Theorem 2 now gives (2.6).

Conditions for equality: (i) Clear.

(ii) This requires, from (2.4 and 5), that

$$M_s(a; I_p) = M_s(a; I_{p+1} - I_p) = a_{p+1}$$
$$M_r(a; I_p) = M_r(a; I_{p+1} - I_p) = a_{p+1}$$

and so the means $M_r(a;\, I_p)$ and $M_s(a;\, I_p)$ are equal. This is only true if all the a_n for $1\leqq n\leqq p$ are equal (see [1] Section 2.9) in which case all the a_n for $1\leqq n\leqq p+1$ are equal and the stated condition follows.

(iii) and (iv) These follow as for (ii) except that only one of (2.4) and (2.5) is required in each case.

5. Generalizations. As is to be expected the above inequalities can be generalized to the case of weighted means (see [1] Section 2.2). The means $M_r(a; I)$ given in (1.1) can be extended to

$$M_r(a; p; I) \equiv N(p; I)^{-1/r}\left\{\sum_{n\in I} p_n a_n^r\right\}^{1/r} \qquad (0 < r < \infty)$$

$$M_0(a; p; I) \equiv \left\{ \prod_{n\in I} a_n^{p_n} \right\}^{N(p;I)^{-1}}$$

where $\{p_n;\ n \geqq 1\}$ is a fixed sequence of strictly positive numbers and

$$N(p; I) = \sum_{n\in I} p_n.$$

There is a corresponding definition for $\rho_{r,s}(a;\ p;\ I)$ and an extension of the inequalities in Theorems 1, 2 and 3 to such weighted means; the proofs are entirely similar.

There are analogies of these inequalities for integrals in place of sums (see [1], Sections 6.6–6.8); the statement and proof of such inequalities is left to the reader.

References

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A DIFFERENTIAL EQUATION WITH NON-UNIQUE SOLUTIONS

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M. Lavrentieff [Math. Zeit. 23 (1925), 197–209] gave an example of a real-valued function $U(x, u)$, continuous on a rectangle R, with the property that for every choice of initial point (x_0, u_0) interior to R, the initial value problem

$$u' = U(x, u), \qquad u(x_0) = u_0, \tag{1}$$

has more than one solution on every interval $[x_0, x_0+\epsilon]$ and $[x_0-\epsilon, x_0]$ for small $\epsilon>0$. The object of this note is to give another, but somewhat simpler, example of this type. The function $U(x, u)$ below will be defined for all (x, u).

Let S_0 be the set of arcs

$$u = 4i + \cos \pi x \text{ and } u = 4i + 2 - \cos \pi x \text{ for } -\infty < x < \infty, i = 0, \pm 1, \cdots, \tag{2}$$

considered to be made up of subarcs defined on the intervals of unit length, $k \leqq x \leqq k+1$ and $k=0, \pm 1, \cdots$.

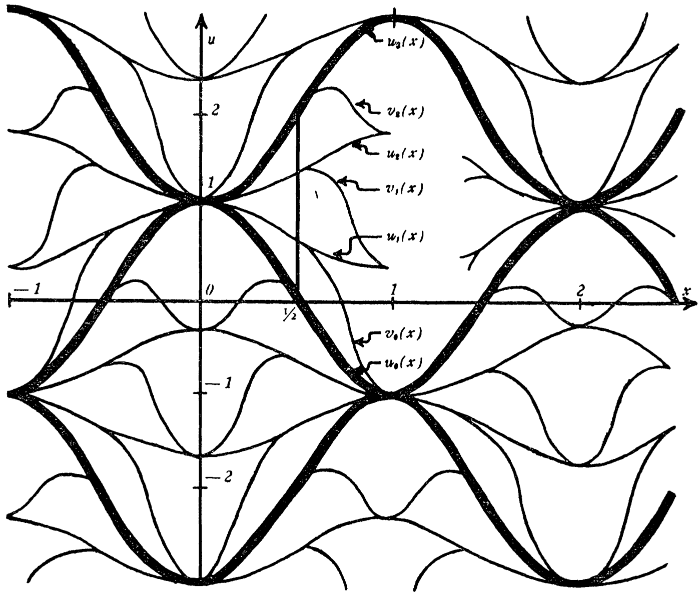
For every $n=0, 1, 2, \cdots$, there will be constructed a set S_n of twice continuously differentiable arcs

$$u = u_{jk}(x),\ k/2^n \leqq x \leqq (k+1)/2^n \quad \text{and} \quad j, k = 0, \pm 1, \cdots. \tag{3}$$

The symbol S_n will denote either the set of arcs (3) or the set of points on these arcs. The set S_n of arcs (3) will have the properties that (i)

$$u_{jk}(x) < u_{j+1,k}(x) \quad \text{for} \quad k/2^n < x < (k+1)/2^n; \tag{4}$$

(ii) the arcs $u=u_{jk}(x)$ and $u=u_{j+1,k}(x)$ have exactly one endpoint in common; (iii) for any pair j, k there is at least one index h such that $u_{h,k-1}=u_{h+1,k-1}=u_{jk}$ at $x=k/2^n$ and an index i such that $u_{i,k+1}=u_{i+1,k+1}=u_{jk}$ at $x=(k+1)/2^n$; (iv) any two arcs of S_n which have a point in common have the same tangent at that point; hence (v) any continuous arc $u=u(x)$, say, on $a\leqq x\leqq b$, which is made up of arcs of S_n can be continued over $-\infty<x<\infty$, not uniquely, so as to have the same property and any such continuation is of class C^1 (and piecewise of class C^2); also (vi) if $U_n(x, u)$ is defined on the point set S_n to be the



In the sketch above, the heavy curved lines represent arcs of S_0. Heavy and light curved lines represent arcs of S_1 if $m_0=3$. The construction of the arcs of S_1, not in S_0, is indicated above: the arcs $u(x)=u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)=v(x)$ are defined on $[a, b]=[0, 1]$, the arcs $v_0(x)$, $v_1(x)$, $v_2(x)$ on $[c, b]=[\frac{1}{2}, 1]$. The sketch makes it clear how S_0 or S_1 divides the plane into sets G.

slope of the tangent at the point $(x, u)\in S_n$, then $U_n(x, u)$ is uniformly continuous on S_n and the arcs of (v) constitute the set of solutions of

$$u' = U_n(x, u); \tag{5}$$

(vii) the sets S_0, $S_1, \cdots$, satisfy $S_n\subset S_{n+1}$, so that $U_{n+1}(x, u)$ is an extension of $U_n(x, u)$; (viii) $S=\bigcup S_n$ and, in fact, the set of endpoints $(k/2^n, u_{jk}(k/2^n))$ for $j, k=0, \pm 1, \cdots$ and $n=0, 1, \cdots$, is dense in the plane; finally, (ix)

$$U(x, u) = \lim_{n\to\infty} U_n(x, u) \tag{6}$$

which is defined on $S=\bigcup S_n$ has a (unique) continuous extension over the plane. Condition (ix) is the only nontrivial condition.

Let $\pi^2 = \epsilon_0 > \epsilon_1 > \cdots$,

$$M_n = \sum_{k=0}^{n} \epsilon_k \quad \text{and} \quad M = \sum_{k=0}^{\infty} \epsilon_k < \infty. \tag{7}$$

Suppose that S_n has already been constructed so that the functions (3) satisfy

$$| u'_{jk}(x) |, \ | u''_{jk}(x) | \leqq M_n; \tag{8}$$

and that if

$$d_n = \sup_{j,k,x} \max (| u_{j+1,k}(x) - u_{j,k}(x) |, | u'_{j+1,k}(x) - u'_{jk}(x) |), \tag{9}$$

then

$$d_n \leqq \epsilon_n, \tag{10}$$

and if $n > 0$ and no arc of S_{n-1} lies between $u = u_{ik}(x)$, $u = u_{hk}(x)$, then

$$| u'_{ik}(x) - u'_{hk}(x) | \leqq d_{n-1} + \epsilon_n. \tag{11}$$

The set of arcs S_{n+1} will be obtained from those of S_n by inserting, on each interval $[k/2^n, (2k+1)/2^{n+1}]$ and $[(2k+1)/2^{n+1}, (k+1/2^n]$, a finite number of arcs between the arcs $u = u_{jk}(x)$, $u = u_{j+1,k}(x)$ of S_n. The arcs of S_n and these inserted arcs will constitute the set S_{n+1}.

For convenience, let $u(x) = u_{jk}(x)$, $v(x) = u_{j+1,k}(x)$, $a = k/2^n$, $b = (k+1)/2^n$, and $c = \frac{1}{2}(a+b)$. Suppose that $u(a) = v(a)$; the construction in the case $u(b) = v(b)$ is similar. Then $u(x)$, $v(x)$ are defined on $[a, b]$, $b - a = 1/2^n$,

$$u(x) < v(x) \text{ on } (a, b], u(a) = v(a), u'(a) = v'(a), \tag{12}$$

$$| u'(x) |, | u''(x) |, | v'(x) |, | v''(x) | \leqq M_n, \tag{13}$$

$$| u(x) - v(x) |, | u'(x) - v'(x) | \leqq d_n \leqq \epsilon_n. \tag{14}$$

Let $m = m_n > 0$ be an integer to be specified below. For $i = 0, 1, \cdots, m$, put

$$u_i(x) = [(m - i)u(x) + iv(x)]/m \tag{15}$$

on $[a, b]$. Then $u_0(x) = u(x)$, $u_m(x) = v(x)$ and

$$u(x) \leqq u_i(x) < u_{i+1}(x) \leqq v(x) \text{ on} (a, b], \tag{16}$$

$$u_i(a) = u(a), \qquad u'_i(a) = u(a) \qquad \text{for } i = 0, 1, \cdots, m. \tag{17}$$

It is clear from (14) that

$$| u_h - u_i | \leqq | h - i | d_n/m \leqq d_n, \tag{18}$$

$$| u'_i | \leqq M_n, \quad | u'_h - u'_i | \leqq | h - i | d_n/m \leqq d_n, \tag{19}$$

$$| u''_i | \leqq M_n. \tag{20}$$

For $i=0, 1, \cdots, m-1$ and $c=\frac{1}{2}(a+b)$, put

$$v_i(x) = u_i(x) \sin^2 2^{n+1}\pi(x-c) + u_{i+1}(x) \cos^2 2^{n+1}\pi(x-c) \tag{21}$$

on $[c, b]$, so that $b-c=1/2^{n+1}$ implies that

$$u_i(x) < v_i(x) < u_{i+1}(x) \text{ on } (c, b), \tag{22}$$

$$v_i = u_{i+1}, \quad v_i' = u_{i+1}' \text{ at } x = c \text{ and } v_i = u_i, \quad v_i' = u_i' \text{ at } x = b. \tag{23}$$

The relations in (23) involving derivatives follow from

$$\begin{aligned} v_i' &= u_i' \sin^2 2^{n+1}\pi(x-c) + u_{i+1}' \cos^2 2^{n+1}\pi(x-c) \\ &\quad + 2^{n+1}\pi(u_i - u_{i+1}) \sin 2^{n+2}\pi(x-c). \end{aligned} \tag{24}$$

From (24) and (18)–(20),

$$|v_i'| \leqq M_n + 2^{n+1}\pi d_n/m, \qquad |v_i''| \leqq M_n + (2^{n+2} + 2^{2n+3}\pi)\pi d_n/m. \tag{25}$$

Also, by (21),

$$v_i - u_i = (u_{i+1} - u_i) \cos^2 2^{n+1}\pi(x-c),$$

$$v_i - u_{i+1} = (u_i - u_{i+1}) \sin^2 2^{n+1}\pi(x-c),$$

so that (18)–(19) give

$$|v_i - u_h| \leqq d_n/m, \quad |v_i' - u_h'| \leqq (1 + 2^{n+1}\pi)d_n/m \qquad \text{for } h = i, i+1. \tag{26}$$

Finally, let $m=m_n$ be chosen so large that

$$(2^{n+2} + 2^{2n+3}\pi)\pi d_n/m < \epsilon_{n+1}/3. \tag{27}$$

In order to obtain S_{n+1} from S_n, let the arcs $u=u_i(x)$, $i=0, \cdots, m$, on $[a, c]$ and the arcs $u=u_i(x)$, $i=0, \cdots, m$, and $u=v_h(x)$, $h=0, \cdots, m-1$, on $[c, b]$ be inserted between $u=u(x)$, $u=v(x)$. It is clear from (19)–(20), (25)–(26) and (27) that the analogues of (8), (10) hold if n is replaced by $n+1$. Also the analogue of (11) follows from (19), (26), (27).

This completes the construction of the sequence $S_0 \subset S_1 \subset \cdots$. It is clear that $S=\cup S_n$ is dense in the (x, u)-plane.

The continuity of $U(x, u)$, given by (6), will now be considered. Let $p \geqq n \geqq 0$. The set of arcs S_n divide the plane into closed sets G of the form $G=\{(x, u): \gamma \leqq x \leqq \delta, u^n(x) \leqq u \leqq v^n(x)\}$, where no point of S_n is interior to G, $u=u^n(x)$ and $u=v^n(x)$ on $[\gamma, \delta]$, $\delta-\gamma=2/2^n$, are arcs each made up of two arcs of S_n, $u^n=v^n$ at $x=\gamma, \delta$ and $u^n<v^n$ on (γ, δ).

Let $(x_0, u_p) \in G \cap S_p$ and let (x^1, u^1) be any point of the boundary of G. The difference

$$\Delta_p = | U_p(x_0, u_p) - U_p(x^1, u^1) |$$

will be estimated. Consider first the case that $p=n$. Then (x_0, u_n) is on the

boundary of G, say $u_n = u^n(x_0)$. Since $U_n(x, u(x)) = u^{n\prime}(x)$, it is seen, by (8), that

$$|U_n(x_0, u_n) - U_n(\gamma, u(\gamma))| \leqq M_n |x_0 - \gamma| \leqq 2M/2^n.$$

Thus, in the case that $p = n$, $\Delta_n \leqq 4M/2^n$.

Let $p > n$. It can be supposed that $(x_0, u_p) \in S_p - S_{p-1}$. Let $u_n = u^n(x_0)$ and $u_n \leqq u_{n+1} \leqq \cdots \leqq u_p$, where (x_0, u_j) is the highest point of the segment $x = x_0$, $u_{j-1} \leqq u \leqq u_p$ which is in S_j, $j = n+1, \cdots, p$. Then, by (11)

$$|U_p(x_0, u_{j+1}) - U_p(x_0, u_j)| \leqq d_j + \epsilon_{j+1} \leqq 2\epsilon_j.$$

Hence

$$|U_p(x_0, u_p) - U_p(x_0, u_n)| \leqq 2 \sum_{j=n}^{\infty} \epsilon_j.$$

If this is combined with $\Delta_n \leqq 4M/2^n$, it follows that

$$\Delta_p \leqq \eta_n, \quad \text{where} \quad \eta_n = 4M/2^n + 2 \sum_{j=n}^{\infty} \epsilon_j.$$

Consider now two points (x_i, u_i), $i = 0, 1$, in S_p, $p \geqq n$. Each of these points (x_i, u_i) is contained in a region $G = G_i$ of the type just considered. There exists a point (x^i, u^i) on the boundary of G_i such that

$$|x^0 - x^1|^2 + |u^0 - u^1|^2 \leqq |x_0 - x_1|^2 + |u_0 - u_1|^2,$$

(where, for example, $(x^0, u^0) = (x^1, u^1)$ if $G_0 = G_1$). Thus the above estimate for Δ_p implies that

$$|U(x_0, u_0) - U(x_1, u_1)| \leqq 2\eta_n + |U_n(x^0, u^0) - U_n(x^1, u^1)|.$$

Since $U_n(x, u)$ is uniformly continuous on S_n, it follows from the last three formula lines that $U(x, u)$ is uniformly continuous on S. Hence $U(x, u)$ has a continuous extension, denoted also by $U(x, u)$, on the (x, u)-plane.

It will now be verified that (1) has the asserted property. It is clear that any continuous arc $u = u(x)$ on an interval $[c, d]$ made of subarcs of S is a solution of (1). If (x_0, u_0) is any point of a set G of the type just considered, then (1) has a solution $u = u(x)$ over $[\gamma, \delta]$ satisfying $u = u^n = v^n$ at $x = \gamma, \delta$. Such a solution can be continued to the left of $x = \gamma$ [right of $x = \delta$] in a nonunique manner by using arcs of S_n. If n is sufficiently large, the interval $[\gamma, \delta]$ containing x_0 is arbitrarily small. This completes the verification.

It may be mentioned that E. R. van Kampen, Amer. J. Math., 59 (1937) 144–152, uses the set of arcs S_0 in a different manner to construct a continuous function $U(x, u)$ with certain properties including the one that all initial value problems (1) with $x_0 = 0$ have nonunique solutions.

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A METHOD FOR OBTAINING POLYNOMIALS OF BERNSTEIN TYPE OF TWO VARIABLES

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1. As is well known, the Bernstein polynomial of mth degree, corresponding to a function $f(x)$ and the interval $I=[0, 1]$, has the following expression

$$(1) \qquad B_m(f; x) = \sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}\right),$$

where

$$p_{n,j}(t) = \binom{n}{j} t^j (1-t)^{n-j}.$$

The polynomial $B_m(f; x)$ possesses many interesting properties which have been studied by numerous mathematicians. A concise presentation of the principal results, obtained up to ten years ago, concerning these polynomials, was given by G. G. Lorentz in his nice book [2].

Their fundamental property is that $B_m(f; x) \to f(x)$, as $m \to \infty$, uniformly on the interval I, if $f(x)$ is continuous on I.

In the following we will use the approximation formula

$$(2) \qquad f(x) \approx B_m(f; x).$$

In the case of two variables one knows the following extensions of the polynomial (1)

$$(3) \qquad B_{m,n}(f; x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x) p_{n,j}(y) f\left(\frac{i}{m}, \frac{j}{n}\right),$$

$$(4) \qquad B_m(f; x, y) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} p_m^{i,j}(x, y) f\left(\frac{i}{m}, \frac{j}{m}\right),$$

where

$$p_m^{i,j}(x, y) = \binom{m}{i}\binom{m-i}{j} x^i y^j (1-x-y)^{m-i-j}.$$

The polynomial (3), which is of mth degree with respect to x and nth degree with respect to y, corresponds to the square S: $0 \leqq x, y \leqq 1$. It can be obtained if we expand the function $f(x, y)$, defined on S, by the formula (2) and then expand the resulting function $f(i/m, y)$ by the same formula with respect to y. The polynomial (4) has the global degree m and corresponds to the isosceles right triangle $\Delta: x+y \leqq 1$, $x \geqq 0$, $y \geqq 0$. To my knowledge this polynomial was first found by A. Dinghas [1]; it was also mentioned by Lorentz [2]. In the

papers [4, 5], where we discussed approximation by this polynomial, we have also shown a probabilistic way to obtain it.

We pose the question: is it possible to deduce the polynomial (4) using the formula (2)? In this short paper we will give a positive answer to this question, and indicate a simple method for extending the polynomial (1) to two variables.

2. Let $\phi_1=\phi_1(x)$ and $\phi_2=\phi_2(x)$ be two polynomials which on the interval I possess the following properties: $\phi_2\geqq\phi_1\geqq 0$. Assuming the function $f(x)$ is defined on the domain D, determined by the equations $y=\phi_1$, $y=\phi_2$, $x=0$, $x=1$, let us make the following change of variable

$$y = [\phi_2 - \phi_1]t + \phi_1. \tag{5}$$

Using the formula (2) we get

$$f(x, y) = f[x, (\phi_2 - \phi_1)t + \phi_1] \approx \sum_{i=0}^{m} p_{m,i}(x)\phi\left(\frac{i}{m}, t\right),$$

where

$$\phi\left(\frac{i}{m}, t\right) = f\left[\frac{i}{m}, \left(\phi_2\left(\frac{i}{m}\right) - \phi_1\left(\frac{i}{m}\right)\right)t + \phi_1\left(\frac{i}{m}\right)\right].$$

Let us attach to the node i/m the natural number n_i. By the formula (2), applied to the variable t, we have

$$\phi\left(\frac{i}{m}, t\right) \approx \sum_{j=0}^{n_i} p_{n_i,j}(t)\phi\left(\frac{i}{m}, \frac{j}{n_i}\right).$$

In this way we obtain the approximation formula $f(x, y)\approx B(f; x, y)$, where

$$B(f; x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} p_{m,i}(x)p_{n_i,j}(t)\phi\left(\frac{i}{m}, \frac{j}{n_i}\right),$$

where t is given by (5). More explicitly,

$$\begin{aligned} B(f; x, y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} p_{m,i}(x)p_{n_i,j}\left(\frac{y-\phi_1}{\phi_2-\phi_1}\right) \\ \cdot f\left[\frac{i}{m}, \left(\phi_2\left(\frac{i}{m}\right) - \phi_1\left(\frac{i}{m}\right)\right)\frac{j}{n_i} + \phi_1\left(\frac{i}{m}\right)\right]. \end{aligned} \tag{6}$$

For this function to represent a polynomial we must conveniently particularize the polynomials ϕ_1 and ϕ_2. The following cases are remarkable:

a) $\phi_1=0$, $\phi_2=1$, $n_i=n$ $(i=\overline{0, m})$. In this case the domain D becomes the square S and we get the polynomial (3) of degree (m, n).

b) $\phi_1=0$, $\phi_2=1-x$, $n_i=m-i$ $(i=\overline{0, m})$. Now we obtain the polynomial (4) corresponding to the triangle Δ.

c) $\phi_1=0$, $\phi_2=x$, $n_i=i$ $(i=\overline{0, m})$. The domain D transforms into the triangle $\bar{\Delta}$: $x\geqq 0$, $y\geqq 0$, $y-x\geqq 0$, and we have

$$\bar{B}_m(f; x, y) = \sum_{i=0}^{m}\sum_{j=0}^{i} \bar{p}_m^{i,j}(x, y) f\left(\frac{i}{m}, \frac{j}{m}\right), \tag{7}$$

where

$$\bar{p}_m^{i,j}(x, y) = \binom{m}{i}\binom{i}{j}(1-x)^{m-i}y^j(x-y)^{i-j}.$$

d) $\phi_1=x$, $\phi_2=1$, $n_i=m-i$. The corresponding Bernstein polynomial is

$$\bar{\bar{B}}_m(f; x, y) = \sum_{i=0}^{m}\sum_{j=0}^{m-i} \bar{\bar{p}}_m^{i,j}(x, y) f\left(\frac{i}{m}, \frac{i+j}{m}\right), \tag{8}$$

where

$$\bar{\bar{p}}_m^{i,j}(x, y) = \binom{m}{i}\binom{m-i}{j} x^i(y-x)^j(1-y)^{m-i-j},$$

the domain D being reduced to the triangle $\bar{\bar{\Delta}}$ determined by the equations $y=x$, $y=1$, $x=0$.

e) $\phi_1=1-x$, $\phi_2=1$, $n_i=i$. In this case we have the triangle $\bar{\bar{\bar{\Delta}}}$ determined by $y=1$, $y=1-x$, $x=1$ and the Bernstein polynomial

$$\bar{\bar{\bar{B}}}_m(f; x, y) = \sum_{i=0}^{m}\sum_{j=0}^{i} \bar{\bar{\bar{p}}}_m^{i,j}(x, y) f\left(\frac{i}{m}, 1-\frac{i}{m}+\frac{j}{m}\right), \tag{9}$$

where

$$\bar{\bar{\bar{p}}}_m^{i,j}(x, y) = \binom{m}{i}\binom{i}{j}(1-x)^{m-i}(x+y-1)^j(1-y)^{i-j}.$$

Remark. The polynomials (4), (7), (8) and (9) have the global degree equal to m. We can pass from one of these to another by a simple transformation. Thus, for instance, applying the transformation: $u=x$, $v=x+y$ to (4) gives us (8).

The properties of convergence of the above mentioned polynomials to the function $f(x, y)$, assumed continuous on the corresponding triangle, can be deduced from the results established in [**4**]. However, we shall give here a short proof for this property of convergence.

Let us consider, for instance, the polynomials (4). The following method seems to be simple and in addition permits us at the same time to evaluate the corresponding order of approximation, by using the notion of modulus of continuity. The modulus of continuity of a function $f(x, y)$ is defined by

$$\omega(\delta_1, \delta_2) = \sup |f(x'', y'') - f(x', y')|, \tag{10}$$

where $\delta_1>0$, $\delta_2>0$ are real numbers, whereas (x', y') and (x'', y'') are points of Δ such that: $|x''-x'|\leqq\delta_1$, $|y''-y'|\leqq\delta_2$.

Taking into account that on Δ we have

$$p(x,y)\geqq 0, \qquad \sum_{i=0}^{m}\sum_{j=0}^{m-i} p_m^{i,j}(x,y)\equiv 1,$$

$$|f(x'',y'')-f(x',y')| \leqq \omega(|x''-x'|, |y''-y'|) \leqq \omega(\delta_1,\delta_2)$$

and the inequality (see the proof of this in [3])

$$\omega(\lambda_1\delta_1, \lambda_2\delta_2) \leqq (\lambda_1+\lambda_2+1)\omega(\delta_1,\delta_2) \qquad (\lambda_1>0, \lambda_2>0),$$

we can write successively

$$|f(x,y)-B_m(f;x,y)| \leqq \sum_{i=0}^{m}\sum_{j=0}^{m-i} p_m^{i,j}(x,y)\left|f(x,y)-f\left(\frac{i}{m},\frac{j}{m}\right)\right|$$

$$\leqq \sum_{i=0}^{m}\sum_{j=0}^{m-i} p_m^{i,j}(x,y)\omega\left(\left|x-\frac{i}{m}\right|, \left|y-\frac{j}{m}\right|\right)$$

$$= \sum_{i=0}^{m}\sum_{j=0}^{m-i} p_m^{i,j}(x,y)\omega\left(\frac{1}{\delta_1}\left|x-\frac{i}{m}\right|\delta_1, \frac{1}{\delta_1}\left|y-\frac{j}{m}\right|\delta_2\right)$$

$$\leqq \sum_{i=0}^{m}\sum_{j=0}^{m-i} p_m^{i,j}(x,y)\left(\frac{1}{\delta_1}\left|x-\frac{i}{m}\right|+\frac{1}{\delta_2}\left|y-\frac{j}{m}\right|+1\right)\omega(\delta_1,\delta_2)$$

$$\leqq \left[\frac{1}{2\sqrt{m}}\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)+1\right]\omega(\delta_1,\delta_2),$$

since

$$\sum_{i=0}^{m}\sum_{j=0}^{m-i} p_m^{i,j}(x,y)\left|x-\frac{i}{m}\right| \leqq \left[\sum_{i=0}^{m}\sum_{j=0}^{m-i} p_m^{i,j}(x,y)\left(x-\frac{i}{m}\right)^2\right]^{1/2}$$

$$= \left[\sum_{i=0}^{m}\binom{m}{i}x^i(1-x)^{m-i}\left(x-\frac{i}{m}\right)^2\right]^{1/2} = \left[\frac{x(1-x)}{m}\right]^{1/2} \leqq \frac{1}{2\sqrt{m}}, \text{ etc.}$$

By choosing $\delta_1=\delta_2=1/\sqrt{m}$ we get the inequality

$$|f(x,y)-B_m(f;x,y)| \leqq 2\omega\left(\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right).$$

Hence, uniformly on Δ: $B_m(x,y)\to f(x,y)$, as $m\to\infty$.

For the polynomials (3) we have the following inequality of Popoviciu-Ipatov (see the proof in [3])

$$|f(x,y)-B_{m,n}(f;x,y)| \leqq \frac{3}{2}\,\omega\left(\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right).$$

f) Let us suppose that $\phi_1=0$, $\phi_2=Ax(1-x)$ (A being a constant), and $n_i \leqq \min\{i, m-i\}=m_i$; in this case the expression (6) will also be a polynomial. We will take $n_i=m_i$ and we obtain

$$\widetilde{B}_m^A(f; x, y) = \sum_{i=0}^{m}\sum_{j=0}^{m_i} A^{-m_i}\binom{m}{i}\binom{m_i}{j} x^{i-m_i}(1-x)^{m-i-m_i}y^j(Ax-Ax^2-y)^{m_i-j}$$
$$\cdot f\left[\frac{i}{m}, A\frac{i}{m}\left(1-\frac{i}{m}\right)\frac{j}{m_i}\right]$$

which corresponds to the function $f(x, y)$ and to the domain bounded by the x-axis and the arc of parabola defined by $y=Ax(1-x)$; $\widetilde{B}_m^A$ is an mth degree polynomial.

Example. If $m=4$ and $A=2$ we have $m_0=0$, $m_1=1$, $m_2=2$, $m_3=1$, $m_4=0$, and

$$\begin{aligned}B_4^2(f; x, y) = {} & (1-x)^4f(0,0) + 2(1-x)^2(2x-2x^2-y)f(\tfrac{1}{4},0) + 2(1-x)^2yf(\tfrac{1}{4},\tfrac{3}{8})\\ & + \tfrac{3}{2}(2x-2x^2-y)^2f(\tfrac{2}{4},0) + 3y(2x-2x^2-y)f(\tfrac{2}{4},\tfrac{1}{4}) + \tfrac{3}{2}y^2f(\tfrac{2}{4},\tfrac{2}{4})\\ & + 2x^2(2x-2x^2-y)f(\tfrac{3}{4},0) + 2x^2yf(\tfrac{3}{4},\tfrac{3}{8}) + x^4f(1,0).\end{aligned}$$

The remainders of the approximation formulae of $f(x, y)$, by one of the polynomials $B(f; x, y)$ considered above, can be obtained if we take into account the expression of the remainder for one variable (see, for instance, [6]) and the procedure used for obtaining the expression (6). Thus, for example, in the case a), we have

$$R(f; x, y) = -\frac{x(1-x)}{2m}f''_{x^2}(\xi, y) - \frac{y(1-y)}{2n}f''_{y^2}(x, \eta) - \frac{x(1-x)y(1-y)}{4mn}f^{\mathrm{IV}}_{x^2y^2}(\xi, \eta),$$

and in the case b) we can show that the remainder has the following form

$$R(f; x, y) = -\frac{x(1-x)}{2m}f''_{x^2}(\xi, \eta) + \frac{xy}{m}f''_{xy}(\xi', \eta') - \frac{y(1-y)}{2m}f^{\mathrm{IV}}_{x^2y^2}(\xi'', \eta''),$$

where the coordinates of the differentiation points belong to the interval I.

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CONGRUENCES FOR $p_r(n)$ AND RAMANUJAN'S τ FUNCTION

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1. Introduction. Define $p_r(n)$ by $[\phi(x)]^r = \sum_{n=0}^{\infty} p_r(n)x^n$, for all r, where $\phi(x) = (1-x)(1-x^2)(1-x^3) \cdots$.

In this paper the following congruences are proved:

(I) *If $R/n = m/t$, $(m, t) = 1$, then $p_{\alpha R}(n) \equiv 0 \pmod{m}$, where $\alpha = \pm 1$.*

(II) *If R is a prime then $p_{\alpha R+1}(n) \equiv \sum_\beta (-1)^\beta p_\alpha(t) \pmod{R}$, where $\alpha = \pm 1$ and t has the integral values given by $t = [2n - \beta(3\beta - 1)]/2R$, β integral, and*

$$p_{\alpha R+3}(n) \equiv \sum_j (-1)^j(2j + 1)p_\alpha(t_1) \pmod{R},$$

where $\alpha = \pm 1$ and t_1 has the integral values given by $t_1 = [2n - j(j+1)]/2R$, $j \geqq 0$.

(III) *If R is a prime then $p_{R+1}(mR+t) \equiv 0 \pmod{R}$ and $p_{-(R-1)}(mR+t) \equiv 0 \pmod{R}$, where $t \not\equiv \beta(3\beta-1)/2 \pmod{R}$.*

(IV) *If R is a prime then $p_{R+3}(mR+t) \equiv 0 \pmod{R}$, and $p_{-(R-3)}(mR+t) \equiv 0 \pmod{R}$, where $t \not\equiv k(k+1)/2 \pmod{R}$ and also when $t = (R^2-1)/8$.*

(V) *If $p_k(mR+t) \equiv 0 \pmod{R}$, R being a prime and t any integer for which the congruence holds, then $p_{k \pm LR}(mR+t) \equiv 0 \pmod{R}$.*

Using the above theorems we have obtained a new proof of MacMahon's congruence for $p_{-1}(n)$ modulo 2. (In our notation $p_{-1}(n) = p(n)$ denotes the number of unrestricted partitions of n.) Some of the congruences for Ramanujan's τ function for the primes 2, 3, 5, 7 and 23 and Ramanathan's congruences for $p_{-2}(n)$ modulo 5 follow as corollaries to the above theorems. Three new congruences for $p_{-1}(n)$ modulo 2 of MacMahon's type and one congruence modulo 3 have also been obtained. Finally the congruences below are discussed.

$$p_{-3}(3m + 2) \equiv 0 \pmod{9}, \qquad p_{-3}(9m + 5) \equiv 0 \pmod{27},$$
$$p_{-3}(9m + 8) \equiv 0 \pmod{81}, \quad \text{and} \quad p_{-4}(4m + 3) \equiv 0 \pmod{8}.$$

2. Some preliminary theorems. Let

$$\phi(x) = (1 - x)(1 - x^2)(1 - x^3) \cdots \tag{2.1}$$

and

$$[\phi(x)]^r = \sum_{n=0}^{\infty} p_r(n)x^n \qquad \text{for all } r. \tag{2.2}$$

Taking the logarithms of both sides of (2.2) we get

$$r \log [\phi(x)] = \log \sum_{n=0}^{\infty} p_r(n)x^n. \tag{2.3}$$

Since

$$\log \phi(x) = -\sum_{n=1}^{\infty} \sigma(n)x^n/n, \tag{2.4}$$

where $\sigma(n)$ is defined as the sum of the divisors of n, (2.3) becomes

$$-r\sum_{n=1}^{\infty} \sigma(n)x^n/n = \log \sum_{n=0}^{\infty} p_r(n)x^{n-1}. \tag{2.5}$$

Differentiating (2.5) with respect to x we obtain, after some rearrangements,

$$r\sum_{n=1}^{\infty} \sigma(n)x^{n-1}\sum_{n=0}^{\infty} p_r(n)x^n = \sum_{n=1}^{\infty} np_r(n)x^n, \tag{2.6}$$

whence, equating the coefficients of x^{n-1}, we get

$$p_r(n) = -(r/n)\sum_{j=1}^{n} p_r(n-j)\sigma(j). \tag{2.7}$$

From (2.7) it is evident that

THEOREM 1. *If $R/n=m/t$, $(m, t)=1$, then $p_{\alpha R}(n)\equiv 0 \pmod m$, for $\alpha=\pm 1$.*

Also it is evident that

$$\sum_{j=1}^{n} p_r(n-j)\sigma(j) \equiv 0 \pmod t. \tag{2.8}$$

By elementary methods we can prove

THEOREM 2. *If R is a prime, then $p_{kR}(mR)\equiv p_k(m) \pmod R$ and $p_{-kR}(mR) \equiv p_{-k}(m) \pmod R$.*

3. In this section some of the applications of Theorems 1 and 2 are discussed. In the first place we prove the following two theorems:

THEOREM 3. *If R is prime then $p_{\alpha R+1}(n)\equiv \sum_\beta (-1)^\beta p_\alpha(t) \pmod R$, where $\alpha=\pm 1$ and t has the integral values given by $t=[2n-\beta(3\beta-1)]/2R$, β integral.*

THEOREM 4. *If R is a prime then $p_{\alpha R+3}(n)\equiv \sum_j (-1)^j(2j+1)p_\alpha(t_1) \pmod R$, where t_1 has the integral values given by $t_1=[2n-j(j+1)]/2R$, $j\geqq 0$.*

Since

$$\phi(x) = \sum_{\beta=-\infty}^{\infty} (-1)^\beta x^{\beta(3\beta-1)/2}, \tag{3.1}$$

it follows that

$$\sum_{n=0}^{\infty} p_{R+1}(n)x^n = \sum_{n=0}^{\infty} p_R(n)x^n \sum_{\beta=-\infty}^{\infty} (-1)^\beta x^{\beta(3\beta-1)/2}. \tag{3.2}$$

Expanding and equating the coefficients of x^n we get

(3.3) $$p_{R+1}(n) = \sum_{\beta} (-1)^{\beta} p_R\left(n - \frac{\beta(3\beta - 1)}{2}\right).$$

Now in view of Theorem 1 we have

(3.4) $p_R(n - \beta(3\beta - 1)/2) \equiv 0 \pmod R$ when R is coprime to $n - \beta(3\beta - 1)/2$.

Also when $n-\beta(3\beta-1)/2$ is divisible by R, from Theorem 2, we get

(3.5) $$p_R(n - \beta(3\beta - 1)/2) \equiv p(t) \pmod R,$$

where t has integral values given by

(3.6) $$t = [2n - \beta(3\beta - 1)]/2R.$$

Using (3.4), (3.5) and (3.6) we get from (3.3) the proof of Theorem 3 when $\alpha = 1$. Similarly Theorem 3 can be proved when $\alpha = -1$. Also since

(3.7) $$[\phi(x)]^3 = \sum_{j=0}^{\infty} (-1)^j (2j + 1) x^{j(j+1)/2}$$

it follows that

(3.8) $$\sum_{n=0}^{\infty} p_{r+3}(n) x^n = \sum_{n=0}^{\infty} p_r(n) x^n \sum_{j=0}^{\infty} (-1)^j (2j + 1) x^{j(j+1)/2}.$$

Expanding and equating the coefficients of x^n, we get

(3.9) $$p_{r+3}(n) = \sum_{j} (-1)^j (2j + 1) p_r(n - j(j + 1)/2).$$

Let $r = R$ be a prime. Then, in view of Theorem 1, we have

(3.10) $$p_R(n - j(j + 1)/2) \equiv 0 \pmod R$$

when $n-j(j+1)/2$ is coprime to R. When $n-j(j+1)/2$ is divisible by R, from Theorem 2 we get

(3.11) $$p_R\left(n - \frac{j(j + 1)}{2}\right) \equiv \sum p(t_1) \pmod R,$$

where t_1 has the integral values given by

(3.12) $$t_1 = [2n - j(j + 1)]/2R.$$

Using (3.10), (3.11) and (3.12) we get the proof of Theorem 4 for $\alpha = 1$, from (3.9). Similarly Theorem 4 can be proved for the case $\alpha = -1$.

Now putting $R = 2$ and $\alpha = -1$, from Theorem 3 we get a congruence for $p_{-1}(n)$ modulo 2, namely

(3.13) $$p_{-1}(n) \equiv \sum_{t} p_{-1}(t) \pmod 2,$$

where t has the integral values given by

$$t = [2n - \beta(3\beta - 1)]/4, \qquad \beta \text{ integral.} \tag{3.14}$$

In (3.9) putting $r=-4$, we see that

$$p_{-1}(n) \equiv \sum_j p_{-4}\left(n - \frac{j(j+1)}{2}\right) \pmod 2. \tag{3.15}$$

Now from Theorems 1 and 2 we know that

$$\begin{aligned} &p_{-4}(2n+1) \equiv 0 \pmod 2, \qquad p_{-4}(2(2n+1)) \equiv 0 \pmod 2, \text{ and} \\ &p_{-4}(4n) \equiv p_{-2}(2n) \equiv p_{-1}(n) \pmod 2. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we get

$$p_{-1}(n) \equiv \sum_t p_{-1}(t) \pmod 2, \tag{3.17}$$

where t runs over the integral values given by $t=[2n-j(j+1)]/8$, $j\geqq 0$, which is a well-known congruence due to MacMahon [6].

Similarly, starting from the identities

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{n=0}^{\infty} p_2(n)x^n \sum_{n=0}^{\infty} p_{-1}(n)x^n, \tag{3.18}$$

and

$$\sum_{n=0}^{\infty} p_2(n)x^n = \sum_{n=0}^{\infty} p_3(n)x^n \sum_{n=0}^{\infty} p_{-1}(n)x^n \tag{3.19}$$

and using the congruences

$$\begin{aligned} &p_2(2m) \equiv p(m) \pmod 2 \\ &p_2(2m+1) \equiv 0 \pmod 2, \end{aligned} \tag{3.20}$$

and the results (3.1) and (3.7), the following congruences can be easily obtained:

$$\sum_\delta p_{-1}(n - \delta(3\delta - 1)) \equiv 1 \text{ or } 0 \pmod 2 \qquad (\delta \text{ integral}), \tag{3.21}$$

according as $n=\beta(3\beta-1)/2$ or $n\neq\beta(3\beta-1)/2$, and

$$\begin{aligned} &\sum_k p\left(2m + 1 - \frac{k(k+1)}{2}\right) \equiv 0 \pmod 2 \qquad (k \geqq 0), \\ &\sum_k p\left(2m - \frac{k(k+1)}{2}\right) \equiv 1 \text{ or } 0 \pmod 2 \qquad (k \geqq 0), \end{aligned} \tag{3.22}$$

according as $m=\beta(3\beta-1)/2$ or $m\neq\beta(3\beta-1)/2$.

Now in Theorem 3, putting $\alpha = -1$ and $R = 3$, we get

$$p_{-2}(n) \equiv \sum_{\beta}' (-1)^{\beta} p_{-1}(t) \pmod 3, \tag{3.23}$$

where t has the integral values given by

$$t = [2n - \beta(3\beta - 1)]/6, \qquad (\beta \text{ integral}). \tag{3.24}$$

Since

$$p_{-1}(n) = \sum_{m} (-1)^m p_{-2}\left(n - \frac{m(3m-1)}{2}\right) \qquad (m \text{ integral}), \tag{3.25}$$

substituting (3.24) in (3.25) we get

$$p_{-1}(n) \equiv \sum_{m,\beta} (-1)^m (-1)^{\beta} p_{-1}(t) \pmod 3, \qquad (m, \beta \text{ integral}), \tag{3.26}$$

where t has the integral values given by

$$t = [2n - m(3m-1) - \beta(3\beta - 1)]/6, \qquad (m, \beta \text{ integral}). \tag{3.27}$$

We now discuss some known congruences for Ramanujan's τ function. Ramanujan's τ function is defined by

$$x[\phi(x)]^{24} = \sum_{n=1}^{\infty} \tau(n) x^n. \tag{3.28}$$

Comparing (3.28) with (2.1) we see that

$$\tau(n) = p_{24}(n-1). \tag{3.29}$$

In view of Theorem 1, we have

$$p_{24}(3n+1),\ p_{24}(3n+2) \equiv 0 \pmod 3, \tag{3.30}$$

and hence

$$\tau(3n+2),\ \tau(3n+3) \equiv 0 \pmod 3, \tag{3.31}$$

which is a well-known congruence due to Lahiri [5]. Now in view of Theorems 1 and 2 we have $p_{24}(8m) \equiv p_{12}(4m) \cdots \equiv p_3(m) \pmod 2$, and $p_{24}(n) \equiv 0 \pmod 2$ if $n \not\equiv 8m$, and hence

$$\tau(8m+1) \equiv p_3(m) \pmod 2 \text{ and} \tag{3.32}$$

$$\tau(n) \equiv 0 \pmod 2 \qquad \text{if } n \not\equiv 8m+1. \tag{3.33}$$

Now it is known that

$$p_3\left(\frac{k(k+1)}{2}\right) \equiv 1 \pmod 2. \tag{3.34}$$

Using (3.32) and (3.34) we get

$$\tau\left(\frac{8k(k+1)}{2}+1\right) \equiv 1 \pmod 2, \text{ or} \tag{3.35}$$
$$\tau((2k+1)^2) \equiv 1 \pmod 2.$$

Combining (3.35) and (3.33) we get the result that $\tau(n)$ is odd only when n is an odd square, which is a well-known result due to Ramanathan.

4. In this section we discuss some more theorems for $p_r(n)$. Since

$$[\phi(n)]^r\phi(n) = \sum_{n=0}^{\infty} p_{(r+1)}(n)x^n, \tag{4.1}$$

we get from (3.1) and (4.1)

$$\sum_{n=0}^{\infty} p_r(n)x^n \sum_{\beta=-\infty}^{\infty} (-1)^\beta x^{\beta(3\beta-1)/2} = \sum_{n=0}^{\infty} p_{r+1}(n)x^n. \tag{4.2}$$

Let $r=R$ be a prime. Then Theorems 1 and 2 imply that

$$\begin{aligned} p_R(n) &\equiv 0 \pmod R \qquad \text{if } n \text{ is coprime to } R \text{ and} \\ p_R(mR) &\equiv p(m) \pmod R. \end{aligned} \tag{4.3}$$

Hence

$$\sum_{m=0}^{\infty} p_R(n)x^n \sum_{\beta=-\infty}^{\infty} (-1)^\beta x^{\beta(3\beta-1)/2} \equiv \sum_{n=0}^{\infty} p_{R+1}(n)x^n \pmod R. \tag{4.4}$$

Now $p_{R+1}(n)$ must be divisible by R for all values of n for which x^n does not occur in the left member of (4.4) and such values of n are given by

$$n \not\equiv \beta(3\beta-1)/2 \pmod R \qquad \text{for any } \beta. \tag{4.5}$$

Hence we have proved

THEOREM 5. *If R is a prime then $p_{R+1}(mR+t) \equiv 0 \pmod R$, where $t \not\equiv \beta(3\beta-1)/2 \pmod R$ for any β.*

Similarly we can prove

THEOREM 6. *If R is a prime then $p_{-(R-1)}(mR+t) \equiv 0 \pmod R$, where $t \not\equiv \beta(3\beta-1)/2 \pmod R$ for any β.*

As an illustration of Theorems 5 and 6, let us consider $R=5$. We then have

$$p_6(5m+t),\ p_{-4}(5m+t) \equiv 0 \pmod 5 \qquad \text{for } t = 3 \text{ and } 4. \tag{4.6}$$

Since we are considering congruences for $p_R(n)$ and $p_{-R}(n)$ for $R=p\pm 1$, where p is a prime, we may regard these as congruences at an interval of one.

We now derive congruences for an interval of 3. We have

$$\sum_{n=0}^{\infty} p_{R+3}(n)x^n = \sum_{n=0}^{\infty} p_R(n)x^n \sum_{n=0}^{\infty} (-1)^n(2n+1)x^{n(n+1)/2}. \tag{4.7}$$

If R is prime then using (4.3) and (4.7) we get

$$\sum_{n=0}^{\infty} p_{R+3}(n)x^n \equiv \sum_{n=0}^{\infty} p(n)x^{nR}, \quad \sum_{n=0}^{\infty} (-1)^n(2n+1)x^{n(n+1)/2} \pmod{R}. \tag{4.8}$$

Then proceeding as before we can prove

THEOREM 7. *If R is a prime then $p_{R+3}(mR+t) \equiv 0 \pmod{R}$, where $t \not\equiv k(k+1)/2 \pmod{R}$, and also when $t=(R^2-1)/8$.*

That t can also take the value $(R^2-1)/8$ can be proved as follows. From (4.8) it is evident that $p_{R+3}(n)$ will also be divisible by an R for which $(2n+1) \equiv 0 \pmod{R}$, meaning thereby that $n=(R-1)/2$ and $t=n(n+1)/2=(R^2-1)/8$. Similarly we can prove:

THEOREM 8. *If R is a prime then $p_{-(R-3)}(mR+t) \equiv 0 \pmod{R}$, where $t \not\equiv k(k+1)/2 \pmod{R}$ and also when $t=(R^2-1)/8$.*

As an illustration let us consider $R=5$. Then from Theorem 8, we get Ramanathan's congruences [12] for $p_{-2}(n)$, namely

$$p_{-2}(5m+2),\ p_{-2}(5m+3), \text{ and } p_{-2}(5m+4) \equiv 0 \pmod{5}. \tag{4.9}$$

Also we have

$$\sum_{n=0}^{\infty} p_{k\pm RS}(n)x^n = \sum_{n=0}^{\infty} p_k(n)x^n \sum_{n=0}^{\infty} p_{\pm RS}(n)x^n. \tag{4.10}$$

Expanding and equating the coefficients of x^n and putting $n=mR+t$ we get

$$p_{k\pm RS}(mR+t) = \sum_{j=0}^{mR+t} p_{\pm RS}(mR+t-j)p_k(j). \tag{4.11}$$

From (4.11) and Theorems 1 and 2 the following theorems can easily be proved.

THEOREM 9. *If $p_k(mR+t) \equiv 0 \pmod{R}$, R being a prime, and t is any integer for which the congruence holds, then $p_{k\pm RS}(mR+t) \equiv 0 \pmod{R}$.*

THEOREM 10. *If $p_{-k}(mR+t) \equiv 0 \pmod{R}$, R being a prime, and t is any integer for which the congruence holds, then $p_{-(k\pm LR)}(mR+t) \equiv 0 \pmod{R}$.*

Below are given some applications of Theorems 9 and 10.

We know that

$$\begin{aligned} &p_{-1}(5m+4) \equiv 0 \pmod{5}, \quad p_{-1}(7m+5) \equiv 0 \pmod{7}, \text{ and} \\ &p_{-1}(11m+6) \equiv 0 \pmod{11}. \end{aligned} \tag{4.12}$$

Then from Theorem 10, considering $L=1$, we have

$$p_{-6}(5m+4) \equiv 0 \pmod 5. \tag{4.13}$$

$$p_{-8}(7m+5) \equiv 0 \pmod 7. \tag{4.14}$$

$$p_{-12}(11m+6) \equiv 0 \pmod{11}. \tag{4.15}$$

And

$$p_{4}(5m+4) \equiv 0 \pmod 5. \tag{4.16}$$

$$p_{6}(7m+5) \equiv 0 \pmod 7. \tag{4.17}$$

$$p_{10}(11m+6) \equiv 0 \pmod{11}. \tag{4.18}$$

It is to be noted that the congruences (4.13) to (4.18) can also be derived from the identities given by M. Newman ([7]–[11]).

We now derive some well-known congruences for Ramanujan's τ function for the primes 5, 7 and 23 using the theorems of this paper.

From Theorem 7 we get

$$p_{10}(7m+t) \equiv 0 \pmod 7 \qquad \text{for } t = 2, 4, 5 \text{ and } 6. \tag{4.19}$$

Then using Theorem 9, with $k=10$ and $R=7$, we get

$$\begin{aligned} p_{10+14}(7m+t) &\equiv p_{24}(7m+t), \qquad \text{for } t = 2, 4, 5 \text{ and } 6. \\ &\equiv 0 \pmod 7, \end{aligned} \tag{4.20}$$

since

$$\tau(n) = p_{24}(n-1). \tag{4.21}$$

Hence from (4.20) we get

$$\tau(7m+t) \equiv 0 \pmod 7 \qquad \text{for } t = 3, 5, 6 \text{ and } 7. \tag{4.22}$$

From Theorem 5, considering $R=23$, we get

$$p_{24}(23m+t) \equiv 0 \pmod{23}, \tag{4.23}$$

where t runs over the nonresidues of $\beta(3\beta-1)/2 \pmod{23}$. Combining (4.21) and (4.23) we get

$$\tau(23m+t) \equiv 0 \pmod{23}, \tag{4.24}$$

where $t=1+$nonresidues of $\beta(3\beta-1)/2 \pmod{23}$.

In view of Theorem 9 with $k=4$ and $R=5$, from (4.16) we have

$$P_{4+20}(5m+4) \equiv P_{24}(5m+4) \equiv 0 \pmod 5. \tag{4.25}$$

Combining (4.25) and (4.21) we get

$$\tau(5m) \equiv 0 \pmod 5. \tag{4.26}$$

5. In this section we discuss the following congruences:

(5.1) $p_{-3}(3m+2) \equiv 0 \pmod 9$ (5.2) $p_{-3}(9m+5) \equiv 0 \pmod{27}$

(5.3) $p_{-3}(9m+8) \equiv 0 \pmod{81}$ and (5.4) $p_{-4}(4n+3) \equiv 0 \pmod 8$.

The author proved (5.1), and conjectured the other congruences; their proofs were supplied by M. Newman.

By the methods of M. Newman's paper [9] it can be proved that

$$\sum_{n=0}^{\infty} p_{-3}(3n+2)x^n = 9\prod_{n=1}^{\infty} \frac{(1-x^{3n})^9}{(1-x^n)^{12}} \tag{5.5}$$

so that

$$\frac{1}{9}\sum_{n=0}^{\infty} p_{-3}(3n+2)x^n \equiv \prod_{n=1}^{\infty} \frac{(1-x^{27n})}{(1-x^{3n})^4} \pmod 3.$$

Hence, if $(n, 3)=1$,

$$p_{-3}(3n+2) \equiv 0 \pmod{27}. \tag{5.6}$$

We also see from (5.5) that

$$\frac{1}{9}\sum_{n=0}^{\infty} p_{-3}(3n+2)x^n = \prod_{n=1}^{\infty} \frac{(1-x^{3n})^9}{(1-x^n)^9(1-x^n)^3} \equiv \prod_{n=1}^{\infty} \frac{(1-x^{3n})^6}{(1-x^n)^3} \pmod 9, \tag{5.7}$$

and hence

$$\frac{1}{9}p_{-3}(3n+2) \equiv \sum p_{-3}(n-3r)p_6(r) \pmod 9. \tag{5.8}$$

Replacing n by $3n+2$, we obtain

$$\frac{1}{9}p_{-3}(9n+8) \equiv \sum p_{-3}(3(n-r)+2)p_6(r) \equiv 0 \pmod 9$$
$$p_{-3}(9n+8) \equiv 0 \pmod{81}. \tag{5.9}$$

As was pointed by M. Newman, the congruence (5.4) follows immediately from O. Kolberg's result [4]:

$$p_{-4}(2n+1) \equiv 4p_{10}(n) \pmod{16}.$$

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A NOTE ON CERTAIN NON-ASSOCIATIVE ALGEBRAS

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1. Introduction. In this note we hope to point up similarities in the structure theory of certain non-associative algebras, all of which satisfy an identity of a particular form. The identities of this form constitute some sort of first order weakening of the associative identity. This will serve to emphasize the rather pathological behavior of a previously studied class of algebras which also satisfy an identity of this form.

2. Main Section. By an algebra A over a field F we shall mean a finite dimensional vector space over F which has a multiplication that distributes over addition. Furthermore, we shall assume throughout that F has characteristic $p \neq 2, 3, 5$. We define the associator (x, y, z) by $(x, y, z) = x(yz) - (xy)z$. Then an associative algebra A is one in which $(x, y, z) = 0$ for all $x, y, z \in A$. In the study of non-associative algebras one does not usually entirely abandon the associative identity; one merely replaces it by some weaker identity or set of identities. It would then seem natural to study algebras which satisfy an identity of the form:

$$\text{I.} \qquad (x, y, z) = \pm (\sigma(x), \sigma(y), \sigma(z)) \qquad \text{for all } x, y, z \in A,$$

where σ is some permutation of the three elements. That is, the associator of any three elements of A is equal to plus or minus the associator of some permutation σ of the three elements. Indeed, many well-known non-associative algebras satisfy an identity of this form. They include the left and right alternative algebras, the algebras of types $(1, 1)$ and $(-1, 0)$, and the flexible algebras.

In the remainder of this note it will be assumed that A is power-associative. Then certainly A satisfies $xx^2 = x^2x$ and thus, the linearization of this identity:

$$\text{II.} \qquad \sum_{\sigma} (\sigma(x), \sigma(y), \sigma(z)) = 0 \qquad \text{for all } x, y, z \in A,$$

where σ runs over the six permutations of the three elements [1, Chapter I]. As is easily seen, there are eleven nontrivial identities of the form I. These we enumerate below:

(1)	$(x, y, z) = -(x, z, y)$	(6)	$(x, y, z) = -(y, z, x)$
(2)	$(x, y, z) = -(y, x, z)$	(7)	$(x, y, z) = -(z, x, y)$
(3)	$(x, y, z) = (x, z, y)$	(8)	$(x, y, z) = (y, z, x)$
(4)	$(x, y, z) = (y, x, z)$	(9)	$(x, y, z) = (z, x, y)$
(5)	$(x, y, z) = -(x, y, z)$	(10)	$(x, y, z) = -(z, y, x)$

$$(11) \qquad (x, y, z) = (z, y, x).$$

The right alternative identity $yx^2 = (yx)x$ is equivalent to (1) and the left alternative identity $x^2y = x(xy)$ is equivalent to (2) for F of characteristic not 2 [2]. Algebras which satisfy II and either (3) or (4) are of type $(1, 1)$ or $(-1, 0)$, respectively [4]. Certainly (5) is equivalent to the associative identity for F of characteristic not 2. Successive applications of (6) yield

$$(x, y, z) = -(y, z, x) = (z, x, y) = -(x, y, z)$$

so that (6) implies (5) and, in a similar manner, (7) implies (5). Suppose A satisfies II and (8). Then

$$(x, y, z) = (y, z, x) = (z, x, y).$$

This along with II implies that $3(x, y, z) = -3(y, x, z)$. By the conditions on F, we have $(x, y, z) = -(y, x, z)$ which is (2). But using (8) we find $(x, y, z) = -(y, x, z) = -(x, z, y)$. Thus A must also satisfy (1) so that A is alternative. Likewise, if A satisfies II and (9), A is alternative. Identity (10) is the linearization of the flexible identity $x(yx) = (xy)x$. Again, for F of characteristic not 2, (10) is equivalent to the flexible identity. Algebras satisfying (11) were introduced in [5].

We define the radical N of a power-associative algebra A to be the maximal nil ideal of A. A is said to be semi-simple whenever $N = 0$. An algebra A is simple if A has no nontrivial ideals and is not a nil algebra.

THEOREM 1. *Let A be a semi-simple algebra satisfying one of the identities (1)–(9). Then A is an alternative algebra.*

Proof. If A satisfies (1) and (2), then the result follows from [2, Theorem 6]. If A satisfies (3) or (4), then the theorem holds by [3, Main Theorem]. The remaining cases are immediate from our earlier remarks.

THEOREM 2. *Let A be a semi-simple algebra satisfying one of the identities* (1)–(10). *Then A has a unity element and is the direct sum of simple algebras.*

Proof. This result is well known for alternative algebras. As for algebras satisfying (10), we refer to [6, Theorem 3.5].

THEOREM 3. *Let A be a simple algebra satisfying one of the identities* (1)–(10). *Then A is one of the following*:

(i) *a commutative Jordan algebra*;

(ii) *a quasi-associative algebra*; *or*

(iii) *an algebra of degree* 1 *or* 2.

Proof. This is Oehmke's classification theorem for simple flexible algebras [6, Theorem 4.2] which is applicable as soon as we note that any alternative algebra is also flexible.

3. Examples. We have purposely omitted those algebras satisfying (11) from our discussion since the above stated results do not hold in this instance.

Let B be the algebra over the field F of characteristic not 2 with a basis e, u, v where e is the unity element of B, $uv=-vu=e$, and $u^2=v^2=0$. Then B is a power-associative algebra satisfying (11). Note that B is not flexible since $u(vu)=-(uv)u=-u$. By direct verification one sees that B has no nontrivial right or left ideals. Thus B is simple.

Now set $C=B+B'$ where B is the algebra defined above and B' is the two dimensional algebra over F with a basis u', v' where $u'^2=v'^2=0$, and $u'v'=-v'u'=e$, e the unity of B. We also set $BB'=B'B=0$. Then C is a power-associative algebra satisfying (11). Any nonzero right or left ideal of C contains the simple ideal B. Thus C has nil radical $N=0$. Hence, C is semi-simple. But, C has no unity element and is not the direct sum of simple algebras.

Using two different characterizations of the Jacobson radical (for associative rings) we find that in one instance C is semi-simple (i.e.C has zero radical) while for the other characterization C is a radical ring. Let R be a ring (non-associative). An element $x\in R$ is right quasi-regular if there is a $y\in R$ such that $x+y-xy=0$. A right ideal of R is quasi-regular if every element of the right ideal is right quasi-regular. We then define the radical J_1 of R to be the maximal quasi-regular right ideal. With this definition of the radical we see that the ring C defined above has radical $J_1=0$ since any nonzero right ideal of C must contain the element e which is not right quasi-regular. We say a right ideal M of a ring R is modular if there is an element $a\in R$ such that for all $x\in R$, $ax-x\in M$. Then define the radical J_2 of R to be the intersection of the modular maximal right ideals of R. In the ring C there are no proper modular right ideals M for any such right ideal must contain the ideal B. Since $C^2=B$, M must be C which is not proper. Thus, the radical J_2 of C is C so that with this definition of the radical C is a radical ring. For further examples of simple power-associative algebras satisfying (11) see [5].

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A GENERALIZED POLARIZATION IDENTITY

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An important consequence of the identity between two complex numbers z, w

$$z\bar{w} = \frac{1}{4}\sum_{m=1}^{4} i^m \left| z + i^m w \right|^2 \tag{1}$$

is its extension to conjugate bilinear functionals defined on a complex linear space. In this note we shall derive the following generalization of (1):

Identity 1. For every $n=1, 2, \cdots$, there exists an integer M_n and complex numbers (depending only on n)

$$a_{jk},\ b_k,\ 1 \leqq j \leqq 2n,\ 1 \leqq k \leqq M_n$$

such that for every set $z_1, \cdots, z_{2n}$ of $2n$ complex numbers

$$z_1 \cdot z_2 \cdots z_n \cdot \bar{z}_{n+1} \cdot \bar{z}_{n+2} \cdots \bar{z}_{2n} = \sum_{k=1}^{M_n} b_k \left| \sum_{j=1}^{2n} a_{jk} z_j \right|^{2n}. \tag{2}$$

Identity 1 has an immediate extension to conjugate multilinear functionals on a complex linear space. For example, in the Lebesgue space L_{2n} define

$$(f_1, \cdots, f_n; f_{n+1}, \cdots, f_{2n}) = \int \prod_{1}^{n} f_k \bar{f}_{k+n} dt. \tag{3}$$

Then from (2)

$$(f_1, \cdots, f_n; f_{n+1}, \cdots, f_{2n}) = \sum_{k=1}^{M_n} b_k \left\| \sum_{j=1}^{2n} a_{jk} f_j \right\|^{2n}, \tag{4}$$

where $\|f\| = (\int |f|^{2n} dt)^{1/2n}$ is the norm of L_{2n}.

From (4) it follows that the spaces L_{2n} have more structure than a general Banach space. There is a generalized scalar product given by (3) which is derived from the norm in exact analogy to the Hilbert space situation. The author

has carried this analogy one step further by obtaining a generalization of the parallelogram law. By an appropriate extension of Identity 3 below we can derive

Identity 2. For every $n=1, 2, \cdots$, there exists an integer R_n and nonzero numbers (depending only on n) a, c_k, d_k, $k=1, \cdots, R_n$ such that for every pair of complex numbers z, w

$$|z|^{2n} + a|w|^{2n} = \sum_{k=1}^{R_n} c_k |z + d_k w|^{2n}. \tag{5}$$

We shall now derive Identity 1 by proving four more identities. Although a somewhat shorter existence proof can be given, we have chosen the longer route since we actually exhibit the constants a_{jk}, b_k, M_n and, moreover, the method may suggest further generalizations (e.g., a parallelogram law for $z_1, z_2, \cdots, z_{2n}$).

Identity 3. Let n be any positive integer, $\omega = \exp(i\pi/2n)$. Then for any complex number z,

$$\sum_{k=1}^{4n} |z + \omega^k|^{2n} = 4n \sum_{s=0}^{n} C_s |z|^{2s},$$

where

$$C_s = n! \sum_{m=0}^{s} ((m!)^2(s-m)!(n-s-m)!)^{-1}, \qquad 0 \leqq s \leqq [n/2],$$

$$C_s = C_{n-s}, \qquad [n/2] < s \leqq n.$$

Proof. Setting $z = r\exp(-i\theta)$

$$\sum_{k=1}^{4n} |z + \omega^k|^{2n} = \sum_{j=0}^{n} \binom{n}{j} (2r)^j (1 + r^2)^{n-j} S_j(\theta), \tag{6}$$

where

$$S_j(\theta) = \sum_{k=1}^{4n} \left(\cos\left(\theta + \frac{\pi k}{2n}\right)\right)^j = 0, \qquad \text{if } j \text{ is odd},$$
$$(7)$$
$$= 4n2^{-j}\binom{j}{j/2}, \qquad \text{if } j \text{ is even}.$$

(6) and (7) combined result in

$$\sum_{k=1}^{4n} |z + \omega^k|^{2n} = 4n\, n! \sum_{s=0}^{n} r^{2s} \sum_{m,j \in A_s} ((m!)^2 j!(n-2m-j)!)^{-1}, \tag{8}$$

where A_s is the set of pairs m, j satisfying the following three conditions simultaneously: $m+j=s$, $0 \leqq m \leqq [n/2]$, $0 \leqq j \leqq n-2m$. The sum over A_s can be converted to the form of the C_s by separate consideration for $s \leqq [n/2]$ and $s > [n/2]$.

Identity 4. Let $P(z)=\sum_{k=1}^{m} a_k z^k$ be any homogeneous polynomial and let b be any complex number. Then

$$a_m\left(\prod_{s=1}^{m-1}(b^m-b^s)\right)z^m=\sum_{j=0}^{m-1}(-1)^j\sigma_j^{(m-1)}(b,b^2,\cdots,b^{m-1})P(b^{m-1-j}z),$$

where $\sigma_j^{(m-1)}(u_1,\cdots,u_{m-1})$ is the jth elementary symmetric function in $m-1$ letters ($\sigma_0^{(m-1)}\equiv 1$ and $\prod_{s=1}^{0}(b^1-b^s)\equiv 1$).

Proof. The proof follows from substituting the polynomial expression for $P(b^{m-1-j}z)$, interchanging summation over k and j and observing that for $1\leqq k\leqq m$

$$\sum_{j=0}^{m-1}(-1)^j\sigma_j^{(m-1)}(b,b^2,\cdots,b^{m-1})b^{(m-1-j)k}=\prod_{s=1}^{m-1}(b^k-b^s)=\delta_{km}\prod_{s=1}^{m-1}(b^m-b^s).$$

Identity 5. Let $n\geqq 2$, $\omega=\exp(i\pi/2n)$. Then for any complex number z and any nonnegative number ρ,

$$\begin{aligned}4n^3|z|^{2(n-1)}&\prod_{s=1}^{n-2}(\rho^{2(n-1)}-\rho^{2s})\\&=\sum_{j=0}^{n-2}\sum_{k=1}^{4n}(-1)^j\sigma_j^{(n-2)}(\rho^2,\cdots,\rho^{2(n-2)})|\rho^{n-2-j}z+\omega^k|^{2n}\\&\quad-4n\prod_{j=1}^{n-2}(1-\rho^{2j})-4n|z|^{2n}\prod_{j=1}^{n-2}(\rho^{2n}-\rho^{2j}).\end{aligned}$$

(For $n=2$ the products are defined to be 1.)

Proof. In Identity 4 set $z=w^2$, $b=\rho^2$ and

$$P(w^2)=G(w)=\sum_{s=1}^{n-1}C_s w^{2s},$$

where the C_s are from Identity 3. Replacing w with $|z|$ in the resulting equation and using Identity 3 for $G(\rho^{n-2-j}|z|)=G(|\rho^{n-2-j}z|)$ we obtain

$$\begin{aligned}4nC_{n-1}|z|^{2(n-1)}&\prod_{s=1}^{n-2}(\rho^{2(n-1)}-\rho^{2s})\\&=\sum_{j=0}^{n-2}\sum_{k=1}^{4n}(-1)^j\sigma_j^{(n-2)}(\rho^2,\cdots,\rho^{2(n-2)})|\rho^{n-2-j}z+\omega^k|^{2n}\\&\quad-4n\sum_{j=0}^{n-2}(-1)^j\sigma_j^{(n-2)}(\rho^2,\cdots,\rho^{2(n-2)})\\&\quad-4n|z|^{2n}\sum_{j=0}^{n-2}(-1)^j\sigma_j^{(n-2)}(\rho^2,\cdots,\rho^{2(n-2)})\rho^{2n(n-2-j)},\end{aligned}\tag{9}$$

where we have used $C_0 = C_n = 1$. Since

$$\sum_{j=0}^{n-2} (-1)^j \sigma_j^{(n-2)}(u_1, \cdots, u_{n-2})x^j = \prod_{s=1}^{n-2} (1 - u_s x)$$

and $C_{n-1} = C_1 = n^2$ then (9) reduces to Identity 5.

Identity 6. For every $n = 2, 3, \cdots$, there exists an integer L_n and complex numbers (depending on n) a_k, b_k, d_k, $1 \leqq k \leqq L_n$, such that for every pair of complex numbers z, w

$$|z|^{2(n-1)} |w|^2 = \sum_{k=1}^{L_n} a_k | b_k z + d_k w |^{2n}.$$

Proof. For $w \neq 0$ replace z with zw^{-1} in Identity 5. Choose $\rho > 0$, $\rho \neq 1$ and multiply through with

$$\frac{1}{4} n^{-3} | w |^{2n} \prod_{s=1}^{n-2} (\rho^{2(n-1)} - \rho^{2s})^{-1}.$$

The result for $w = 0$ then follows by continuity.

We now prove Identity 1. For $n = 1$ we have (1). Assume the identity for $n-1$, $n \geqq 2$. Then

$$z_1 \cdots z_n \cdot \bar{z}_{n+1} \cdots \bar{z}_{2n}$$

$$= \sum_{k=1}^{M_{n-1}} b_k^{(n-1)} z_n \bar{z}_{2n} \left| \sum_{j=1}^{n-1} a_{jk}^{(n-1)} z_j + \sum_{j=n}^{2(n-1)} a_{jk}^{(n-1)} z_{j+1} \right|^{2(n-1)}, \tag{10}$$

where we have denoted the dependence of the constants on $n-1$ with superscripts.

Replace z with z_n and w with z_{2n} in (1) and substitute the result in the right hand side of (10). Identity 1 is then obtained for n by applying Identity 6.

We would like to end by making a conjecture. In [1] it was shown that if L is a normed linear space with norm $\| \ \|$ then a necessary and sufficient condition that there exist a scalar product (,) on L such that $\|f\|^2 = (f, f)$ is

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2 \qquad \text{for all } f, g \in L. \tag{11}$$

Now it is easy to abstract the various properties of the generalized scalar product given by (3) and these properties will define our notion of an nth order scalar product. Now we ask: What is a necessary and sufficient condition on the norm $\| \ \|$ of a normed complex linear space so that there exists an nth order scalar product (;) such that

$$\|f\|^{2n} = (f, \cdots, f; f, \cdots, f)?$$

It is conjectured that the condition is either Identity 2 (with $| \ |$ replaced by $\| \ \|$) or else an appropriate generalization of (5) for $z_1, \cdots, z_{2n}$.

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AN ALTERNATIVE DERIVATION OF THE Z-TRANSFORM

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Introduction. The Z-transform, originally developed for use in the analysis of sampled data and digital control systems, has been historically developed by "modulating an impulse train," which is a mathematically unsatisfying approach. The present work extends that of H. A. Helm, of the Bell Telephone Laboratories to include nonzero initial conditions and functions that are discontinuous at the sampling instants. It is hoped that the derivation given here will encourage the study of the Z-transform and its adoption in the analysis of situations to which it has not yet been applied.

A general transform expression. Using the Stieltjes integral, we may define a transform, which we call the Laplace-Stieltjes transform and denote by

$$L_s\{f;\alpha\} = \int_0^\infty f(t)e^{-st}d\alpha(t), \tag{1}$$

where $f(t)$ is a real valued function defined for all t and equal to zero for $t<0$, and where $s=\sigma+j\omega$.

This expression is a generalization of the familiar Laplace transform. It reduces to the Laplace transform for $\alpha(t)=t$. If $\alpha(t)$ is continuous and has a continuous first derivative, $\alpha'(t)$, (1) reduces to the Laplace transform of $f(t)\alpha'(t)$.

The Z-transform. The function $\alpha(t)$ need not be a continuous function. We will consider one discontinuous integrator in detail, for it will lead, after a change of variable, to the Z-transform expression.

Consider the real-valued function $\alpha_1(t)$ defined for all t to be the unique integer satisfying the inequalities

$$\alpha_1(t) < t \leqq \alpha_1(t) + 1. \tag{2}$$

We note that $\alpha_1(t)$ differs from the "greatest-integer function," $[t]$, only in that

$\alpha_1(t)$ is continuous from the left at each discontinuity, while $[t]$ is continuous from the right at each discontinuity.

This integrating function is a special case of the class of integrating functions known as step function integrators. For this class of integrators the integral in (1) reduces to a sum. To illustrate, consider the integral

$$\int_a^b f(t)d\alpha(t), \tag{3}$$

where $\alpha(t)$ is defined on $[a, b]$ as follows:

$$\begin{aligned}
&\alpha(a),\ \alpha(c),\ \alpha(b) \text{ are arbitrary,}\\
&\alpha(t) = \alpha(a) \text{ if } a \leqq t < c,\\
&\alpha(t) = \alpha(b) \text{ if } c < t \leqq b.
\end{aligned}$$

It can be shown that if $f(t)$ is any function defined on $[a, b]$ in such a way that at least one of the functions f and α is continuous from the left at $t=c$ and at least one is continuous from the right at $t=c$, then f is Stieltjes integrable with respect to α on $[a, b]$ and

$$\int_a^b f(t)d\alpha(t) = f(c)[\alpha(c+) - \alpha(c-)], \tag{4}$$

where $\alpha(c-)$ and $\alpha(c+)$ are the left and right hand limits of α with respect to t at c.

From the linearity of the Stieltjes integral we may write for the expression (1), with $\alpha=\alpha_1$,

$$\int_0^\infty f(t)e^{-st}d\alpha_1(t) = \sum_{k=0}^{\infty} f(k)e^{-sk}. \tag{5}$$

This follows directly from (4), for the integral may be written as a sum of integrals taken over intervals containing only one discontinuity of α_1, and the jump in α_1 at each discontinuity is a unit jump.

Now if the scale of α_1 is changed from t to t/T, we have the integrator $\alpha_1(t/T)$ which is discontinuous at $t=0, T, 2T, \cdots$ and otherwise identical to $\alpha_1(t)$. We then have the transformation

$$L_s\{f; \alpha_1\} = \int_0^\infty f(t)e^{-st}d\alpha_1(t/T). \tag{6}$$

Writing this as a series, we have

$$L_s\{f; \alpha_1\} = \int_0^\infty f(t)e^{-st}d\alpha_1(t/T) = \sum_{n=0}^{\infty} f(nT)e^{-snT} = F(e^{sT}). \tag{7}$$

Letting $e^{sT}=z$ we have

$$L_s\{f;\alpha_1\} = \int_0^\infty f(t)z^{-t/T}d\alpha_1(t/T) = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z). \tag{8}$$

$F(z)$ is the Z-transform of $f(t)$.

If the series in (8) converges at all for some z, it has a finite radius of convergence δ, and converges absolutely and uniformly for all $|z| \geqq \delta_1 > \delta$. Using $|z| = |e^{sT}| = e^{\sigma T}$, this gives $\sigma \geqq 1/T \log \rho_1 > 1/T \log \rho$, where $1/T \log \rho$ is the abscissa of convergence of the transform (6).

In writing the integrals in (7) and (8) as infinite series, special care must be taken when the function f is not continuous at the discontinuities of α_1. Note that f and α_1 must be such that at least one of these functions is continuous from the left at each discontinuity of α_1 and at least one is continuous from the right at each discontinuity of α_1.

If we do not define $f(0)$ as $f(0^+)$, but rather maintain that $f(0)=0$, then we must re-define the Z-transform as

$$F(z) = f(0^+) + \int_T^\infty f(t)z^{-t/T}d\alpha_1(t/T). \tag{9}$$

A similar situation exists wherever f and α_1 are both discontinuous.

The inverse Z-transform. The inverse Z-transform may be readily derived from the series expression in (8) by multiplying through by z^{n-1} and integrating term-by-term around any closed contour in the z-plane which encloses the origin and for which $|z| \geqq |z_0| = \delta_1$. For most transforms a convenient contour is a circle about the origin enclosing all poles of $F(z)$. The development of the inversion integral can be made in this way quite readily, with the result

$$f(nT) = \frac{1}{2\pi i}\oint_\Gamma z^{n-1}F(z)dz, \tag{10}$$

where Γ is a contour as described above. It is to be noted that inversion of the Z-transform only gives the values of $f(t)$ for $t=nT$. This is not really surprising, for these are the only values of $f(t)$ used in arriving at $F(z)$.

For convenience, the Z-transform pair is given below.

$$F(z) = L_s\{f;\alpha_1\} = \int_0^\infty f(t)z^{-t/T}d\alpha_1(t/T) = \sum_{n=0}^{\infty} f(nT)z^{-n}, \tag{11}$$

$$f(nT) = \frac{1}{2\pi i}\oint_\Gamma z^{n-1}F(z)dz. \tag{12}$$

Tables of transform pairs for specific functions and tables of the properties of the Z-transform can be found in many places in the literature **[1, 2, 3, 4]**.

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LARGEST SUBTOPOLOGIES WITH SPECIAL PROPERTIES

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Let (X, τ) be a topological space and P any topological property. We shall be interested in the following question: Is there a largest P-subtopology of τ (i.e., a P-subtopology for X which contains all other P-subtopologies for X)?

Remark 1. Let (X, τ) be a topological space and $\{\tau_\alpha\}_{\alpha\in\Delta}$ a family of subtopologies. It is well known that there exists a smallest subtopology τ^* of τ which contains every τ_α. We term τ^* the topology generated by $\{\tau_\alpha\}_{\alpha\in\Delta}$.

THEOREM 1. *Let (X, τ) be a topological space and suppose τ has a T_3-subtopology. Then τ has a largest T_3-subtopology. (T_3 means regular and T_1.)*

Proof. Let $\{\tau_\alpha\}_{\alpha\in\Delta}$ be the family of all T_3-subtopologies of τ. Then $\Delta\neq\emptyset$. Let τ^* be the topology generated by $\{\tau_\alpha\}_{\alpha\in\Delta}$. It suffices to show that τ^* is T_3. But τ^* is T_1 since it contains a T_1 topology. We show now that τ^* is a regular topology. Let $x \in U \in \tau^*$. Then $x \in U_1 \cap \cdots \cap U_n \subset U$, where $U_i \in \tau_{\alpha_i}$, $i=1, \cdots, n$. Since τ_{α_i} is regular, there exists for each i a $V_i\in\tau_{\alpha_i}$ such that $x\in V_i\subset\overline{V}_i\subset U_i$. Let $V=V_1\cap \cdots \cap V_n$. Then

$$x \in V \subset \overline{V} = \overline{V_1\cap \cdots \cap V_n} \subset \overline{V}_1 \cap \cdots \cap \overline{V}_n \subset U_1 \cap \cdots \cap U_n \subset U.$$

Thus (X, τ^*) is regular.

Remark 2. The methods of the above proof, together with the fact that $\{\emptyset, X\}$ is a regular-subtopology, show that every topology has a largest regular-subtopology.

THEOREM 2. *Let (X, τ) be a topological space and let τ have a Tychonoff-subtopology. Then τ has a largest Tychonoff-subtopology. (Tychonoff means completely regular and T_1.)*

Proof. Let $\{\tau_\alpha\}_{\alpha\in\Delta}$ be the family of all Tychonoff-subtopologies of τ. Then $\Delta\neq\emptyset$. Let τ^* be the topology generated by $\{\tau_\alpha\}_{\alpha\in\Delta}$. τ^* is clearly T_1 since it contains a subtopology which is T_1. It suffices then to show that τ^* is completely regular. Let $z \in U \in \tau^*$. Then $z \in U_1 \cap \cdots \cap U_n \subset U$ where $U_i \in \tau_{\alpha_i}$ for $i=1, \cdots, n$. Since τ_{α_i} is a completely regular-subtopology of τ, for each i there exists an $f_i\colon X\to[0, 1]$ continuous relative to τ_{α_i} and hence τ^* such that $f_i(z)=0$ and $f_i(\mathcal{C}U_i)=1$. Let $h(x)=\max\{f_i(x)\}_{i=1}^n$. Then $h\colon X\to[0, 1]$ is continuous relative to τ^*. It clearly suffices to prove this for $n=2$. But this follows immediately from the well-known identity $\max\{f_1, f_2\}=(f_1+f_2+|f_1-f_2|)/2$. Now $h(z)=0$ and

$$\begin{aligned} h(\mathcal{C}U) \subset h(\mathcal{C}(U_1\cap\cdots\cap U_n)) &= h(\mathcal{C}U_1\cup\cdots\cup\mathcal{C}U_n) \\ &= h(\mathcal{C}U_1)\cup\cdots\cup h(\mathcal{C}U_n) \\ &= f_1(\mathcal{C}U_1)\cup\cdots\cup f_n(\mathcal{C}U_n) = \{1\}. \end{aligned}$$

Remark 3. Using the above methods, together with the fact that $\{\emptyset, X\}$ is a completely regular-subtopology of τ, we can show that every topology has a largest completely regular-subtopology.

THEOREM 3. *Let (X, τ) be a noncompact topological space. Then there does not exist a largest compact-subtopology of τ.*

Proof. Let τ^* be any compact-subtopology of τ. Then $\tau^*\neq\tau$. Take $U\in\tau-\tau^*$. Then $\{\emptyset, U, X\}$ is a compact-subtopology of τ, but $\{\emptyset, U, X\}\not\subset\tau^*$. Thus τ does not have a largest compact-subtopology.

Remark 4. Theorem 3 remains valid if compact is replaced by first axiom, Lindelöf, connected, normal, locally compact, separable, and no doubt many other properties.

Finally the author wishes to thank the referee whose suggestions improved the exposition and simplified some of the proofs.

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MATHEMATICAL NOTES

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A THEOREM OF LEVITZKI

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THEOREM. *Let S be a ring with maximum condition for left ideals. Then every nil one-sided ideal of S is nilpotent.*

This theorem is well known. (See Levitzki's "On Multiplicative Systems," Comp. Math. 8 (1950), and N. Jacobson, "Structure of Rings," p. 199, and I. N. Herstein "A Theorem of Levitzki," Proc. AMS 13 (1962), where a proof by A. W. Goldie is also cited.) The following proof is believed to be simpler than any known proof.

We denote the left annihilator of an element x by $l(x)$, and the (two-sided) ideal generated by x by $(x)_t$.

LEMMA. *Let S be a semigroup with maximum condition for annihilator left ideals. If S has a nonzero nil left or right ideal, then S contains a nonzero nilpotent ideal.*

Proof. Let A be a nonzero nil right ideal, and let $l(a)$, $0 \neq a \in A$, be a maximal one among $l(x)$ for all $0 \neq x \in A$. If $au \neq 0$, then $(au)^n = 0$, $(au)^{n-1} \neq 0$ for some $n > 1$. Hence $au \in l((au)^{n-1}) = l(a)$ and $aua = 0$. Thus, $aSa = 0$ and $(a)_t^3 = 0$.

In case S has a nonzero nil left ideal B, Sb is nil for any $b \in B$, and hence bS also is nil. Let $0 \neq b \in B$. If $bS = 0$, $(b)_t^2 = 0$. If $bS \neq 0$, we may apply the above argument to $A = bS$.

Proof of the theorem. Let N be the maximal nilpotent ideal of S, and C a nil right (left) ideal of S. If C were not contained in N, then $(C+N)/N$ would be a nonzero nil right (left) ideal of S/N, and hence S/N would contain a nonzero nilpotent ideal by the lemma, contradicting the maximality of N. Thus, $C \subset N$, whence any nil one-sided ideal C is nilpotent, as desired.

COMPACTNESS IN FUNCTION SPACES

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It is well known that a family F of continuous functions from a compact uniform space to a compact uniform space has compact closure in the uniform topology if and only if F is equicontinuous (Theorem 11.13 [3]). Irving Glicksberg [2] has shown that a topological space X is pseudocompact if and only if every pointwise bounded equicontinuous family of real valued continuous functions on X has compact closure in the uniform topology. In this note we consider families of functions from a pseudocompact space X to a uniform space

$(Y, \mathcal{V})$. A topological space X is pseudocompact if every real valued continuous function on X is bounded. We do not require that X be completely regular. It is shown that a family F of continuous functions from a pseudocompact space to a compact uniform space is equicontinuous if and only if F has compact closure in the uniform topology. If X is a topological space and $(Y, \mathcal{V})$ is a uniform space, "$C(X, (Y, \mathcal{V}))$" will denote the space of continuous functions from X to Y with the uniform topology.

LEMMA 1. *A topological space (X, τ) is pseudocompact if and only if (X, p) is compact for each pseudo-metric p such that the topology corresponding to p is contained in τ.*

Proof. If (X, τ) is pseudocompact and p is a pseudo-metric whose topology is contained in τ, then (X, p) is pseudocompact. Since a pseudo-metric space is completely regular and paracompact, it follows from Theorems 3 and 8 [1] that (X, p) is compact. For the reverse implication suppose that (X, p) is compact for each pseudo-metric whose topology is contained in τ. Let f be a continuous real valued function on (X, τ). The smallest uniformity on X which makes f uniformly continuous has a countable base and is consequently pseudometrizable. The corresponding topology τ_0 is contained in τ since f is continuous on (X, τ). Now (X, τ_0) is compact by hypothesis and f is continuous on (X, τ_0). Therefore f is bounded.

LEMMA 2. *If $(X, \mathcal{U})$ is a uniform space and the pseudometric space (X, p) is compact for each pseudo-metric p in the gage P of $\mathcal{U}$, then $(X, \mathcal{U})$ is totally bounded.*

Proof. This follows immediately since $(X, \mathcal{U})$ is uniformly isomorphic to a subspace of the compact space $\chi\{(X, p) \mid p \in P\}$. (Cf. [4, p. 187].)

COROLLARY. *If (X, τ) is a pseudocompact space and $\mathcal{U}$ is a uniformity whose topology is contained in τ, then $(X, \mathcal{U})$ is totally bounded.*

From the corollary we see that a topologically complete, completely regular space is compact if and only if it is pseudocompact.

LEMMA 3. *If X is a pseudocompact space, (Y, p) a pseudometric space and F is an equicontinuous family of functions on X to Y such that $\overline{F(x)}$ is compact in Y for each x in X, then F has compact closure in $C(X, (Y, p))$.*

Proof. Let $\mathcal{U}$ be the smallest uniformity on X which makes the family F uniformly equicontinuous. $(X, \mathcal{U})$ is pseudocompact since the $\mathcal{U}$-topology is contained in the original topology on X. Also $(X, \mathcal{U})$ is pseudo-metrizable. Thus $(X, \mathcal{U})$ is compact as in the proof of Lemma 1. The conclusion now follows from Theorem 21 [4, p. 236].

LEMMA 4. *If X is a topological space and $(Y, \mathcal{V})$ is a complete uniform space, then the family of all continuous functions from X to $(Y, \mathcal{V})$ is complete in the uniform topology.*

Proof. See 11.05, page 91 [3].

Let X be a topological space and $(Y, \mathcal{V})$ a pseudocompact uniform space. A set of pseudo-metrics P is a subbase for the gage of the uniformity $\mathcal{V}$ if the collection of all sets of the form $\{(y_1, y_2) \mid p(y_1, y_2) < r\}$ where $r > 0$ and $p \in P$ is a subbase for $\mathcal{V}$. Let P be a set of pseudo-metrics on Y which forms a subbase for the gage of $\mathcal{V}$ and let $p \in P$. We define a pseudo-metric p^* on Y^X as follows.

$$p^*(f, g) = \underset{x \in X}{\text{l.u.b.}} \{p(f(x), g(x))\}.$$

The set $\{p^* \mid p \in P\}$ is a subbase for the gage of the uniformity $\mathcal{V}^*$ of uniform convergence in Y^X. For $F \subset Y^X$ we will denote by F^p the closure of F in the topology determined by p^*.

THEOREM 1. *If X is a pseudocompact space, $(Y, \mathcal{V})$ is a pseudocompact uniform space and F is a family of continuous functions from X to Y, then F is equicontinuous if and only if F is totally bounded in $C(X, (Y, \mathcal{V}))$. In addition, if F is equicontinuous, then $\chi\{(F^p, p^*) \mid p \in P\}$ is a compactification (not Hausdorff) of F with the uniform topology.*

Proof. Assume that F is equicontinuous. Since Y is pseudocompact (Y, p) is compact for each $p \in P$, a base for the gage of $\mathcal{V}$. Thus by Lemma 3 and the Tychonoff product theorem $\chi\{(F^p, p^*) \mid p \in P\}$ is compact. Now $(F, \mathcal{V}^*)$ is uniformly isomorphic to a subset of this compact space and is consequently totally bounded. Since X, with the smallest uniformity which makes each member of F uniformly continuous, is totally bounded the reverse implication follows easily. (Cf. 11.12 [3].) The remaining conclusion follows from the definition of F^p.

It should be noted that the hypothesis on $(Y, \mathcal{V})$ can be weakened to get the equivalence of equicontinuity and total boundedness of F. It is sufficient to assume that $F(X)$ is totally bounded in $(Y, \mathcal{V})$.

As an immediate consequence of Lemma 4 and Theorem 1 we have the following.

THEOREM 2. *If X is a pseudocompact space, $(Y, \mathcal{V})$ is a compact uniform space and F is a family of continuous functions from X to Y, then F is equicontinuous if and only if F has compact closure in $C(X, (Y, \mathcal{V}))$.*

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COMPLEX FIBONACCI NUMBERS AND FIBONACCI QUATERNIONS

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1. Introduction. In [1], [2] and [3], properties of the generalised Fibonacci sequence

$$\begin{array}{lllllllll} & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \cdots & H_n, \cdots, \\ (1)\quad H(\equiv H_{pq}): & p & p+q & 2p+q & 3p+2q & 5p+3q & 8p+5q \cdots & (p-q)F_n+qF_{n+1} \cdots, \end{array}$$

were discussed (p, q arbitrary integers), where F_n is the nth term of the classical Fibonacci sequence $F(\equiv H_{10})$. The Lucas sequence is $G \equiv H_{12}$.

Here, we extend the results of [1] and [2] to the complex case. Because of the various types of sequences dealt with, it is convenient to distinguish the corresponding p, q, l, m, e of [1] by an appropriate suffix. Thus, in (1), we must now write $p_H, \cdots, e_H$ for $p, \cdots, e$.

2. Generalised complex Fibonacci sequence. Define the *nth generalised complex Fibonacci number*

$$(2)\qquad D_n = H_n + iH_{n+1} = (p_H - q_H + iq_H)F_n + (q_H + ip_H)F_{n+1}$$

using (1), whence

$$(3)\qquad p_D = (1+i)p_H + iq_H, \qquad q_D = q_H + ip_H$$

i.e., the *generalised complex Fibonacci sequence* is $D \equiv H_{(1+i)p_H+iq_H, q_H+ip_H}$.

Special cases: (a) $p_H = 1$, $q_H = 0$: complex Fibonacci sequence $C \equiv H_{1+i,i}$, (b) $p_H = 1$, $q_H = 2$: complex Lucas sequence $H_{1+3i,2+i}$.

From (a), we have

$$(4)\qquad e_C = 2p_C - q_C = 2(1+i) - i = 2 + i,$$

and from [1], (3) and (4),

$$\begin{aligned} e_D &= \{(1+i)p_H + iq_H\}^2 - \{(1+i)p_H + iq_H\}\{q_H + ip_H\} - \{q_H + ip_H\}^2 \\ &= (2+i)\{p_H^2 - p_Hq_H - q_H^2\} = e_Ce_H. \end{aligned}$$

Another sequence, among many, for which $e_C = 2+i$ is $H_{1,-i}$ implying $H_n = -iC_{n-1}$. The conjugate $\overline{C}_n$ of $C_n = F_n + iF_{n+1}$ leads to the sequence $\overline{C} \equiv H_{1-i,-i}$ or, alternatively, $H_{1,i}$ for which $H_n = i\overline{C}_{n-1}$. Also $\bar{e}_C = e_{\overline{C}}$.

Parallelling [1], we have, for C,

$$(5)\qquad \begin{cases} C_{n-1}C_{n+1} - C_n^2 = (-1)^n e_C \\ C_n^2 + C_{n+1}^2 = ie_CF_{2n+2} \\ C_{n+1}^2 - C_{n-1}^2 = ie_CF_{2n+1} \\ (2C_{n+1}C_{n+2})^2 + (C_nC_{n+3})^2 = (2C_{n+1}C_{n+2} + C_n^2)^2 \quad \text{(Pythagorean theorem)} \\ \dfrac{C_{n+r} + (-1)^rC_{n-r}}{C_n} = F_{r+1} + F_{r-1} = a^r + b^r \quad \text{(independent of } n\text{)}. \end{cases}$$

Also, for moduli, writing $|C_n|^2 = c_n$, we find

$$(6)\qquad \begin{cases} c_n = F_n^2 + F_{n+1}^2 = F_{2n+1} \\ c_n - c_{n-1} = F_{n+1}^2 - F_{n-1}^2 = F_{2n} \end{cases}$$

i.e., terms $F_1, F_2, F_3, F_4, \cdots, F_{2n}, F_{2n+1}, \cdots$ of F are expressible as $c_0, c_1 - c_0, c_1, c_2 - c_1, \cdots, c_n - c_{n-1}, c_n, \cdots$ respectively. Results (5) and (6) may be extended to the general sequence without any difficulty, though some of the formulae begin to look a bit awkward. In the case of $|D_n|^2 = H_n^2 + H_{n+1}^2 = (2p_H - q_H)H_{2n+1} - e_H F_{2n+1}$, we see that there is no simple analogue to the moduli-sequence above for F.

Whereas [3] the geometrical meaning of e_H is the area of a certain parallelogram, in the complex cases e_C and e_D are merely complex numbers. Notice that $\arg C_n = \tan^{-1}(F_{n+1}/F_n) \to \tan^{-1}(1+\sqrt{5})/2 \doteq 58°17'$ and $\pi/4 \leqq \arg C_n \leqq \tan^{-1} 2 (\doteq 63°26')$, i.e., C_n moves in a very restricted region of the complex plane. Similarly for D_n.

3. Mixed complex Fibonacci number. It is tempting to experiment with a complex number $F_n + iG_{n+1}$ but nothing very interesting seems to develop. However, we may define the *mixed complex Fibonacci number*

$$(7)\qquad M_n = F_n + iG_n = (1 - i)F_n + 2iF_{n+1}$$

whence $p_M = 1 + i (= p_C)$, $q_M = 2i (= 2q_C)$, $e_m = 6$, i.e., $M \equiv H_{1+i,2i}$. Note that $\arg M_n = \tan^{-1}(G_n/F_n) \to \tan^{-1}\sqrt{5}$.

Generalising for H_n, H_n' (nth terms of H_{pq}, H_{rs} respectively) we have

$$(8)\qquad \mathit{H}_n = H_n + iH_n'$$

whence $N \equiv H_{p+ir,q+is}$ and $e_N = e_H - e_{H'} + i(2pr - 2qs - qr - ps)$, i.e., e_N is purely real if $2(pr - qs) = qr + ps$ and purely imaginary if $e_H = e_{H'}$. In particular, if $r = q$, $s = p$, then $e_N = 2(p^2 - q^2) - i(p^2 + q^2) = p^2\bar{e}_C - q^2 e_C$.

4. Fibonacci quaternions. Define the *nth Fibonacci quaternion* to be

$$(9)\qquad Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

(where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$) with conjugate quaternion $\bar{Q}_n = F_n - iF_{n+1} - jF_{n+2} - kF_{n+3}$, so that from [1],

$$(10)\qquad Q_n\bar{Q}_n = \sum_{i=0}^{3} F_{n+i}^2 = 3F_{2n+3}$$

and

$$(11)\qquad Q_n^2 = 2F_nQ_n - Q_n\bar{Q}_n$$

which, on division by $Q_n \neq 0$, yields the obviously true relation $2F_n = Q_n + \bar{Q}_n$.

Furthermore, we find

$$\sum_{i=0}^{3} Q_{n+i} = G_{n+3} + iG_{n+4} + jG_{n+5} + kG_{n+6} \tag{12}$$

which is the quaternion Q_{n+3} for the Lucas sequence.

Generalising, we define the *nth generalised Fibonacci quaternion*

$$P_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} \tag{13}$$

yielding, in particular, $P_n\overline{P}_n = 3\{(2p_H - q_H)H_{2n+3} - e_H F_{2n+3}\}$.

It is not intended to carry the theory any further at this stage.

Our definitions of complex Fibonacci sequences and quaternions do lead to some reasonably neat relationships. However, the question remains: are there other definitions which are more effective?

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THE INVERSION OF A CONVOLUTION TRANSFORM WHOSE KERNEL IS A LAGUERRE POLYNOMIAL

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Recently Ta Li [2] has shown that a certain convolution transform whose kernel involves a Tchebycheff polynomial can be inverted by another similar convolution. R. G. Buschman [1] has proved a similar result for Legendre polynomials. The present author has given new proofs of these two inversions by the method of the Laplace transform. That method reveals how similar formulas may be invented, and we apply it here to invert the transform

$$f(x) = \int_0^x L_n(x-t)g(t)dt, \tag{1}$$

where $L_n(t)$ is the Laguerre polynomial

$$L_n(x) = \frac{1}{n!} D^n(e^{-x}x^n).$$

We prove the following theorem.

THEOREM. *If $g(x) \in C^1$ for $0 \leqq x < \infty$, $g(0) = 0$, and $f(x)$ is defined by equation (1), then $f(x) \in C^2$ for $0 \leqq x < \infty$, $f(0) = f'(0) = 0$, and*

$$g(x) = \int_0^x L_{n+1}(t-x)e^{x-t}(D-1)^2 f(t)dt. \tag{2}$$

We give two proofs. By differentiating equation (1) we have

$$f'(x) = g(x) + \int_0^x L_n(x - t)g(t)dt,$$

so that $f(0) = f'(0) = 0$.

It is known that

$$\int_0^\infty e^{-st}L_n(t)dt = \frac{(s-1)^n}{s^{n+1}} \qquad s > 0,$$

and from this it is easy to see that

$$\int_0^\infty e^{-st}L_n(-t)dt = \frac{(s+1)^n}{s^{n+1}} \qquad s > 0. \tag{3}$$

If $F(s)$ and $G(s)$ are the unilateral Laplace transforms of $f(t)$ and $g(t)$, respectively, then equation (1) transforms into

$$F(s) = \frac{(s-1)^n}{s^{n+1}} G(s),$$

whence

$$G(s) = \frac{s^{n+1}}{(s-1)^{n+2}} (s-1)^2 F(s). \tag{4}$$

But by (3) the function $s^{n+1}/(s-1)^{n+2}$ is the generating function for $e^tL_{n+1}(-t)$. Since $f(0) = f'(0) = 0$, the function $(s-1)^2F(s)$ is the Laplace transform of $(D-1)^2f(t)$. Hence the theorem follows from (4).

For the second proof we need a lemma.

LEMMA. *For* $n = 0, 1, 2, \cdots$, *and* $x \geqq 0$, $\int_0^x L_{n+1}(t-x)L_n(t)e^{-t}dt = x$.

This follows in an obvious way from the identity

$$\frac{(s+1)^{n+1}}{s^{n+2}} \frac{s^n}{(s+1)^{n+1}} = \frac{1}{s^2} \cdot$$

Now to prove (2), we note first that in the presence of the conditions $f(0) = f'(0) = 0$, the integral (2) is equal to $(D-1)^2h(x)$, where

$$h(x) = \int_0^x L_{n+1}(t - x)e^{x-t}f(t)dt \tag{5}$$

$$= \int_0^x L_{n+1}(-t)e^tf(x - t)dt. \tag{6}$$

This follows by direct differentiation of the integral (6).

Let us now substitute the integral (1) in (5) to obtain

$$\begin{aligned} h(x) &= \int_0^x L_{n+1}(t-x)e^{x-t}dt \int_0^t L_n(t-y)g(y)dy \\ &= \int_0^x g(y)dy \int_0^x L_{n+1}(t-x)e^{x-t}L_n(t-y)dt \\ &= \int_0^x g(y)dy \int_0^{x-y} L_{n+1}(y+t-x)e^{x-y-t}L_n(t)dt \\ &= \int_0^x e^{x-y}(x-y)g(y)dy. \end{aligned}$$

Here we have used the lemma for the inner integral. But now direct differentiation yields

$$(D-1)^2 h(x) = g(x),$$

and the proof is complete.

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NOTE ON "AN EXTENSION OF FERMAT'S THEOREM"

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In a paper entitled "An Extension of Fermat's Theorem," this MONTHLY, 57 (1950) 87–89, A. A. Trypanis proves the following: If a is an integer not divisible by the prime p, then $a^{(p-1)/p^n} \equiv 1 \pmod{p^{1/p^n}}$. This result is to be understood in the sense that, for any one of the p^n determinations of $a^{(p-1)/p^n}$ and p^{1/p^n} as complex numbers, we have $a^{(p-1)/p^n} - 1 = p^{1/p^n} \cdot \alpha$, where α is an *algebraic integer*. The word "extension" in the title of Trypanis's paper results from the fact that Euler's theorem gives, in rational number theory, $a^{\phi(p^n)} = a^{p^{n-1}(p-1)} \equiv 1 \pmod{p^n}$. It is the purpose of this note to show that Trypanis's result is very easy to demonstrate and does not require the rather formidable proof involving cyclotomic fields given in his work. The only facts we require for a proof are

1. The set of algebraic integers $\mathfrak{A}$ is closed under addition, subtraction and multiplication and $\mathfrak{A}$ contains the rational integers. In other words, the set of algebraic integers is an integral domain containing the domain of rational integers.

2. If $\beta \neq 0$ is an algebraic integer, then any one of the k determinations of $\beta^{1/k}$, where k is a natural number, is an algebraic integer. This is an immediate corollary of the fact that the domain of algebraic integers is integrally closed, i.e., if α is a root of the monic polynomial $x^k + \beta_{k-1}x^{k-1} + \cdots + \beta_1 x + \beta_0 = 0$, where the β_i are in $\mathfrak{A}$, then α is itself in $\mathfrak{A}$.

Proofs of 1 and 2 can be found in any standard account of algebraic numbers, for example, in Volume II, Chapter 2, of W. J. LeVeque's book "Topics in Number Theory."

THEOREM. *Suppose a and p are rational integers, p a prime, and $p \nmid a$. If n is any natural number, then* $a^{(p-1)/p^n} \equiv 1 \pmod{p^{1/p^n}}$.

Proof. We will show that, for any of the p^n determinations of $a^{(p-1)/p^n}$ as a complex number, $(1/p)(a^{(p-1)/p^n}-1)^{p^n}$ is an algebraic integer. Granting this for a moment, we have $(a^{(p-1)/p^n}-1)^{p^n}=p\cdot\beta$, where β is an algebraic integer. Taking p^nth roots gives $a^{(p-1)/p^n}-1=p^{1/p^n}\cdot\beta^{1/p^n}$ for some determination of β^{1/p^n}. But β^{1/p^n} is an algebraic integer by No. 1, above, with $k=p^n$, $\beta_{k-1}=\cdots=\beta_1=0$, $\beta_0=-\beta$. Hence the result of Trypanis.

To show that $(1/p)(a^{(p-1)/p^n}-1)^{p^n}$ is an algebraic integer, assume first that $p>2$ and expand $(a^{(p-1)/p^n}-1)^{p^n}$ by the binomial theorem:

$$(*)\qquad \begin{aligned}(a^{(p-1)/p^n}-1)^{p^n} = (a^{p-1}-1) - \binom{p^n}{1}a^{(p-1)(p^n-1)/p^n}+\binom{p^n}{2}a^{(p-1)(p^n-2)/p^n}\\ -\cdots+\binom{p^n}{p^n-1}a^{(p-1)/p^n}.\end{aligned}$$

The rational integers $\binom{p^n}{j}$ are divisible by p for $1\leqq j\leqq p^n-1$. Now $a^{(p-1)/p^n}$ is an algebraic integer by No. 2, above, and hence so are all the $(a^{(p-1)/p^n})^j$ by No. 1. Since $a^{p-1}-1$ is divisible by p (as a rational integer) by Fermat's theorem, we see that $(1/p)(a^{(p-1)/p^n}-1)^{p^n}$ is a sum of algebraic integers and hence is itself an algebraic integer (No. 2).

In the case of the prime $p=2$ (a case which Trypanis overlooks), the term $a^{p-1}-1$ in (*) becomes $a+1$ which is divisible by 2 since a is odd for $p=2$. Thus $\frac{1}{2}(a^{1/2^n}-1)^{2^n}$ is an algebraic integer and this completes the proof.

ON QUADRILATERALS OF MINIMAL PERIMETER INSCRIBED IN A GIVEN QUADRILATERAL

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1. A well-known result of H. A. Schwarz ([1], p. 344) states that among all triangles inscribed in an acute-angled triangle the pedal triangle is the unique one having minimal perimeter. In this note we wish to consider a generalization of this result to quadrilaterals, a problem apparently first posed by Steiner ([2], p. 45, 9).

In view of the fact that the pedal triangle has a "reflecting" property—viz., it provides a closed path that a billiard ball could follow, rebounding once from each side of the given triangle—we shall study the properties of quadrilaterals admitting inscribed "reflecting" quadrilaterals.

It should be made clear at this point that we consider only convex quadrilaterals together with inscribed convex reflecting quadrilaterals. The situation is typified by the case of the rectangle, which admits a one-parameter family of inscribed reflecting parallelograms.

2. DEFINITION. *Let $ABCD$ be a convex quadrilateral with $PQRS$ inscribed as indicated in Figure* 1. *We shall say that $PQRS$ is reflecting in $ABCD$ if and only if* $\angle SPA = \angle QPB$, $\angle PQB = \angle RQC$, $\angle QRC = \angle SRD$, *and* $\angle RSD = \angle PSA$.

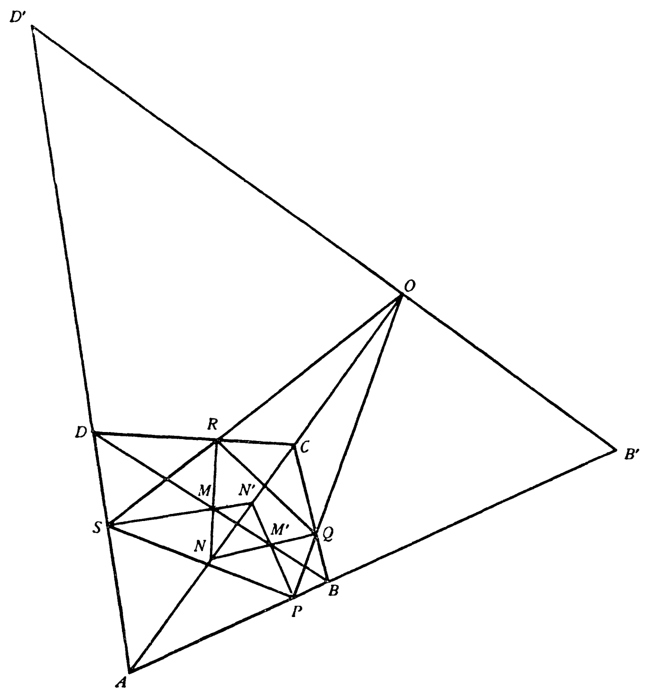

FIG. 1

Remark. In the following we assume $PQRS$ is nondegenerate, so, for example, P and Q do not coincide with B.

THEOREM 1. *Let $PQRS$ be reflecting in $ABCD$ (Figure* 1). *Then SR intersects PQ in O on the (extended) diagonal AC (or else $SR//PQ//AC$), and the analogous statements hold for RQ and SP.*

Proof. (a) Suppose SR is not parallel to PQ, and let O be their intersection (Figure 1). Construct line $B'D'$ through O such that $\angle QOB' = \angle ROD'$. Then $\triangle SPO$ is reflecting in $\triangle AB'D'$ and thus is the pedal triangle of $\triangle AB'D'$. Thus $OA \perp B'D'$ and OA bisects $\angle ROQ$. Hence lines RC, QC, and OA, being the angle bisectors of $\triangle RQO$, meet in C; that is, O lies on the diagonal AC, as was to be proved.

(b) To complete the proof, we show that if $RQ//SP$, then $SP//DB$, or equivalently, $\angle PSA = \angle BDA$. It is easily seen that $\angle ADC = \angle ABC = 90°$, so that $ABCD$ is cyclic. If also $SR//PQ$, then $ABCD$ is a rectangle and the theorem is obvious. So suppose SR intersects PQ at O. Then $\triangle SPO$ is the pedal triangle of $\triangle AB'D'$ (from (a)); hence the perpendiculars to DA and BA at S and P respectively meet in a point N' on AC. Since $ABCD$ is cyclic, $\angle BDA = \angle BCA$.

Since $\angle ABC = 90°$, $BCA = PN'A$. Since $APN'S$ is cyclic, $\angle PN'A = \angle APS$. Thus $\angle BDA = \angle PSA$, completing the proof.

Remark. Part (b) of the proof of Theorem 1 shows that (if the situation is as depicted in Figure 1) the perpendiculars to DA and BA at S and P respectively meet in a point N' on AC. But it is also the case that the perpendiculars to DC and BC at R and Q respectively meet in a point N on AC. To see this, let N be the intersection of those perpendiculars. Let Q' and R' be points such that $Q \in NQ'$, $R \in NR'$, $O \in R'Q'$, and $\angle ROR' = \angle QOQ'$. Then $\triangle ROQ$ is reflecting in $\triangle NQ'R'$ and hence is its pedal triangle. This implies that ON bisects $\angle ROQ$. But it was shown above (part (a), Theorem 1) that OA bisects $\angle ROQ$. Thus N lies on OA, which coincides with AC.

Even in case $SR//PQ$, it is readily established that the perpendiculars to DA and BA at S and P respectively meet on AC, as do the perpendiculars to DC and BC at R and Q respectively. Hence we have the following theorem.

THEOREM 2. *Let $PQRS$ be reflecting in $ABCD$. Then the perpendiculars to DC and BC at R and Q respectively meet in a point N on AC; the perpendiculars to DA and BA at S and P respectively meet in a point N' on AC. Similarly the perpendiculars NR and $N'S$ meet in M on DB, and NQ and $N'P$ meet in M' on DB.*

Remark. One now sees that in case $SR//PQ$, not only is $SR//PQ//AC$, as asserted in Theorem 1, but also SR and PQ are equidistant from AC, since then $RNQC$ is a rectangle whose diagonal lies on AC.

THEOREM 3. *Let $ABCD$ admit the reflecting convex quadrilateral $PQRS$. Then $ABCD$ is cyclic and contains its circumcenter in its interior.*

Proof. Using the obvious cyclic quadrilaterals, we have $\angle ACB = \angle NCQ = \angle NRQ = \angle SRM = \angle SDM = \angle ADB$. Hence D lies on the circle through A, B, and C, so $ABCD$ is cyclic.

To show that the center of the circumcircle lies interior to $ABCD$, note that if $\angle ACB \geqq 90°$, then the intersection of QN and RN could not be on AC. Hence $ACB < 90°$, so the circumcenter lies in the same open half-plane bounded by AB as the quadrilateral $ABCD$. The same argument applied to each of the other sides shows finally that the circumcenter lies interior to $ABCD$.

THEOREM 4. *Let $ABCD$ be a convex cyclic quadrilateral containing its circumcenter in its interior. Then $ABCD$ admits a one-parameter family of reflecting convex quadrilaterals.*

Proof. The conditions on $ABCD$ imply that some adjacent pair of its angles, say $\angle DAB$ and $\angle CBA$, are greater than or equal to 90°. We shall show that each point P interior to segment AB is the vertex of exactly one reflecting quadrilateral.

Let the perpendicular to AB at P intersect AC in N' and BD in M' (Figure 1). Possibly $M' = N'$. Then the perpendicular dropped on BC from M' lands at Q interior to segment BC, because $\angle DCB$ is acute (being opposite obtuse

$\angle DAB$) and $\angle DBC$ is acute (since $ABCD$ contains its circumcenter). Similarly the perpendicular dropped on DA from N' lands at S interior to segment DA. Let M be the intersection of SN' and BD. Then the perpendicular dropped on DC from M lands at R interior to segment DC.

We now observe that MR intersects $M'Q$ on AC. For let N'' be the intersection of MR and $M'Q$ and let N be the intersection of MR and AC. Since $\angle N''MN' = \angle SMR = 180° - \angle SDR = \angle ABC = 180° - \angle PM'Q = 180° - \angle N''M'N'$, we have that quadrilateral $M'N'MN''$ is cyclic. Thus $\angle MN''N' = \angle N'M'M$. But $\angle MNN' = \angle RNC = 90° - \angle DCA = 90° - \angle DBA = \angle PM'B = \angle N'M'M$. Thus $\angle MN''N' = \angle MNN'$, so $N = N''$ as asserted.

Moreover, $PQRS$ is the unique reflecting quadrilateral with vertex at P. For, using the obvious cyclic quadrilateral at each stage, we have $\angle N'PS = \angle N'AS = \angle CAD = \angle CBD = \angle QBM' = \angle QPM'$, so that $PQRS$ reflects at P. The same holds for each of the other vertices, so $PQRS$ is reflecting. The uniqueness is immediate from Theorem 2.

Thus we can associate with each point of the open segment AB exactly one convex quadrilateral reflecting in $ABCD$, which proves the theorem.

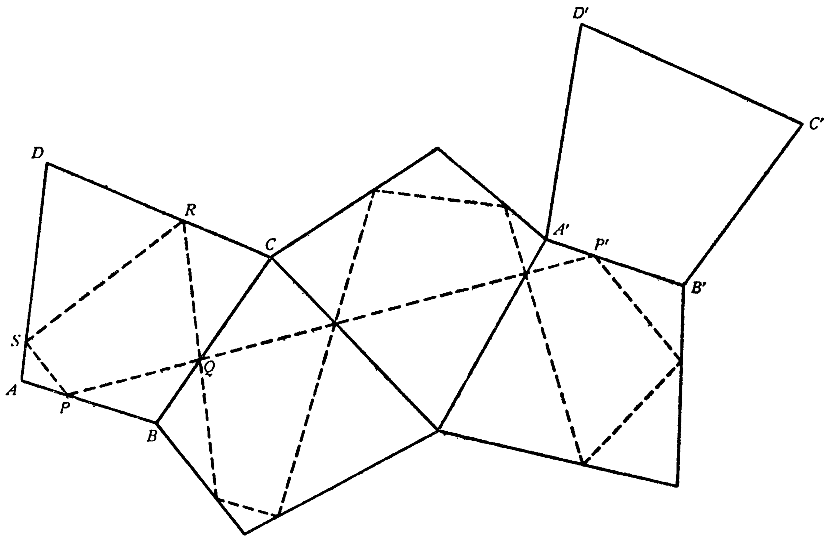

FIG. 2

Remark. An interesting special case of the construction used in the proof of Theorem 4 occurs when $M = M' = N = N' =$ intersection of the diagonals of $ABCD$. Then $PQRS$ could well be termed the "pedal quadrilateral" of $ABCD$. Note that all our reflecting quadrilaterals are obtained as inscribed quadrilaterals "parallel" to the pedal quadrilateral.

THEOREM 5. *Let $ABCD$ satisfy the conditions of Theorem 4. Then among all the convex quadrilaterals inscribed in $ABCD$, the reflecting quadrilaterals have minimum perimeter.*

Proof. We employ the method used by Schwarz ([1], p. 344) for the case of the triangle, reflecting $ABCD$ successively in sides BC, CD, DA, and AB, as indicated in Figure 2. Since $ABCD$ is cyclic, its final position will be simply a translation of its initial position. The perimeter of the reflecting quadrilateral with vertex at P is just $\overline{PP'}$. The theorem follows.

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A THEOREM ON FIXED POINTS UNDER ISOMETRIES

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1. Introduction. In this note we prove the following:

THEOREM I. *Let A be an arbitrary subset of a uniformly convex Banach space B and $f: A \to A$ an isometry of A into itself. If a point $a \in A$ exists with the property that $C = \overline{\text{co}}\{f^n(a)\}$ is a bounded subset of A then $c \in C$ exists such that $f(c) = c$.*

We recall that a normed linear space is uniformly convex [1] if to any $\epsilon > 0$ an $\eta(\epsilon) > 0$ exists such that

$$\|x\| = \|y\| = 1, \qquad \|x - y\| \geqq \epsilon \Rightarrow \tfrac{1}{2}\|x + y\| \leqq 1 - \eta(\epsilon). \tag{1}$$

1.2. The above theorem is closely related to a result of Brodskiĭ and Milman [2] concerning the existence of fixed points under isometries of convex sets possessing the so-called normal structure. Although the present result seems to be obtainable from [2] we prefer the direct, rather simple proof given here.

2. For the proof of Theorem I we need the following three lemmas.

2.1. LEMMA 1. *Let K be a convex closed and bounded subset of a uniformly convex Banach space B. Let $R(x)$ denote the family of all rays issuing from x, $x \in K$; let $l(r, x)$ denote the length of the segment $r \cap K$, where $r \in R(x)$. Let, further, $\delta = \delta(K)$ be the diameter of K. Then there exists an α, $0 \leqq \alpha < 1$, so that*

$$\rho = \inf_{K} \sup_{R(x)} l(r, x) \leqq \alpha\delta. \tag{2}$$

Proof. Let $p, q, s \in K$ with $\|p - q\| \geqq \delta/2$, s arbitrary. Put $(p - s)/\delta = x$, $(q - s)/\delta = y$. Then $\|x\| \leqq 1$, $\|y\| \leqq 1$ and $\|x - y\| = \|p - q\|/\delta \geqq \frac{1}{2}$.

Now (1) is equivalent, as pointed out by I. Berstein, to the following condition

$$\|x\| \leqq 1, \quad \|y\| \leqq 1, \quad \|x - y\| \geqq \epsilon \Rightarrow \tfrac{1}{2}\|x + y\| < 1 - \eta'(\epsilon). \tag{1'}$$

Let, then, $t = \frac{1}{2}(p + q)$. By convexity of K, $t \in K$; and by (1′)

$$\tfrac{1}{2}\|x + y\| = \frac{1}{\delta}\left\|\frac{p + q}{2} - s\right\| = \frac{1}{\delta}\|t - s\| < 1 - \eta'(\tfrac{1}{2}).$$

Hence, putting $\alpha=1-\eta'(\frac{1}{2})$, we have $\sup_{R(t)} l(r, t)\leqq\alpha\delta$ implying $\rho\leqq\alpha\delta$ as asserted.

2.2. LEMMA 2. *Let S be an arbitrary subset of a uniformly convex Banach space B and g an isometry of the convex hull* co$S$ *into B. Then*

$$g(\text{co}S) \subset \text{co}\, g(S). \tag{3}$$

Proof. If $x\in\text{co}S$ then there exists a positive integer n so that $x=\sum_{i=1}^{n}\lambda_i x_i$, where $\{x_1, x_2, \cdots, x_n\}\subset S$, $\lambda_i>0$, $(i=1, 2, \cdots, n)$ and $\sum_{i=1}^{n}\lambda_i=1$. We have to show that $g(x)\in\text{co}\, g(S)$. For $n=1$ the assertion is trivial. Suppose it holds for $n-1$ and write $x=\sum_{i=1}^{n-1}\lambda_i x_i+\lambda_n x_n=\alpha y_1+(1-\alpha)y_2$, where $\alpha=\sum_{i=1}^{n-1}\lambda_i$, $y_1=(\sum_{i=1}^{n-1}\lambda_i)^{-1}\sum_{i=1}^{n-1}\lambda_i x_i$, $y_2=x_n$. Now x belongs to the open segment (y_1, y_2); hence by the isometry property of g, $\|g(y_1)-g(x)\|+\|g(x)-g(y_2)\|=\|g(y_1)-g(y_2)\|$. As shown by Clarkson [1], however, this implies that $g(x)\in(g(y_1), g(y_2))$. The last fact and the inductive assumption on $n-1$, amounting to $g(y_1)\in\text{co}S$, yield $g(x)\in\text{co}S$ thus establishing the lemma.

2.3. LEMMA 3. *Let S be an arbitrary bounded subset of a normed linear space. Then*

$$\delta(\text{co}S) = \delta(S). \tag{4}$$

For a proof of this lemma see e.g. [3], p. 24. (The corresponding lemma there is stated for the case where the containing space is Euclidean; its proof, however, applies to the present case.)

3. Proof of the theorem. By (2) a point $c_0\in C$ exists with the property that $\sup_{R(c_0)} l(r, c_0)\leqq\alpha\delta$, where $\delta=\delta(C)$ and $0\leqq\alpha<1$. We consider the sequence $\{f^n(c_0)\}$. Since f is an isometry on C we have

$$\|f^m(c_0) - f^n(c_0)\| = \|f^{m-n}(c_0) - c_0\|, \qquad (m \geqq n).$$

Hence

$$\begin{aligned}\delta(\{f^n(c_0)\}) &= \sup_{m,n}\|f^m(c_0) - f^n(c_0)\| = \sup_{k}\|c_0 - f^k(c_0)\| \\ &\leqq \sup_{x\in C}\|c_0 - x\| = \sup_{R(c_0)} l(r, c_0) \leqq \alpha\delta.\end{aligned}$$

By (4), then, $\delta(\text{co}\{f^n(c_0)\})\leqq\alpha\delta$. Hence for the closure $C_0=\overline{\text{co}\{f^n(c_0)\}}$ we obtain

$$\delta_0 = \delta(C_0) \leqq \alpha\delta. \tag{5}$$

Furthermore we have

$$f(C_0) \subset C_0. \tag{6_0}$$

Indeed applying Lemma 2 we obtain

$$f(\text{co}\,\{f^n(c_0)\}) \subset \text{co}\, f(\{f^n(c_0)\}) = \text{co}\,\{f^{n+1}(c_0)\} \subset \text{co}\,\{f^n(c_0)\}.$$

This and the obvious fact that f is continuous on C_0 yield

$$f(C_0) = f(\overline{\text{co}\,\{f^n(c_0)\}}) \subset \overline{f(\text{co}\,\{f^n(c_0)\})} \subset \overline{\text{co}\,\{f^n(c_0)\}} = C_0$$

proving (6_0).

We observe that C_0 is convex, closed, bounded and mapped into itself by f. Thus the above procedure may be applied to C_0 to obtain $C_1 \subset C_0$ satisfying

$$\delta_1 = \delta(C_1) \leqq \alpha\delta_0 \quad \text{and} \quad f(C_1) \subset C_1.$$

In an analogous fashion we can define inductively C_i and $\delta_i = \delta(C_i)$ for all $i = 1, 2, \cdots$. Clearly $\delta_i \leqq \alpha^{m+1}\delta \to 0$, $m \to \infty$; and

$$(6_i) \qquad f(C_i) \subset C_i, \qquad \text{for } i = 0, 1, 2, \cdots.$$

The sequence $\{C_i\}$ satisfies all the assumptions of Cantor's well-known theorem to the effect that a decreasing sequence of closed sets, in a complete metric space, whose diameters tend to zero, have an intersection which consists of a single point.

Let $\bigcap_{i=1}^{\infty} C_i = \{c\}$; then $f(c) = c$. Indeed should $f(c) \neq c$ then $\delta\{f^n(c)\} > 0$ which is impossible since $\delta_i \to 0$ and C_i is mapped into itself.

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REMARKS ON A PAPER OF RHOADES

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B. E. Rhoades [1] defined two infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ to be almost identical if there exists a natural number q such that $a_{nk} = b_{nk}$ for $k > q$. He raised the question whether the almost identity of two matrices is necessary or sufficient for total equivalence of the matrices. (Two matrices are totally equivalent if each sequence evaluated by one of the matrices is evaluated by the other matrix to the same value—finite or infinite.)

We begin with the following definition:

Two matrices A and B are ultimately identical if there exist natural numbers N and q such that $a_{nk} = b_{nk}$ when $n > N$, $k > q$.

We note that two almost identical matrices are ultimately identical.

In this note we shall prove:

1. *If two regular summation matrices are ultimately identical, then they are totally equivalent. There exist two totally regular matrices which are strictly stronger than the identity matrix, and which are totally equivalent but not ultimately identical.*

2. *Let A and B denote two regular, positive, row-finite matrices and let* $A_n(x) = \sum_k a_{nk}x_k$. *Then* $A_n(x) - B_n(x) = O(1)$ *for each sequence x such that* $A_n(x) \to \infty$, *if and only if A and B are ultimately identical.*

Proof of 1. If the regular matrices are ultimately identical, then for each positive number ϵ, there exist two natural numbers N and q, such that if $n > N$,

$$a_{nk} = b_{nk} \qquad k > q,$$

$$|a_{nk}| < \epsilon, \qquad |b_{nk}| < \epsilon, \qquad k = 0, 1, \cdots, q.$$

If x is a sequence which has A and B transforms, then $|A_n(x) - B_n(x)| < 2M(q+1)\epsilon$ for $n > N$, where $M = \max(|x_0|, |x_1|, \cdots, |x_q|)$. Since ϵ is arbitrary, $A_n(x) - B_n(x) \to 0$, that is, A and B are totally equivalent.

To see that ultimate identity is not necessary for total equivalence, we let the triangles A and B be defined by $a_{00} = 1$, $a_{n,n-1} = a_{n,n} = 1/2$ for $n \geqq 1$, $a_{nk} = 0$ otherwise, and $b_{00} = b_{11} = 1$, $b_{10} = 0$, $b_{n,n-2} = b_{n,n-1} = 1/2$, for $n \geqq 2$, $b_{nk} = 0$ otherwise. The matrices A and B are strictly stronger than the identity matrix, totally equivalent, but not ultimately identical.

Proof of 2. The "if" part is fairly clear. To show the "only if" part, we let A and B denote two regular positive, row-finite matrices. If A and B are not ultimately identical, there exist arbitrarily large values of n such that $a_{nk} \neq b_{nk}$ for arbitrarily large values of k. There exists a sequence $\{n_j, k_j\}$, $n_j \to \infty$, $k_j \to \infty$ such that $a_{n_j,k_j} \neq b_{n_j,k_j}$. Since A and B are row-finite, the numbers k_j may be chosen so that $a_{n_j,k} = b_{n_j,k} = 0$ for $k > k_j$. Either $a_{n_j,k_j} > b_{n_j,k_j}$ or $a_{n_j,k_j} < b_{n_j,k_j}$ for arbitrarily large values of k_j; without loss in generality we may assume that the first condition holds. Then $\{n_j, k_j\}$ has a subsequence $\{m_j, q_j\}$, $m_j \to \infty$, $q_j \to \infty$, such that $a_{m_j,q_j} > b_{m_j,q_j}$ while $a_{m_j,k} = b_{m_j,k} = 0$ for $k > q_j$.

Let $x = \{x_n\}$ be a sequence increasing to infinity such that

$$x_{q_j} > \left[m_j + \sum_{k=0}^{q_j-1} (a_{m_j,k} - b_{m_j,k})x_k \right] \Big/ (a_{m_j,q_j} - b_{m_j,q_j})$$

for $j = 1, 2, \cdots$. Since A and B are totally regular, $A_n(x) \to \infty$, $B_n(x) \to \infty$. On the other hand, $A_{m_j}(x) - B_{m_j}(x) > m_j$ for each value of j, that is, $A_n(x) - B_n(x)$ is unbounded.

In conclusion, we raise the following question: In order that two regular matrices which are strictly stronger than the identity matrix be totally equivalent, is it necessary or sufficient that there exist two natural numbers N_1, N_2 such that for each $n > N_1$ the nth row of one matrix appear as the mth row of the other, for some $m > N_2$?

Let A and B be two positive, regular matrices. What are necessary and sufficient conditions on A and B in order that each sequence x such that $A_n(x) \to \infty$ satisfies $A_n(x)/B_n(x) \to 1$?

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THE COMPUTATION OF THE GENERALIZED INVERSE OF SINGULAR OR RECTANGULAR MATRICES

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Let A be an arbitrary, real, $m\times n$ matrix of rank r. Recently Ben Israel and Wersan [1] showed that the unique $n\times m$ matrix X such that (i) $AXA=A$, (ii) $XAX=X$, (iii) $(AX)'=AX$, (iv) $(XA)'=XA$, is the matrix obtained as the solution to the extremum problem: Minimize tr XX', subject to

$$A'AX = A'. \tag{1}$$

As is well known, the above result holds also for matrices over the complex field, if one interprets the transpose as the conjugate transpose.

This suggests the following computation, which requires only one $r\times r$ matrix inversion, unlike the formulas suggested before in [1] and [2], requiring two such inversions.

Since the system (1) will always be consistent, we need consider only r of the n equations in (1). Assume, without loss of generality, that the first r rows are independent. This is a theoretical convenience, which, when it comes to computation in practice, may require a (routine) computer program to exhibit *some* r independent rows. Write $B'X=C'$, where the $r\times n$ matrix B' is formed by the first r rows of $A'A$, and the $r\times m$ matrix C' is the matrix formed by the first r rows of A'. Then

$$X = B(B'B)^{-1}C'. \tag{2}$$

Proof. We will write x_i for the ith column of X, and c_i for the ith column of C'. The vector x_i minimizing $x_i'x_i$ subject to $B'x_i=c_i$ is, using the familiar Lagrangian technique, given by the solution of the system

$$\begin{bmatrix} I & B \\ B' & 0 \end{bmatrix}\begin{bmatrix} x_i \\ u_i \end{bmatrix} = \begin{bmatrix} 0 \\ c_i \end{bmatrix},$$

where u_i is a column vector of r Lagrangians. Since

$$\begin{bmatrix} I - B(B'B)^{-1}B' & B(B'B)^{-1} \\ (B'B)^{-1}B' & -(B'B)^{-1} \end{bmatrix}\begin{bmatrix} I & B \\ B' & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

the solution is given by $x_i=B(B'B)^{-1}c_i$. Hence (2) follows immediately.

Addendum. The referee pointed out an alternative proof. Write $A'=DC'$, and $A'A=DB'$, where A, B and C are as before, and D is an $n\times r$ matrix of rank r.

Write A^+ for the generalized inverse of A. Using the fact that $A=BC$ implies $A^+=C^+B^+$, cf. [2], and $A^+=(A'A)^+A'$ cf. [3], and $A^+=A^{-1}$, whenever A is square and of full rank, we get:

$$A^+ = (A'A)^+A' = (B')^+D^+DC'. \tag{3}$$

Now $B(B'B)^{-1} = B(B'B)^+ = BB^+(B')^+$. Premultiplying $(B')^+D^+D$ and $BB^+(B')^+$ by B^+, and postmultiplying by D^+, establishes $(B')^+D^+D = B(B'B)^{-1}$. Hence (3) is equivalent to (2).

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AN EXTENSION OF TAYLOR'S FORMULA

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Taylor's formula with remainder and many of its extensions require the existence of derivatives. Since "enough" derivatives may fail to exist, it seems desirable to have relations which, nevertheless, may serve us. The purpose of this note is to extend Taylor's formula to certain functions whose derivatives have summable (integrable in the sense of Lebesgue) one-sided upper and lower derivates ([1] p. 73 and [2] p. 354). Our results contain as a very special case Hummel and Seebeck's generalization of Taylor's formula ([3] p. 243).

In our expansion, m and n are nonnegative integers and we use the convention that the binomial coefficient $\binom{m}{k}$ vanishes when $k>m$. Also, we follow the usual conventions for derivatives and derivates at the end points of intervals.

THEOREM 1. *Let the function f and its first $m+n$ derivatives be defined and continuous on the finite closed interval $[a, x]$ and let $f^{(m+n)}$ have a derivate $D_*f^{(m+n)}$ which is infinite only at the points of a reducible set S in $[a, x]$ and is summable in $[a, x]$ when the points of S are ignored. If there are numbers L and U such that $L \leqq D_*f^{(m+n)}(t) \leqq U$ almost everywhere on $[a, x]$, then there is a number M, $L \leqq M \leqq U$, such that*

$$f(x) = f(a) + \sum_{k\geqq 1} \frac{(m+n-k)!}{(m+n)!}\left[\binom{m}{k} f^{(k)}(a) - (-1)^k \binom{n}{k} f^{(k)}(x)\right](x-a)^k + R,$$

where

$$R = \frac{(-1)^n m!n!(x-a)^{m+n+1}}{(m+n)!(m+n+1)!} M.$$

Proof. Let $\phi(t) = t^n(1-t)^m$ and define F by the relation

$$F(t) = \sum_{k=1}^{m+n} (-1)^{m+n+k}(x-a)^k \phi^{(m+n-k)}(t) f^{(k)}(a + t(x-a)).$$

Then (cf. [4] p. 125)

$$\int_0^1 D_*F(t)dt = \int_0^1 (-1)^{m+n+1}(x-a)\phi^{(m+n)}(t)f'(a+t(x-a))d$$
$$+\int_0^1 (-1)^m(x-a)^{m+n+1}\phi(t)D_*f^{(m+n)}(a+t(x-a))dt.$$

Hence ([2] p. 601)

$$\sum_{k\geqq 1}(m+n-k)!\left[\binom{m}{k}f^{(k)}(a)-(-1)^k\binom{n}{k}f^{(k)}(x)\right](x-a)^k$$
$$=(m+n)![f(x)-f(a)]$$
$$+(-1)^{n+1}(x-a)^{m+n+1}\int_0^1 t^n(1-t)^m D_*f^{(m+n)}(a+t(x-a))dt.$$

We get the desired expression by solving for $f(x)$ and noting ([1] p. 231) that for some number M, with $L\leqq M\leqq U$,

$$\int_0^1 t^n(1-t)^m D_*f^{(m+n)}(a+t(x-a))dt$$
$$=M\int_0^1 t^n(1-t^m)dt=\frac{m!n!}{(m+n+1)!}M.$$

THEOREM 2. *Let the function f and its first $m+n$ derivatives be defined and continuous on the closed interval $[a, x]$. If $f^{(m+n)}$ is of bounded variation on $[a, x]$ and if it has a right derivate $D_rf^{(m+n)}$ and a left derivate $D_lf^{(m+n)}$ each of which is finite at each point of $[a, x]$, then there exist numbers p, q, and ξ with $p\geqq 0$, $q\geqq 0$, $p+q=1$, and $a<\xi<x$ such that*

$$f(x)=f(a)+\sum_{k\geqq 1}\frac{(m+n-k)!}{(m+n)!}\left[\binom{m}{k}f^{(k)}(a)-(-1)^k\binom{n}{k}f^{(k)}(x)\right](x-a)^k+R,$$

where

$$R=\frac{(-1)^n m!n!(x-a)^{m+n+1}}{(m+n)!(m+n+1)!}[pD_rf^{(m+n)}(\xi)+qD_lf^{(m+n)}(\xi)].$$

Proof. Let ϕ and F be defined as in the proof of Theorem 1 and let p and q be two nonnegative numbers such that $p+q=1$. Then ([2] p. 596; [4] p. 125)

$$\int_0^1 [pD_rF(t)+qD_lF(t)]dt$$
$$=\sum_{k\geqq 1}(-1)^{n+1}(m+n-k)!\left[\binom{m}{k}f^{(k)}(a)-(-1)^k\binom{n}{k}f^{(k)}(x)\right](x-a)^k$$

$$= (-1)^{n+1}(m+n)![f(x)-f(a)]$$
$$+ (x-a)^{m+n+1}\int_0^1 t^n(1-t)^m[pD_rf^{(m+n)}(a+t(x-a))$$
$$+ qD_lf^{(m+n)}(a+t(x-a))]dt.$$

Solving for $f(x)$ gives

$$f(x) = f(a) + \sum_{k\geq 1}\frac{(m+n-k)!}{(m+n)!}\left[\binom{m}{k}f^{(k)}(a) - (-1)^k\binom{n}{k}f^{(k)}(x)\right](x-a)^k$$
$$+ (-1)^n\frac{(x-a)^{m+n+1}}{(m+n)!}\int_0^1 t^n(1-t)^m[pD_rf^{(m+n)}(a+t(x-a))$$
$$+ qD_lf^{(m+n)}(a+t(x-a))]dt.$$

But there is a number M such that the integral term in the above expression can be written

$$\frac{(-1)^n m!n!(x-a)^{m+n+1}}{(m+n)!(m+n+1)!}M.$$

Hence to complete the proof we need to show only that there exist suitable numbers p, q, and ξ such that $pD_rf^{(m+n)}(\xi)+qD_lf^{(m+n)}(\xi)=M$.

If one of the derivates under consideration, say $D_rf^{(m+n)}$, is such that $M\leqq D_rf^{(m+n)}(a+t(x-a))$ almost everywhere on $[0, 1]$, then $D_rf^{(m+n)}(a+t(x-a)) = M$ except at the points of a set S of measure zero. Let θ be any point of the open interval $(0, 1)$ which is not in S. Then suitable choices for p, q, and ξ are 1, 0, and $a+\theta(x-a)$, respectively. A similar argument may be used when one of the derivates is essentially bounded above by M.

If neither derivate is essentially bounded above or below, there are points α and β in $(0, 1)$ such that at least one of the derivates is greater than M at α, and at least one of the derivates is less than M at β. Hence the function $G(t) = f^{(m+n)}(a+t(x-a))-Mt$ has an extremum (possibly relative) at a point θ, $0<\theta<1$. Then $D_rG(\theta)\cdot D_lG(\theta)\leqq 0$. If $D_rG(\theta)=0$, we have $D_rf^{(m+n)}(a+\theta(x-a)) = M$ and we choose $p=1$ and $q=0$. If $D_rG(\theta)\neq 0$, we choose

$$p = \frac{D_lG(\theta)}{D_lG(\theta)-D_rG(\theta)} \quad\text{and}\quad q = \frac{D_rG(\theta)}{D_rG(\theta)-D_lG(\theta)}\cdot$$

In either case $p\geqq 0$, $q\geqq 0$, $p+q=1$ and $pD_rG(\theta)+qD_lG(\theta)=0$. Setting $\xi=a+\theta(x-a)$ and expressing the derivates of G in terms of the derivates of f give

$$pD_rf^{(m+n)}(\xi) + qD_lf^{(m+n)}(\xi) - M = 0$$

which completes the proof.

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ON A SINE FUNCTIONAL EQUATION

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It is well known and has been rediscovered several times that if $f(x)$ is a real-valued finite-valued function of a real variable, continuous at a point, and satisfying the functional equation

$$f(x+y)f(x-y) = f^2(x) - f^2(y) \tag{1}$$

for all real x and y then f must be continuous everywhere and be one of the following three functions: $f(x)=k_1x$, $f(x)=k_2 \sin k_3x$, $f(x)=k_4 \sinh k_5x$ where the k_i are arbitrary real constants. [**1**, **6**, **8**, **9**; for a complete bibliography see **1**, pp. 114–115.]

The functional equation (1) is closely connected with the Cauchy functional equation

$$\phi(x+y) = \phi(x) + \phi(y) \tag{2}$$

for real-valued functions of a real variable. In fact clearly every solution of (2) is a solution of (1). By a well-known theorem due independently to Ostrowski [**5**] and Kestelman [**4**] if ϕ is a solution of (2) which is bounded above on a set of positive measure, then ϕ is continuous and hence the linear function $\phi(x)=cx$. Clearly any analogous result cannot hold for (1) since, if we let $g(x)$ be any finite-valued everywhere discontinuous solution of (2) constructed by using a Hamel basis for the real numbers [**3**], then the function $f(x)=\sin(g(x))$ is bounded on the whole real line, satisfies (1) and is everywhere discontinuous. We will show below, however, that the only solutions of (1) measurable on some interval are continuous.

Besides weakening the "continuity conditions" on f one might also consider the functional equation (1) with $=$ replaced by $\leqq$. The analogous substitution in (2) gives the important class of subadditive functions; in (1), however, nothing new is obtained.

In brief we have the following

THEOREM. *If $f(x)$ is a real-valued (finite-valued) function of a real variable, measurable on some interval, and satisfying the functional inequality*

$$f(x+y)f(x-y) \leqq f^2(x) - f^2(y) \tag{3}$$

for all real x and y, then f(x) is one of the following three functions: $f(x)=k_1x$, $f(x)=k_2 \sin k_3x$, $f(x)=k_4 \sinh k_5x$, *where the* k_i *are arbitrary real constants.*

Proof. We first prove that the functional inequality (3) for all real x and y implies that (1) holds for all real x and y.

Putting $x=y=0$ in (3) gives $f^2(0)\leqq 0$ and hence

$$f(0) = 0. \tag{4}$$

Putting $x=-y$ in (3) gives, using (4), $f^2(y)\leqq f^2(-y)$. Hence by (3) and (4)

$$0 \geqq f^2(y) - f^2(-y) \geqq f(0)f(2y) = 0$$

and so

$$f^2(y) = f^2(-y). \tag{5}$$

It follows that for each real number y, $f(-y)=$either $-f(y)$ or $f(y)$. Suppose for some real y_0, $f(y_0)=f(-y_0)$; then, putting $x=0$ and $y=y_0$ in (3) we get by (4)

$$f(y_0)f(-y_0) \leqq -f^2(y_0)$$

or since by hypothesis $f(-y_0)=f(y_0)$

$$2f^2(y_0) \leqq 0.$$

Hence $f(y_0)=f(-y_0)=0$ and so for all real y,

$$f(-y) = -f(y). \tag{6}$$

Hence from (3), for all real x and y

$$\begin{aligned} f^2(y) &\leqq f^2(x) - f(x+y)f(x-y) = f^2(x) + f(x+y)f(y-x) \\ &\leqq f^2(x) + f^2(y) - f^2(x) = f^2(y) \end{aligned} \tag{7}$$

and hence equation (1) holds for all real x and y.

We may note that the central idea used in the above proof was that by utilizing the quadratic character of (3) we could prove that f was an odd function. It may be seen similarly that an odd subadditive function is additive.

Now suppose f measurable on the interval (a, b). We will show that f is continuous at zero, whence the theorem follows.

The essential idea in our method is due to Banach [2] and would seem to be adaptable to many functional equations to prove that if a solution is measurable on some interval, it is continuous at some point.

Suppose f measurable on the interval (a, b). By Lusin's Theorem, given $\sigma>0$, there is a function F such that F is continuous on (a, b) and $f(x)=F(x)$ for all $x\in(a, b)$ except for a set of measure $<\sigma$. Let $\sigma=(b-a)/6$.

Since $F(x)$ is continuous, given $\epsilon>0$, there exists $\delta=\delta(\epsilon)>0$ (where δ clearly may be taken $<\sigma$) such that for all h with $|h|<\delta$ and for $x\in(a, b)$,

$$|F(x+h)F(x-h) - F^2(x)| < \epsilon. \tag{8}$$

Since

$$F(x) = f(x) \tag{9}$$

for all $x \in (a, b)$ except for a set of measure $<\sigma$,

$$F(x + h) = f(x + h) \tag{10}$$

for all $x \in (a, b)$ except for a set of measure $<\sigma + |h| < \sigma + \delta$. Similarly

$$F(x - h) = f(x - h) \tag{11}$$

for all $x \in (a, b)$ except for a set of measure $<\sigma + \delta$.

Hence by (9), (10), (11)

$$F(x + h)F(x - h) - F^2(x) = f(x + h)f(x - h) - f^2(x) \tag{12}$$

for all $x \in (a, b)$ except for a set of measure $<(\sigma+\delta)+(\sigma+\delta)+\sigma = 3\sigma+2\delta < 5\sigma = 5(b-a)/6 < b-a$.

Hence given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that given h with $|h| < \delta$, there exists $x = x(h) \in (a, b)$ such that $|f(x+h)f(x-h) - f^2(x)| < \epsilon$. But by equation (1) $f(x+h)f(x-h) - f^2(x) = -f^2(h)$ for all real x and h.

Hence given ϵ, there exists $\delta = \delta(\epsilon) > 0$ such that if $|h| < \delta$ then $|f^2(h)| < \epsilon$. Hence $\lim_{h \to 0} f^2(h) = 0$. And so by (4) f is continuous at 0 which proves the theorem, as remarked above.

In conclusion, it should be noted that our proof, since it applies Lusin's Theorem, depends on the axiom of choice. On the other hand, there is a proof by Sierpiński [7] without using the axiom of choice that the only measurable solutions of (2) are linear. Whether such a proof for the analogous theorem about (1) exists remains an open question.

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CORRECTION

David Zeitlin has pointed out some errata in the paper by Imanuel Marx, "Transformations of series by a variant of Stirling's numbers," this MONTHLY, vol. 69, 1962, pp. 530–532.

For $n=1, 2, \cdots$, one has

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = n! \quad \begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = 0, \quad \sum_{i=0}^{n} i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} = 0$$

and in the definition of $B_j(z)$ as well as in the final formula, $1/j!$ is to be replaced by $j!$

Dr. Zeitlin also calls attention to his paper, "Two methods for the evaluation of $\sum_{k=0}^{\infty} k^n x^k$," this MONTHLY, vol. 68, 1961, pp. 986–989, and the references given there.

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University and A. L. SHIELDS, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, Mathematics Department, University of Michigan, Ann Arbor, Michigan.

NOTE ON MULTIPLE POWER SERIES

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The purpose of this note is to clarify a point in the theory of multiple power series. For simplicity we shall treat only the two-variable case. The double series

$$\sum u_{mn} \qquad (m, n = 1, 2, \cdots) \tag{1}$$

is said to converge to s in the sense of Pringsheim (and this is the only type of convergence with which we shall deal here) if $\lim S_{mn} = s, (m, n \to \infty)$, where S_{mn} is the general rectangular partial sum

$$S_{mn} = \sum u_{jk} \qquad (j = 1, \cdots, m; k = 1, \cdots, n). \tag{2}$$

In other words, (1) converges to s if and only if to each $\epsilon > 0$ correspond M, N such that from $m \geqq M$, $n \geqq N$ follows $|S_{mn} - s| < \epsilon$.

In such case the Cauchy criterion holds: Series (1) converges when and only when to each $\epsilon > 0$ correspond M, N such that

$$|S_{mn} - S_{pq}| < \epsilon \tag{3}$$

for all $m, p \geqq M$; $n, q \geqq N$. From (3) it is seen that convergence implies $u_{mn} \to 0$ $(m, n \to \infty)$. However, it does not follow that the double sequence $\{u_{mn}\}$ is then bounded.

Consider now the double power series

$$\sum a_{mn}x^m y^n \qquad (m, n = 0, 1, 2, \cdots). \tag{4}$$

$\{a_{mn}\}$ may be real or complex, and x, y may be real or complex variables. For (4) there is the well-known result: *If for some point* (x_0, y_0) *with* $x_0y_0 \neq 0$ *there is a number* A *such that*

$$|a_{mn}x_0^m y_0^n| \leqq A \qquad \text{(all } m, n) \tag{5}$$

then series (4) *converges in the domain* R: $|x| < |x_0|$, $|y| < |y_0|$ (which is a rectangle for x, y real and a bicylinder when they are complex). Moreover from (5) follow many of the nice properties of series (4) in R: absolute convergence, uniform convergence (on closed subsets), continuity of the sum, validity of term-wise differentiation and integration, and so on.

It is consequently of interest to know when a result like (5) holds, or an equivalent relation

$$|a_{mn}| \leqq A\mu^m\nu^n \qquad \text{(all } m, n), \tag{6}$$

in terms of knowledge of points of convergence of series (4). On this matter there appears to be a misconception. Thus, one finds in Pierpont ([1], page 144) the assertion: Let the power series (4) converge at (x_0, y_0); then it converges for all (x, y) in $|x| < |x_0|$, $|y| < |y_0|$.

That this statement is incorrect is shown by the following example: Let $x_0y_0 \neq 0$. Take

$$a_{00} = a_{01} = a_{10} = a_{11} = 0; \qquad a_{0n} = x_0 n!, \qquad a_{1n} = -n! \qquad (n > 1);$$

$$a_{m0} = y_0 m!, \qquad a_{m1} = -m! \quad (m > 1); \qquad a_{mn} = 0 \qquad (m, n > 1).$$

Then for $m, n > 1$,

$$S_{mn} = (x_0 - x)\sum_{k=2}^{n} k!y^k + (y_0 - y)\sum_{j=2}^{m} j!x^j.$$

Hence S_{mn} converges only at the points $(0, 0)$, (x_0, y_0). (Here and in the proof of Theorem 1 we use the result: lim $(u_m + v_n)$ exists if and only if u_m and v_n separately have limits.) The "proof" that Pierpont gives uses the argument that (5) holds for all m, n. In point of fact, convergence at (x_0, y_0) only guarantees that (5) holds for all m, n that are sufficiently large simultaneously.

The following two theorems explore the relation between convergence of series (4) and the satisfaction of a relation of type (5) or (6).

THEOREM 1. *Given k points (x_j, y_j) with $x_jy_j \neq 0$ $(j=1, \cdots, k)$. Let there be s distinct values among the x_j's and t among the y_j's. Then there exists a double series* (4) *that converges at and only at the $st+1$ points* $(0, 0)$, (x_i, y_j) $(i, j=1, \cdots, k)$.

Proof. Take $a_{mn}=0$, $m=0, 1, \cdots, s$; $n=0, 1, \cdots, t$. Let $B(x)=b_sx^s+\cdots+b_0$, $C(y)=c_ty^t+\cdots+c_0$ be polynomials having the s distinct x_j's and the t distinct y_j's as their respective zeros. Choose

$$a_{qn} = n!b_q(q = 0, 1, \cdots, s; n > t); \qquad a_{mr} = m!c_r(r = 0, 1, \cdots t, m > s).$$

Finally, take $a_{mn}=0(m>s, n>t)$. Then for $m>s$, $n>t$ we have

$$S_{mn}(x, y) = B(x) \sum_{j=t+1}^{n} j!y^j + C(y) \sum_{j=s+1}^{m} j!x^j. \tag{7}$$

For series (4) to converge at (x, y), $S_{mn}(x, y)$ must have a limit as $m, n\rightarrow\infty$, and this requires that separately each of the two terms in (7) have a limit. The first has a limit if and only if $B(x)=0$ or $y=0$, and the second when and only when $C(y)=0$ or $x=0$. Since $x_jy_j\neq 0$, this means that (4) converges only at (x, y) for which either $B(x)=C(y)=0$ or $x=y=0$; that is, only at the points $(0, 0)$, (x_i, y_j) $(i, j=1, \cdots, k)$.

THEOREM 2. *Let Q be an infinite set of points (x, y) having the property that no two have the same x or the same y, and no x or y is zero. If series* (4) *converges on the set Q, then there exist constants A, μ, ν such that* (6) *holds for all m, n.*

Proof. Take $(x_0, y_0)\in Q$. By convergence at (x_0, y_0) there exists an integer N (which we now fix) such that $|a_{mn}x_0^my_0^n| <1$ for all $m, n>N$; i.e.,

$$|a_{mn}| < |x_0|^{-m}|y_0|^{-n} \qquad (m > N, n > N). \tag{8}$$

Now choose $N+1$ points $(x_j, y_j)\in Q$. Since there is convergence at these points there can be found an integer $M>N$ such that

$$|S_{mn}(x_j, y_j) - S_{qr}(x_j, y_j)| < 1 \qquad (j = 1, \cdots, N + 1; m, n, q, r \geqq M). \tag{9}$$

Let M be fixed.

Take $m=q=M$, $r=M+s-1$, $n=M+s$ $(s\geqq 1)$ in (9):

$$\left| \sum_{k=0}^{M} a_{k,M+s}x_j^k y_j^{M+s} \right| < 1 \qquad (j = 1, \cdots, N + 1);$$

that is,

$$\sum_{k=0}^{M} a_{k,M+s}x_j^k = \omega_{j,s}y_j^{-(M+s)},$$

where $|\omega_{js}| < 1$. So

$$\sum_{k=0}^{N} a_{k,M+s} x_j^k = \omega_{js} y_j^{-(M+s)} - \sum_{p=N+1}^{M} a_{p,M+s} x_j^p. \tag{10}$$

Using (8) we see that the right side is of magnitude not exceeding

$$|y_j|^{-(M+s)} + |y_0|^{-(M+s)} \sum_{p=N+1}^{M} |x_j/x_0|^p;$$

so if we set

$$Y = \min\{|y_0|, |y_1|, \cdots, |y_{N+1}|\}, \quad B = \max\left\{1 + \sum_{p=N+1}^{M} |x_j/x_0|^p\right\}$$
$$(j = 1, \cdots, N+1)$$

then

$$\sum_{k=0}^{N} a_{k,M+s} x_j^k = \epsilon_{js} B Y^{-(M+s)} \qquad (j = 1, \cdots, N+1), \tag{11}$$

with $|\epsilon_{j,s}| \leqq 1$.

Now (11) is a system of $N+1$ linear equations in the a's, with a nonvanishing determinant of coefficients (since it is a Vandermonde determinant with distinct x_j's). We can therefore solve by the Cramer rule:

$$a_{k,M+s} = \sum_{j=1}^{N+1} C_{kj} \epsilon_{js} B Y^{-(M+s)} \qquad (k = 0, 1, \cdots, N).$$

Moreover, and this is important, C_{kj} is independent of s. Let

$$C = \max \sum_{j=1}^{N+1} |C_{kj}| \qquad (k = 0, 1, \cdots, N).$$

Then

$$|a_{k,M+s}| \leqq BCY^{-(M+s)} \qquad (k = 0, 1, \cdots, N; s = 1, 2, 3, \cdots). \tag{12}$$

In obtaining (12) we worked with the first $N+1$ rows of series (4). We can argue similarly with the first $N+1$ columns, to obtain the analogous result:

$$|a_{M+s,k}| \leqq B'C'X^{-(M+s)} \qquad (k = 0, 1, \cdots, N; s = 1, 2, 3, \cdots) \tag{13}$$

where B', C' are certain constants and

$$X = \min\{|x_0|, |x_1|, \cdots, |x_{N+1}|\}.$$

Now (8), (12), (13) can all be represented in the form (6). And we have accounted for all coefficients save a finite number (namely where m, n simultaneously are less than or equal to M). So there do exist A, μ, ν satisfying (6) for all m, n, as was to be shown.

Remark. In the above proof the only use made of the hypothesis that Q is an infinite set was to insure the existence of $N+1$ points (x_j, y_j), once N had been selected. We may therefore assert that the conclusion of Theorem 2 holds if: (i) Q is a finite set (x, y) of r points, with no two x's equal, no two y's equal, and no x or y zero. (ii) Series (4) converges on Q. (iii) For some point $(x_0, y_0)\epsilon Q$ the integer N for which (8) is satisfied can be chosen less than r.

DEFINITION. *Let us say that the real- or complex-valued function $f(x, y)$ is "analytic" at* $(0, 0)$ *if it has the power series representation*

$$f(x, y) = \sum a_{mn}x^m y^n \qquad (m, n = 0, 1, \cdots), \tag{15}$$

convergent in some neighborhood of $(0, 0)$.

We then have the corollary of Theorem 2: *If $f(x, y)$ is "analytic" at* $(0, 0)$ *then $\{a_{mn}\}$ in* (15) *satisfies a relation* (6) *for all m, n; and series* (15) *has, in some neighborhood of* $(0, 0)$, *the properties* (*absolute convergence, etc.*) *stated after relation* (5).

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AN OLD THEOREM IN A NEW SETTING

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The reader is no doubt familiar with Euclid's classic proof that there are infinitely many prime numbers among the rational integers. While the analogous result for quadratic number fields may be proved almost trivially by factoring the rational primes, the purpose of this note is to obtain that theorem by a direct generalization of Euclid's method. (See Harry Pollard, *The Theory of Algebraic Numbers*, Carus Monograph No. 9, page 73, paragraph following Corollary 7.6.)

Let $R(\sqrt{d})$ denote the field of all numbers of the form $a+b\sqrt{d}$, where a and b are rational numbers and d is a square-free integer. By $R[\sqrt{d}]$ we mean the ring of all algebraic integers in $R(\sqrt{d})$, that is, roots of monic irreducible polynomials with (rational) integral coefficients. (Henceforth the word *integer* will refer to an element of $R[\sqrt{d}]$ unless explicitly stated otherwise.) If $\alpha=a+b\sqrt{d}$, we denote by $\bar{\alpha}$ the *conjugate* $a-b\sqrt{d}$ of α. The *norm* of α is defined to be

$$N(\alpha) = \alpha\bar{\alpha} = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2.$$

A simple calculation shows that $N(\alpha\beta)=N(\alpha)N(\beta)$. A *unit* ϵ is defined to be an integer such that $N(\epsilon)=\pm 1$. (Alternately, an integer ϵ is a unit if and only if $1/\epsilon$ is an integer.) If $\alpha=\beta\gamma$ where α, β and γ are integers, we say β *divides* α; if $\alpha=\epsilon\gamma$ where ϵ is a unit, γ is called an *associate* of α. A nonunit integer which has only units and associates as divisors is called a *prime* in $R(\sqrt{d})$. Since the norm of an integer is a rational integer, it is clear (using the fact $N(\alpha\beta)=N(\alpha)N(\beta)$) that an integer θ will be prime in $R(\sqrt{d})$ if $N(\theta)$ is a rational prime. By definition,

a prime π is not a unit, hence $|N(\pi)| \geqq 2$. If λ is a nonunit, nonzero integer, then λ factors into a *finite* number of primes. For if λ is not itself prime, write $\lambda=\beta\gamma$, where neither β nor γ is a unit. If either β or γ is not prime repeat the process. This procedure must terminate since a factorization $\lambda=\tau_1 \cdots \tau_n$ induces the factorization $|N(\lambda)|=|N(\tau_1)| \cdots |N(\tau_n)|$ where each factor $|N(\tau_i)|$ exceeds one.

With these preliminaries out of the way, we may proceed with the proof of the lemma upon which the theorem will depend.

LEMMA. *Let α be an algebraic integer in $R[\sqrt{d}]$ such that $N(\alpha)>2$. Then α^2+1 is not a unit in $R[\sqrt{d}]$.*

Proof. Case 1: $d<0$. In this case $|\alpha^2|=|\alpha|^2=\alpha\bar{\alpha}=N(\alpha)>2$, hence $|\alpha^2|-1>1$. Therefore

$$N(\alpha^2+1) = (\alpha^2+1)\overline{(\alpha^2+1)} = |\alpha^2+1|^2 \geqq (|\alpha^2| - 1)^2 > 1,$$

and so α^2+1 is not a unit.

Case 2: $d>0$. Both $\alpha=a+b\sqrt{d}$ and $\bar{\alpha}=a-b\sqrt{d}$ are real, therefore α^2, $\bar{\alpha}^2$ are positive. (Note $N(\alpha)>2$ implies $\alpha\neq 0$.) Hence $N(\alpha^2+1)=(\alpha^2+1)(\bar{\alpha}^2+1)>1$. Thus α^2+1 is not a unit in this case either and the lemma is proved.

THEOREM. *There are infinitely many primes in $R(\sqrt{d})$.*

Proof. We first show that at least one prime exists in $R(\sqrt{d})$.

If $d=-1$, then $2+i$ is prime in $R(\sqrt{-1})=R(i)$ since $N(2+i)=5$, a rational prime.

If $|d|>1$, then $\sqrt{d}$ is an integer (it is a root of $x^2-d=0$), but not a unit of $R(\sqrt{d})$, since $N(\sqrt{d})=-d\neq\pm 1$. Therefore $\sqrt{d}$ factors (not necessarily uniquely) into primes of $R(\sqrt{d})$.

In either case, primes exist. From now on we parallel Euclid's old proof.

Suppose $\pi_1, \cdots, \pi_k$ is a collection of nonassociated primes in $R(\sqrt{d})$ such that every prime ξ in $R(\sqrt{d})$ is associated with π_i for some i. Define

$$\alpha = \pi_1^2 \cdots \pi_k^2.$$

As noted previously, $|N(\pi_i)| \geqq 2$. Therefore

$$N(\alpha) = (N(\pi_1))^2 \cdots (N(\pi_k))^2 \geqq 4 > 2.$$

Thus we may apply the lemma and get

$$\lambda = \alpha^2 + 1 = \pi_1^4 \cdots \pi_k^4 + 1$$

is not a unit. Being a nonunit integer, λ factors into primes of $R(\sqrt{d})$, say

$$\lambda = \xi_1 \cdots \xi_s, \qquad (\xi_i \text{ prime}, i = 1, \cdots, s).$$

Hence

$$\frac{\lambda}{\xi_1} = \frac{\pi_1^4 \cdots \pi_k^4}{\xi_1} + \frac{1}{\xi_1}$$

is an integer. By assumption, ξ_1 must be associated with π_i for some i. Therefore $(\pi_1^4 \cdots \pi_k^4)/\xi_1$ is an integer, which in turn implies $1/\xi_1$ is an integer. It follows that $N(1/\xi_1) = N(1)/N(\xi_1) = 1/N(\xi_1)$ is a rational integer, and so $N(\xi_1) = \pm 1$, i.e. ξ_1 is a unit. However, this contradicts the fact that ξ_1 was a prime, and the contradiction proves the theorem.

INVERSES IN RINGS WITH UNITY

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It is believed that the following proof of a theorem by Kaplansky is noteworthy.

If u is an element of a ring R with unit 1 such that u has more than one right inverse, then u has infinitely many right inverses.

Define

$$S = \{x \mid ux = 1\},$$
$$T = \{xu - 1 + s \mid x \in S\},$$

where s is some fixed member of S. Note that by hypothesis S contains at least two distinct elements. Then $T \subseteq S$, since if $xu-1+s \in T$ then $x \in S$, and $u(xu-1+s) = uxu-u+us = 1u-u+1 = 1$.

The mapping of S onto T given by $x \to xu-1+s$ is one-to-one, since if $xu-1+s = yu-1+s$, where $x, y \in S$, then $xu = yu$ and $x = y$ due to the fact that u has a right inverse. If S is finite then $T = S$. In particular $s \in T$ so that for some $x \in S$ we have $xu-1+s = s$ or $xu = 1$. Consequently, for $t \in S$, distinct from x, we arrive at the following contradiction: $x = x(ut) = (xu)t = t$.

Therefore S is infinite.

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REDUCTION OF THE SIMSON LINE THEOREM TO THE THEOREM OF PASCAL

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I. Pascal's theorem. Pascal's theorem may be stated as follows: If the three pairs of sides of a triangle, passing through the three vertices are cut by three transversals, the six points of intersection lie on a conic, if, and only if, the three transversals cut the respective third sides of the triangle in three collinear points.

The three transversals may have a point in common; whence the proposition: if three collinear points, taken on the sides of a triangle, are projected from

a point, the three projecting lines (transversals) cut the remaining sides of the triangle in six points lying on a conic.

II. Miquel's theorem. Now the point common to the three transversals may lie on the circumcircle of the given triangle. And if the transversals cut the respective third sides of the triangle in three collinear points, then the transversals make equal angles in cyclic order with the respective sides. (The Miquel triad for a point on the circumcircle are collinear. Hence it is seen that this "Miquel line theorem" may be interpreted as a special case of the theorem of Pascal.)

III. Simson's theorem. The Simson line is the particular case of the Miquel triad when the transversals from a point on the circumcircle are respectively orthogonal to the sides of the given triangle. Hence it is seen that the Simson line theorem may also be interpreted as a special case of the theorem of Pascal.

The author is grateful to the referee for valuable suggestions.

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THE EXISTENCE AND UNIQUENESS OF THE EXPONENTIAL FUNCTION AS THE SOLUTION OF $f'=f$, $f(0)=1$

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Introduction. The existence of a function f such that $f'=f$, $f(0)=1$ is established, first for f defined on $[0, \infty)$ as the maximal member of a certain family of functions, and then on all real numbers by a simple extension. The uniqueness of such a function is then obtained as a specialization of the uniqueness theorem for a first order differential equation. A proof of the uniqueness theorem is included for completeness sake.

Existence theorem.

THEOREM. *There is a function* exp *defined on the real numbers such that* $\exp'=\exp$ *and* $\exp(0)=1$.

Proof. We define a class of functions of which the restriction of exp to $[0, \infty)$ will be the maximal member. The proof consists in verifying the existence and desired properties of this maximum.

We call a positive function *subexponential* if it is nondecreasing and every difference quotient is bounded by the value of the function at the upper number. In other words, f is subexponential if f is positive and $0 \leqq f(x)-f(y) \leqq f(x)(x-y)$ whenever x, y are in the domain of f and $x>y$. Using the fact that f is nondecreasing and splitting into cases $x-1\leqq y<x$ and $x<y\leqq x+1$, we can use this

defining inequality to show that $|f(x)-f(y)| \leqq A|x-y|$ whenever $0<|x-y| \leqq 1$, where $A=$the sup of f on $[x-1, x+1]$; consequently, a subexponential function is continuous. We also note that positive constant functions are subexponential.

Let F be the family of those subexponential functions which are defined on $[0, \infty)$ and which have value 1 at 0. F is nonempty since it contains the constant function 1.

Until we arrive at the point where we wish to extend exp to negative numbers, we assume that the real numbers x, y below are nonnegative.

We show first that each set $\{f(x)|f\in F\}$ is bounded above. If f is subexponential we have, taking $x=y+\frac{1}{2}$ in the above inequality, transposing, and multiplying by 2, $f(y+\frac{1}{2}) \leqq 2f(y)$. Taking into account that f is nondecreasing, we can show inductively that if $x \leqq n$, n an integer, then $f(x) \leqq 2^{2n}f(0)$. Thus the l.u.b. of each set must exist, so we can define a function exp on $[0, \infty)$ by $\exp x = \text{l.u.b. } \{f(x)|f\in F\}$.

If f and g are real valued functions we define their *join* $f\vee g$ on the intersection of their domains by $f\vee g(x)=\max(f(x), g(x))$.

The rest of the proof is outlined by a sequence of lemmas with sketches of proofs.

LEMMA 1. *If f, $g\in F$ then $f\vee g\in F$.*

Proof. $f\vee g(0)=1$ is trivial. If $x>y$, let us suppose that $f(x)\geqq g(x)$, so that $f\vee g(x)=f(x)$. Then $f\vee g(x)-f\vee g(y) \leqq f(x)-f(y) \leqq f(x)(x-y)=f\vee g(x)(x-y)$. That the join of nondecreasing functions is nondecreasing is well known and easily proved.

LEMMA 2. *If $f\in F$ and for some pair x, y with $x>y$ we have $f(x)-f(y) < f(y)(x-y)$ then there is $g\in F$ such that $f(x)<g(x)$.*

Proof. Let h be the straight line through $(y, f(y))$ with slope $f(y)$. Then it is easily verified that $g=f\vee h$ works.

LEMMA 3. *If x, $y\in[0, \infty)$ then there is a sequence $\{f_n\}$ of functions in F such that* $\lim f_n(x)=\exp(x)$ *and* $\lim f_n(y)=\exp(y)$.

Proof. By the definition of exp there are sequences $\{g_n\}$, $\{h_n\}$ of functions in F such that $\{g_n(x)\}$ and $\{h_n(y)\}$ are nondecreasing to $\exp(x)$ and $\exp(y)$, respectively. Thus if we define for each positive integer n $f_n=g_n\vee h_n$, then each f_n is in F by Lemma 1 and each sequence $\{f_n(x)\}$ and $\{f_n(y)\}$ has the desired limit.

LEMMA 4. $\exp\in F$.

Proof. The desired inequality (for $x>y$) follows from taking the limit of $0\leqq f_n(x)-f_n(y) \leqq f_n(x)(x-y)$, where $\{f_n\}$ is as in Lemma 3.

LEMMA 5. *If* $x>y$ *then* $\exp(y)(x-y) \leqq \exp(x)-\exp(y)$.

Proof. The function exp cannot satisfy the conclusion of Lemma 2 as does f because of the maximality of exp; consequently, exp cannot satisfy the hypothesis of Lemma 2. Of this hypothesis, exp satisfies the first part, namely, Lemma 4, and so must satisfy the negation of the second part, namely, Lemma 5.

LEMMA 6. $\exp' = \exp$.

Proof. For every x, y the inequalities in Lemma 5 and implicit in Lemma 4 can be divided by $x-y$ to give

$$\exp(y) \leqq (\exp(x) - \exp(y))/(x - y) \leqq \exp(x).$$

By using this and the continuity of exp, we prove that for every y the right hand derivative $\exp'_+(y) = \exp(y)$ letting x approach y from above; we prove that for every positive x the left hand derivative $\exp'_-(x) = \exp(x)$ by letting y approach x from below. Equality of the left and right derivatives then implies $\exp'(x)$ exists and equals the common value $\exp(x)$ for $x>0$; for 0 we have only $\exp'_+(0) = \exp(0) = 1$.

It remains to extend exp to all real numbers. We do this by requiring $\exp(-x) = 1/\exp(x)$. A simple computation using the chain rule and the rule for the derivative of a quotient shows that we still have $\exp' = \exp$. At 0 the left and right derivatives again match.

The usual properties of exp follow easily from the differential equation. For example, the addition rule follows from differentiating

$$[\exp(x + y) - \exp(x)\exp(y)]/\exp x$$

with respect to x, getting 0, so the value of this expression must be constantly what it is at $x=0$, namely, 0. By inductive arguments, using the addition rule, there follows

$$\exp(r) = (\exp(1))^r \text{ for every rational number } r.$$

Uniqueness theorem. This proof of the uniqueness theorem for a first order equation is a modification of the proof given in Courant, *Differential and Integral Calculus*, Vol. II, p. 459, so as to avoid the use of integrals.

THEOREM. *Let* f *be defined on the rectangle* $a \leqq x \leqq b$, $c \leqq y \leqq d$, *and let there be positive* M *such that for every* $x \in [a, b]$ *and* $y_1, y_2 \in [c, d]$, *the Lipschitz condition* $|f(x, y_1) - f(x, y_2)| \leqq M|(y_1 - y_2|$ *is satisfied. Let* x_0 *and* y_0 *be numbers such that* $a \leqq x_0 \leqq b$, $c \leqq y_0 \leqq d$. *Then there is at most one differentiable function* g *defined on* $[a, b]$ *such that* $g(x_0) = y_0$ *and, for every* $x \in [a, b]$, $g(x) \in [c, d]$ *and* $g'(x) = f(x, g(x))$.

(We take $g'(a)$ and $g'(b)$ to be one-sided derivatives. I am indebted to the referee for pointing out that the hypothesis of continuity of f is unnecessary.)

Proof. Suppose h also satisfies these conditions. Then $k=g-h$ would satisfy $k(x_0)=0$ and

$$(1) \qquad |k'(x)| = |f(x, g(x)) - f(x, h(x))| \leqq M|k(x)|.$$

We show that k is 0 on the interval $I=[x_0-1/2M,\ x_0+1/2M]\cap[a, b]$. Since $|k|$ is continuous on I it will have a maximum, K, assumed at some $x_1\in I$, $x_1\neq x_0$. (If $x_1=x_0$ then $K=0$, which is what we want.) Then by the mean value theorem and (1), there is x_2 between x_0 and x_1 such that $K=|k(x_1)-k(x_0)| = |k'(x_2)(x_1-x_0)| \leqq MK\cdot 1/2M$, which shows $K=0$ as desired.

By taking a new x_0 to be an endpoint of a largest interval on which k is 0 and repeating the above process, we obtain a contradiction unless such an endpoint is a or b, so k is 0 on $[a, b]$.

As a corollary to these two theorems we get the first one with an important extra word: "There is a *unique* function exp $\cdots$," since the function f defined by $f(x, y)=y$ obviously satisfies the Lipschitz condition (with $M=1$) on every rectangle.

MATHEMATICAL EDUCATION NOTES

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AN INTERESTING ISOMORPHISM

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1. Introduction. As is well known, the excitement which followed the discovery of Boswell's diaries has led numerous scholars to ransack Old World archives and attics in the hope of making a similar find. Some interesting manuscripts have thus come to light and are now under study. The writers' attention has been called in particular to one from an old tavern in Plymouth, Devonshire. Tentatively dated 1718, it claims to be an unpublished continuation of the memoirs of Captain Lemuel Gulliver, relating how his fifth and last voyage took him across the Atlantic Ocean to the unknown island of Acirema. Its authenticity is dubious; most of Gulliver's biographers still accept his statement that after he returned from the country of the Houyhnhnms he traveled no more.

That question lies beyond the purview of mathematics. But whatever its authorship, the journal has considerable intrinsic value. So remarkably detailed a picture does it give of Acireman society that we can analyze sociological characteristics as if we were dealing with a contemporary civilization. Certain of them may be especially interesting to mathematicians, in view of the current controversy over proposals to modify the requirements for the doctor's degree.

2. Mathematics in Acirema. According to the manuscript, mathematics is the most highly regarded intellectual activity in Acirema, including even the traditional fine arts. Mathematical knowledge and interest are widespread on every level of society. Publication in this field is therefore enormous.

Of course, the majority of these productions are intended for the less intellectual classes and rarely rise above elementary cohomology theory. There is a brisk demand for arithmetical puzzles, the more perverse of which are officially condemned but in practice condoned. However, the well-educated prefer works of great sophistication and elegance. Such books and essays are eagerly discussed, both in the "high-brow" magazines and *viva voce*. Indeed, no one is considered cultured unless he is, or pretends to be, well-informed about them. Mentioning the latest theorem is a standard ploy of social one-upmanship, whether or not one has actually studied it.

All types of people attend the theater, where mathematical contests stimulate the audience to loud cheering for a favorite team, or professional expositors declaim classic and modern mathematical papers in appropriately sonorous tones. It is still more astonishing what a high degree of mathematical competence is taken for granted in daily life by both rich and poor. This is due to the generally excellent secondary school training in the subject. Another result of this is that the student enters college with skill in computation and a tasteful appreciation of mathematical literature.

Naturally, creative mathematicians are among the chief celebrities in Acirema. Not only the large incomes they often earn from royalties and fees, but the glamour attached to their profession, make many young men and women wish to emulate them. But this is not easy. Sufficient ability to earn a living exclusively by devising new mathematics is rare. Most of the ambitious young lack it. Those who have the gift, and are perhaps destined to success in later life, find the initial years to be ones of hardship. Hence a steady income from other sources is essential. Thus many would-be creative mathematicians go temporarily or permanently into applied mathematics or into teaching. We are here concerned with the state of college teaching in Acirema.

Being as respected within the educational system as without, mathematics is required of all students. Most courses are organized around a period or a famous mathematician of the past, e.g., "Readings in Hellenistic Geometry" or "Aspects of Archimedes' Thought." To be sure, there are also some courses in creative mathematics. But the paradox is that, while the successful creative mathematician is so much admired and rewarded, the teaching or practice of

his art within the academic world is discouraged. The courses in question are given disparaging titles such as "Heuristics," "Writing Workshop," and "Problem Solving." It is difficult, and in most universities impossible, to get a higher degree in mathematics by doing inventive work.

Instead, the Ph.D. is awarded for "an original contribution to knowledge about mathematics"—that is, a contribution based on research into the vast literature of the subject. A typical thesis might be entitled, "Considerations in Support of Braunschweiger's Hypothesis that Fermat's Last Theorem was Suggested by a Passage in his Correspondence with Pascal."

Once secure in his position, the mathematics professor characteristically publishes more papers of this nature, or else critical commentaries on pieces of genuine pioneering. The elaborate jargon of these criticisms, and their frequent descent to highly personal remarks about the supposed character of the author under study, have alienated most creative mathematicians and a good many laymen.

It is true that certain professors produce original works without loss of prestige, and a few institutions have established special chairs for "resident mathematicians" on the theory that distinguished outsiders will prove intellectually stimulating to the student. But approval of such a double life is reserved for the most famous. Professors who published an occasional creative work on a somewhat lower plane, for instance in a popular mathematical magazine, are quite bashful about it. Many resort to pseudonyms.

In the hope of encouraging mathematical imagination, one or two universities have recently taken the radical step of awarding the Ph.D. for creating new mathematics. This has been generally deplored by scholars, whose usual remark is: "Of course, creative mathematics is the most important thing in our civilization; but after all, it isn't *research*." Albeit a good deal of bibliographical work is sometimes required to collect background information for a creative thesis, traditionalists deny that it represents true scholarship.

The more revolutionary professors retort that, while scholarly work about mathematics is essential to an orderly development, the dominance of the academic field by critics has tended to stifle originality. Some have gone so far as to say that the creative mathematicians ought to take over higher education altogether. But defenders of the status quo can point to the deplorable results of this policy in the field of literature.

3. Literature in Acirema. Gulliver, assuming that he is actually the author of the journal, waxes as indignant over the state of letters in Acirema as is to be expected of a cultivated eighteenth-century Englishman. He reports that not only the common people, but most intellectuals of the country are virtually illiterate. They have little interest or ability in reading, and as a rule are unable to use the literary language to solve the simplest practical problems. They seldom have any clear notion of what literature is about. When introduced to one of the rare literaticians, the average person makes some such remark as,

"I never could spell," or "You must have no trouble writing a correct sentence." He fails entirely to understand the other's protest that spelling and grammar are not the whole of literature.

Natural scientists are no better. They often admit that they are baffled by the words used in their periodicals, that they just read the formulas and take the rest on faith.

The cause of this situation is obvious enough. The general public feels alienated from literature because there is almost none that is readable, let alone interesting. Creative writers produce works only for each other, in so crabbed a style that not even the professors of literature can understand them. It is not considered respectable to carry on any literary activity except "research," which is defined by literaticians as the writing of original poems, stories, and novels of the most esoteric kind. Literary critics do not exist. The leading creative writers occupy every important position at the universities and in special research institutions. They disdain teaching, and when forced to do it make no effort to interest the student or clarify difficult points.

Nevertheless, the demand for literaticians is high and rapidly increasing. They are needed to write for news media, industry, the government, and wherever else non-mathematical ideas have to be communicated. The thinking public has lately become aware of this need and of the acute shortage of literaticians. It is well understood that they are wanted above all in teaching, so that there may be enough graduates to meet the demands of society. This has led to a reexamination of the whole literary educational system.

At present the Ph.D. is awarded only for "research" as defined above. Since academic advancement depends on continued publication of the same kind of material, there is a vast output of tasteless writing, which almost nobody reads. This tends to repel young people who might otherwise go into the field. They are equally discouraged by university requirement that are quite inappropriate to their intended careers.

Accordingly, it has been suggested that higher degrees in literature be given for scholarly work, as they are in mathematics. Advocates of this change argue that it would bring literature into line with other academic disciplines, increase the number of teachers, and raise the quality of original work. They even hope that it might start a trend toward literacy in the populace at large. Their opponents insist that such proposals are "a direct strike at creative literature." They say that to equate a degree intended to identify the trained novelist or poet with one intended to meet a social need for literary scholars and teachers threatens the standard of excellence hitherto maintained.

4. Conclusion. Unfortunately, Gulliver's stay in Acirema seems to have had an adverse effect upon his style. Toward the end the manuscript becomes incomprehensible, and no student has thus far been able to discover the outcome of the controversy he has described. Perhaps some of the money now being spent on space exploration should go to outfitting an expedition to Acirema.

The authors acknowledge support from grant number M 17 of the R.P.S.F

tional numbers, is asked, "What is the derivative?" There goes a two hour meeting, but the derivative must be explained.

Clearly, both purposes of Remedial Mathematics: laying the groundwork for understanding in a systematic way, and helping the students with specialized difficulties, often require a pace so slow that the lessons take their toll of the mathematician's nerves. No step can be left out; when "$(39-2\cdot 8)x=23x$" is written, then some students will "lose" their instructor. He has to explain that $39-2\cdot 8=39-16=23$. But compensation for this nerve-grating is in the mature attitude of the students who grasp related concepts quite easily, after one or two are sufficiently explained, and in the gratefulness that they show.

The double purpose of the course also leads to a twofold result: on one hand the students will be able to substitute for colleagues whose field is mathematics or to teach biology, chemistry, even physics with the use of elementary mathematics; on the other hand, they will more readily understand classes in their fields. For the instructor it may not be very exciting to teach the rules for finding highest common multiples or lowest common denominators, but there is satisfaction in thinking that through his present students, he will teach classes of high school youngsters in the future.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
Collaborating Editor: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1571. *Proposed by J. L. Pietenpol, Columbia University*

How many times in a twelve hour period are the hands of a clock interchangeable (i.e., such that interchanging the positions of the hands yields a possible clock reading)?

E 1572. *Proposed by Anders Bager, Hjørring, Denmark*

Enumerate the number of (1) squares, (2) rectangles, on an $n\times n$ "chess" board.

E 1573. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Prove that the arithmetic mean of the angle bisectors of a triangle T never exceeds the sum of the distances of the circumcenter from the three sides of T, with equality if and only if T is equilateral.

E 1574. *Proposed by Simon Vatriquant, Brussels, Belgium*

If, in triangle ABC, $\sin^2 A \cos B = \sin^2 C \cos C$, show that the circumdiameter through A, the median through B, and the angle bisector through C are concurrent.

E 1575. *Proposed by Ronald Alter, University of Pennsylvania*

Find all pairs of positive integers m, p such that $\phi(m) = m/p$, where p is prime and ϕ is the Euler function.

E 1576. *Proposed by L. R. Heinen and W. C. Waterhouse, Harvard University*

Find all pairs of positive integers n, s such that $n^{\phi(s)} = s$, where ϕ is the Euler function.

E 1577. *Proposed by Marjorie Bicknell, San Jose State College*

Show that for any integer n one can construct a symmetric fourth order determinant whose elements are ten consecutive integers and whose value is n.

E 1578. *Proposed by E. O. Thorp, New Mexico State University*

Consider the series $\sum_{n=1}^{\infty} |\sin n|^a$, where a is an arbitrary positive number. For which values of a does the series converge and for which values does it diverge?

E 1579. *Proposed by Azriel Rosenfeld, Yeshiva University*

Call a subgroup H of a group G *strongly normal* if every subgroup K of H which is normal in H is normal in G. Prove that supplemented implies strongly normal implies normal, but that neither reverse implication holds.

E 1580. *Proposed by Guy Torchinelli, State University College at Buffalo*

Prove or disprove: If p is prime then $2^{p-1} \not\equiv 1$, mod p^2.

SOLUTIONS

A Non-Elliptical Oval

E 1521 [1962, 566]. *Proposed by Frank Hawthorne, Education Department, State of New York*

It has been claimed that if one "draws a circle" with a pair of compasses on a piece of paper rolled on a tin can, the result, when unrolled, will be an ellipse. Prove or disprove this.

Solution by D. C. B. Marsh, Colorado School of Mining. Establish coordinate axes so as to assume the paper to be rolled on the cylinder $x^2+y^2=a^2$ and $(a, 0, 0)$ to be the center of the curve described. If we unroll the paper onto the plane $x=a$, the general point $(a \cos \theta, a \sin \theta, z)$ on the cylinder will fall into position $(a, a\theta, z)$ on the plane. However, with a constant compass setting we consider only those points on the cylinder for which $(x-a)^2+y^2+z^2=k^2$ for a constant k. This condition may be written as $4a^2 \sin^2 (\theta/2)+z^2=k^2$, from which we observe that the plane locus is $4a^2 \sin^2 (y/2a)+z^2=k^2$, which is *not* an ellipse.

Also solved by Joseph Basile, Joseph Beer, Walter Bluger, Robert Bowen and Robert Spira (jointly), Norman Brenner, Warren Brisley, Frederick Carty, P. R. Chernoff, R. J. Cormier, Frank Dapkus, Gus Di Antonio, Dan Erickson and David Van Essen (jointly), Michael Goldberg, L. D. Goldstone, S. H. Greene, H. W. Guggenheimer, V. E. Hoggatt, Jr. and I. D. Ruggles (jointly), A. R. Hyde, Roman Kaluzniacki, M. S. Klamkin, J. D. E. Konhauser, Wallace Manheimer, Erich Marchand, C. S. Ogilvy, M. J. Pascual, J. L. Pietenpol, E. H. Primoff, H. D. Ruderman, U. V. Satyanarayana, Perry Scheinok, R. Shorrock, R. Sibson, Jr., H. P. Smith, E. L. Spitznagel, Jr., Rory Thompson, Guy Torchinelli, G. W. Walker, Charles Wexler, and Raymond Whitney.

Goldstone pointed out that this problem is stated and solved in Wilson-Tracy, *Analytic Geometry*, 3rd ed., Probs. 11, 13, p. 291. Klamkin pointed out that the problem is contained in the solution to Problem 87, May–June 1952 issue of *Mathematics Magazine*, pp. 282–3. Ogilvy pointed out that the problem is virtually the same as Problem 302, Nov. –Dec. 1957 issue of *Mathematics Magazine*, pp. 115–6. In this last reference it is remarked that the only cylindrical surfaces for which the construction will yield an ellipse is a plane or two planes intersecting at the center of the "circle," and that, of course, the ellipse is then a circle.

Groups, Domains, Fields

E 1522 [1962, 566]. *Proposed by Azriel Rosenfeld, Budd Electronics, Long Island City, New York*

(1) Prove that a group which has only finitely many subgroups must be finite. (2) Prove the corresponding statement about commutative integral domains with identity. (3) Show by example that the corresponding statement about commutative fields is false.

Solution by J. L. Pietenpol, Columbia University. (1) Suppose the group is infinite. If every element has finite order then the cycle of each element constitutes a subgroup, and there are infinitely many of these. If some element has infinite order, its cycle is isomorphic to the group of integers under addition, which has infinitely many subgroups.

(2) The corresponding statement about a commutative integral domain with identity is false, since, for example, the domain of integers has no proper subdomain with identity.

(3) The field of rationals constitutes a commutative field with no proper subfield.

Also solved by H. L. Abbott, Joseph Beer, J. L. Brenner, Brother Joseph Heisler, G. C. Bush, P. R. Chernoff, Leonard Feldman, G. S. Glazer, Gerald Janusz, V. H. Keiser, Jr., Philip Marcus, M. G. Murdeshwar, L. L. Scott, Jr., H. P. Smith, Guy Torchinelli, Dennis Travis, Alan Weinstein, and the proposer.

Editorial Note. While the statement, "A commutative integral domain with identity which has only finitely many subdomains with identity must be finite," is false, the statement, "A commutative integral domain with identity which has only finitely many subdomains must be finite," can be shown to be true.

A Finite Model of the Plane

E 1523 [1962, 566]. *Proposed by J. G. Hocking, Michigan State University*

The transformation of the euclidean plane (referred to complex coordinates) onto the open unit disk given by $z \to z/(1+|z|)$ is one-to-one and continuous. It provides a finite model of the plane. Describe the image of a straight line under this transformation.

Solution by Rory Thompson, Navy Electronics Laboratories, San Diego, California. A line through the origin transforms into a segment of itself; a line not through the origin transforms into that portion within the open unit disk of the conic having the line as directrix, the origin as corresponding focus, and the reciprocal of the positive distance of the line from the origin as eccentricity. To see this let us write the transformation as

$$re^{i\theta} \to [r/(1+r)]e^{i\theta} = r_1 e^{i\theta_1}.$$

We need consider only lines $y=b$, $b>0$, or, in polar coordinates, lines $r=b \csc \theta$' $0<\theta<\pi$. Since $r=r_1/(1-r_1)$, $\theta=\theta_1$, the image of such a line is found to be

$$r_1 = b/(b+\sin\theta_1) = 1/(1-b^{-1}\sin\theta_1),$$

justifying our initial statement.

Also solved by Joseph Beer, Robert Bowen, Norman Brenner, G. C. Bush, Frederick Carty, P. R. Chernoff, M. S. Demos, J. A. Faucher, David Friedman, H. W. Guggenheimer, Fritz Herzog, V. E. Hoggatt, Jr. and I. D. Ruggles (jointly), M. S. Klamkin, T. J. Lee, Jiang Luh, D. C. B. Marsh, G. M. Merriman, J. L. Pietenpol, Perry Scheinok, J. A. Tierney, Guy Torchinelli, W. C. Waterhouse, Charles Wexler, Raymond Whitney, and J. S. W. Wong.

Generalization of a Theorem on the Distribution of Primes

E 1524 [1962, 566]. *Proposed by Albert Wilansky, Lehigh University*

Let $\beta(n)$ be a sequence of positive integers such that $\beta(1)=1$, and $\beta(n+1)-\beta(n)$ is either 0 or 1 for $n=1, 2, 3, \cdots$. Let I be the set of all $n/\beta(n)$ which are integers. Show that I is a segment of the integers.

Solution by W. C. Waterhouse, Harvard University. For $m>1$, suppose that for some n, $n/\beta(n)\geqq m$; let p be the first such n. Then $(p-1)/\beta(p-1)<m \leqq p/\beta(p)$. This implies $\beta(p-1)=\beta(p)$ and hence $m=p/\beta(p)\in I$. Thus I is a segment.

Also solved by Jack Abad, Daniel Ashler, Joseph Beer, Robert Bowen, J. L. Brenner, Norman Brenner, Brother Joseph Heisler, Frederick Carty, P. R. Chernoff, D. I. A. Cohen, N. J. Fine, Dee Fuller, Michael Goldberg, Ralph Greenberg, C. V. Heuer, V. E. Hoggatt, Jr., M. R. Kirch, Jiang Luh, D. C. B. Marsh, J. B. Muskat, Sam Newman, R. J. Oberg, J. L. Pietenpol, John Rainwater, A. S. S. Sastry, U. V. Satyanarayana, Perry Scheinok, Donna J. Seaman, D. L. Silverman, H. P. Smith, E. L. Spitznagel, Jr., Guy Torchinelli, Dennis Travis, John Vinson, N. F. Williamson, J. E. Yeager, Larry Zalcman, and the proposer.

Editorial Note. This problem generalizes the theorem: "Let $\pi(n)$ be the number of primes $\leqq n$. Then for $n>1$, the ratio $n/\pi(n)$ takes on every integer value $m>1$." See S. W. Golomb, On the ratio of n to $\pi(n)$, this MONTHLY, vol. 69, 1962, pp. 36–7.

Inverse of a Triangular Matrix

E 1525 [1962, 566]. *Proposed by Sidney Heller, Brookhaven National Laboratory, Upton, New York*

Find the inverse of the lower triangular matrix of order $n+1$ whose triangle is Pascal's triangle.

I. *Solution by J. E. Yeager, Temple University.* Denote the given matrix by $A=[a_{ij}]$. Define a matrix B by $b_{ij}=(-1)^{i+j}a_{ij}$. Then a typical element of AB is

$$\sum_{k=1}^{n+1}(-1)^{k+j}a_{ik}a_{kj}=\sum_{k=1}^{n+1}(-1)^{k+j}\binom{i-1}{k-1}\binom{k-1}{j-1}=\delta_{ij}.$$

Note: If matrix A is premultiplied and postmultiplied by the matrix

$$\operatorname{diag}\{1,-1,1,-1,\cdots\},$$

the result is matrix B.

II. *Solution by Wallace Givens, Argonne National Laboratory.* The mapping $t^i\to(1+t)^i$ has as its inverse $t^i\to(-1+t)^i$ in the $(n+1)$-dimensional vector space of polynomials. Relative to the basis $1, t, t^2, \cdots, t^n$, the first mapping has as its matrix P a lower triangular matrix with nonzero elements the Pascal triangle. Hence $(P^{-1})_{ij}=(-1)^{i+j}P_{ij}$, $(i, j=0, 1, \cdots, n)$.

Also solved by Jack Abad, R. G. Albert, Raymond Balbes, Marjorie Bicknell, Robert Bowen, D. A. Breault, J. L. Brenner, Norman Brenner, P. R. Chernoff, D. I. A. Cohen, Martin Cohn, Romae J. Cormier, G. C. Dodds, N. J. Fine, David Forslund, S. H. Greene, J. C. Hickman, V. E. Hoggatt, Jr. and I. D. Ruggles (jointly), P. L. Kingston, M. S. Klamkin, Jack Latimer and Lee Thomson (jointly), T. J. Lee, Jiang Luh, D. C. B. Marsh, R. A. Melter, J. B. Muskat, R. J. Oberg, J. L. Pietenpol, Eugene Primoff, G. S. Rogers, G. A. Saatdjian, E. M. Scheuer, J. E. Schneider, H. P. Smith, R. P. Tapscott, Guy Torchinelli, W. C. Waterhouse, David Zeitlin, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College
Collaborating Editors: LEONARD CARLITZ, Duke University; H. S. M. COXETER, University of Toronto and ALBERT WILANSKY, Lehigh University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5082. *Proposed by the Junior Research Seminar for High School Students of Summer 1962, Lehigh University*

Let R be a ring in which, if either $x+x=0$ or $x+x+x=0$, it follows that $x=0$. Suppose that a, b, c and $a+b+c$ are all idempotents in R. Does it follow that $ab=0$?

5083. *Proposed by Joshua Barlaz, Rutgers, The State University*

Prove that there is no set E in the interval (0, 1) having Lebesgue measure $m(E)$ and satisfying the conditions: (1) $0 < m(E) < 1$, (2) $m(E) \cap (0, c) = r \cdot c$ for all c, $0 \leqq c \leqq 1$ and a fixed r.

5084. *Proposed by Robert Spira, University of California, Berkeley*

Find a one-to-one continuous function on the unit disk into the plane which is angle preserving, yet not analytic. (If no such function exists, then we have a simple geometric characterization of analytic functions. If such a function does exist, then from it we will obtain a clue as to a further geometric restriction necessary to characterize the idea of analytic function.)

5085. *Proposed by E. R. Gentile, Universidad del Sur, Argentina*

Let G be any abelian group and $G^2=G\oplus G$ the direct sum of two copies of G. G^2 admits in a natural way structures of left and right modules over the ring Z_2 of 2×2 matrices over the integers Z. We recall that for every abelian group H the mapping $\phi: G^2\otimes_{Z_2}H^2\rightarrow G\otimes_Z H$ defined by $\phi((g_1\oplus g_2)\otimes_{Z_2}(h_1\oplus h_2)) = (g_1+g_2)\otimes_Z(h_1+h_2)$ is a canonical isomorphism.

This permits an incorrect proof of the statement, *If G^2 is isomorphic to H^2, then G is isomorphic to H*, viz.

$$G \cong G \otimes_Z Z \cong G^2 \otimes_{Z_2} Z^2 \cong H^2 \otimes_{Z_2} Z^2 \cong H \otimes_Z Z \cong H.$$

Where is the error?

5086. *Proposed by B. H. Bissinger, Lebanon Valley College, and Conrad Siegel, F.S.A., Harrisburg, Pa.*

A different integer is written on the face of each of 1000 slips of paper and the slips are placed face down. A player (who has no knowledge of the particular integers used) turns over and reads as many of the slips as he wishes. Success occurs when the integer on the last slip turned over is the largest of the 1000 integers.

If only one slip is turned over, the chance of success is obviously .001, as it is when all slips are turned over. Devise a system for playing the game which gives maximum probability of success. Determine this probability.

5087. *Proposed by Seth Warner, Duke University.*

Given two rings each having m elements including a unity element. If m is a square-free integer, prove the rings are isomorphic.

5088. *Proposed by Joe Lipman, Queen's University, Canada*

In Meschkowski, Unsolved and Unsolvable Problems of Geometry (Vieweg & Son, 1960) a function defined on a convex set C of reals is called convex if $2f((x_1+x_2)/2) \leqq f(x_1)+f(x_2)$ everywhere in C.

a) Show that if f is bounded above on some subinterval of C, then this definition agrees with the usual one, viz: for any t in $(0, 1)$, $f(tx_1+(1-t)x_2) \leqq tf(x_1)+(1-t)f(x_2)$ everywhere in C.

b) Find a function on $(-\infty, \infty)$ which is not convex (in the usual sense) but satisfies $2f((x_1+x_2)/2)<f(x_1)+f(x_2)$ for all $x_1 \neq x_2$.

5089. *Proposed by R. D. Sinkhorn, R. R. Gordon and Reid June, Boeing Airplane Co., Wichita, Kan.*

Let a sequence of real functions f_n, $n=1, 2, \cdots$, be generated by

$$f_1(x) = \begin{cases} 1/2a & \text{if } |x| < a \\ 1/4a & \text{if } |x| = a \\ 0 & \text{if } |x| > a \end{cases}, \qquad f_{n+1}(x) = \frac{1}{2a}\int_{x-a}^{x+a} f_n(t)dt,$$

where a is a positive constant. Prove that, independent of x,

$$\lim_{n\to\infty} n^{1/2} f_n(x) = \frac{1}{a}\sqrt{\frac{3}{2\pi}} \cdot$$

5090. *Proposed by Fred Suvorov, Princeton University*

Let $\{x_n\}$ be a sequence of positive real numbers. Consider the set A of all real numbers a such that $\{x_n\}$ converges to 0 (mod a), and show that this set has measure 0. The sequence $\{x_n\}$ is said to converge to 0 (mod a) if the residue classes of the x_n on the circle R/Ra converge to 0, where R is the real numbers considered as an additive group, and Ra is the subgroup generated by a.

5091. *Proposed by W. E. Briggs, University of Colorado*

For integral $k>1$, evaluate:

$$\lim_{x\to\infty}\left[\frac{k}{\phi(k)}\sum_{\substack{n\leq x\\(n,k)=1}}\frac{1}{n}-\sum_{n\leq x}\frac{1}{n}\right].$$

SOLUTIONS

A Trigonometric Inequality

5004 [1962, 63]. *Proposed by Yoshio Matsuoka, Kagoshima-shi, Japan*

If $0\leqq\beta\leqq\alpha<\pi$, prove that

$$\frac{1-\cos\beta+2\cos\alpha}{1+\cos\beta}\leqq\cosh\sqrt{(\alpha^2-\beta^2)}\leqq\frac{1-\cos\alpha+2\cos\beta}{1+\cos\alpha}.$$

Solution by Marlow Sholander, Western Reserve University. The first inequality is immediate once we note that its left hand side is not greater than one. The second inequality is simplified by adding one to each side, dividing by two, letting $\alpha=2x$ and $\beta=2y$, applying half-angle formulas, taking positive square roots, and multiplying by $\cos x$. This leads to the equivalent inequality

$$P(x, y)\equiv\cos x\cosh\sqrt{(x^2-y^2)}\leqq\cos y$$

for $0\leqq y\leqq x\leqq\pi/2$. For $0<t=\sqrt{(x^2-y^2)}$, the partial derivative $P_1(x, y)=-\sin x\cosh t+x\cos x\sinh t/t$ is negative in consequence of the known bounds $\tanh t/t<1<\tan x/x$. Hence $P(x, y)\leqq P(y, y)=\cos y$.

Also solved by L. Carlitz, Jane Evans, Lothar Koschmieder, W. S. Lawton, G. B. Parrish, Maurice Scheier, C. D. Sutherland and W. Myint and M. J. Pascual, and the proposer.

Rank of a Matrix

5005 [1962, 63]. *Proposed by Reuben Hersh, New York University*

Suppose A is any nonsingular n by n matrix with elements a_{ij} and its inverse is B with elements b_{kl}. Then form a new matrix C in the following way: Take the ordered pairs composed of the natural numbers up to n, and in any arbitrary manner number them from 1 to n^2. Let the symbol $[i,j]$ stand for the number of the ordered pair i, j. Then the elements of C can be indexed by the bracket symbol. Now define C by $c_{[i,j],[k,l]}=a_{il}b_{jk}$. Prove that the matrix $(C-I)$ has rank $n(n-1)/2$.

Solution by Marvin Marcus, University of British Columbia. Let R be an n-dimensional vector space with a basis $v_1,\cdots,v_n$ over a field not of characteristic 2. Let R_0^2 be the space of contravariant tensors of rank 2 (see N. Jacobson, *Lectures in Abstract Algebra*, V. II, p. 217). Let α be the linear transformation on R represented by A in the above basis of R. Then the matrix $C-I$ is easily seen to represent the linear transformation $\phi=(\alpha\times\alpha^{-1})\sigma-e$ in the basis $v_i\times v_j$,

$i, j=1, \cdots, n$ of R_0^2 [*ibid.*, p. 214]. We remark that the ambiguity about the ordering is just a matter of deciding on the order of the basis $v_i \times v_j$: this is usually lexicographic. Here $\alpha \times \alpha^{-1}$ is the Kronecker product of the indicated transformations [*ibid.*, p. 211], σ is the interchange operator, $\sigma(u \times v) = v \times u$, and e is the identity on R_0^2. Since α and σ are both nonsingular on R and R_0^2 respectively it follows that the dimension of the range of ϕ is the same as that of $\beta = (i \times \alpha)\phi\sigma = \alpha \times i - (i \times \alpha)\sigma$ where i is the identity on R. Now, $\beta(u \times v) = \alpha(u) \times v - v \times \alpha(u)$ for any u and v in R. In particular, if $1 \leqq s \leqq t \leqq n$ then $\beta(\alpha^{-1}(v_s) \times v_t) = v_s \times v_t - v_t \times v_s$. It is proved in [*ibid* p. 219] that these $n(n-1)/2$ vectors form a basis for the symmetry class of skew-symmetric tensors $S^{(2)}$ in R_0^2. But we saw that the range of β is contained in $S^{(2)}$ and thus it must coincide with $S^{(2)}$. It follows immediately that the dimension of the range of ϕ in R_0^2 is also $n(n-1)/2$, the desired result.

Also solved by G. M. Bergman and by the proposer.

Plane Section of a Tetrahedron

5006 [1962, 63]. *Proposed by E. Ehrhart, Lycée Kleber, Strasbourg, France*

Is the following proposition valid: Every plane section of a tetrahedron is smaller (in area) than the largest face?

Solution by J. J. A. M. Brands and G. Laman, Technological University, Eindhoven, Netherlands. The following observations provide an affirmative answer.

I. A convex function on a closed interval takes its maximal value in one of the endpoints.

II. The sum of two convex functions is convex. In the sequel, "smaller than" will mean having the same area or less.

III. If A and B are fixed points in Euclidean 3-space and if C is on a line l then the area of the triangle ABC is a convex function of any linear parameter λ on l.

Proof. The area is proportional to the distance from C to AB and this distance is the square root of a positive semidefinite quadratic form in λ. This is a convex function.

IV. If A and B are fixed points in Euclidean 3-space and if C and D are on lines l and m respectively in such a way that (a) CD intersects AB in S, (b) S is between C and D, and (c) AB, l and m have mutually no common point; then the area of the quadrangle $ACBD$ is a convex function of any linear parameter λ on l.

Proof. The area is the sum of the areas of the triangles ABC and ABD. Any linear parameter μ determining the place of D on m is a linear function of a linear parameter λ determining the place of C on l. Thus the area of the quadrangle is the sum of two convex functions of λ and by II is itself a convex function.

V. Any plane section of the tetrahedron through one of the edges is smaller than the largest face.

Proof. By I and III this section is smaller than at least one of the faces through the same edge and so smaller than the largest face.

VI. Any plane section through one of the vertices is smaller than the largest face.

Proof. By I and III this section is smaller than either a plane section through an edge or part of a face (or both). The result now follows from V.

VII. Any plane triangular section not passing through a vertex is smaller than the largest face.

Proof. By I and III it is smaller than either a section through a vertex or part of a face (or both). The result now follows from VI.

VIII. Any plane quadrangular section is smaller than the largest face.

Proof. By I and IV it is smaller than at least one of two triangular sections each passing through an edge and so by V smaller than the largest face.

The cases V through VIII yield the desired result.

Editorial Note. The present problem is a restatement of no. E 1298 [1958, 43] by H. Grossman. No solutions of the earlier number were received.

A Known Summation

5007 [1962, 171]. *Proposed by A. V. Boyd and D. A. Higgs, University of the Witwatersrand, Johannesburg, South Africa.*

Sum:

$$\operatorname{sech} \tfrac{1}{2}\pi - \tfrac{1}{3} \operatorname{sech} \tfrac{3}{2}\pi + \tfrac{1}{5} \operatorname{sech} \tfrac{5}{2}\pi - \cdots .$$

Editorial Note. Many of our readers have called attention to the fact that this problem has appeared previously. It was proposed as no. 4191 in this Department [1947, 347–8] by H. F. Sandham and solved by H. E. Fettis using Fourier series. In an accompanying note C. D. Olds mentioned that the formula is a special case of a result due to Ramanujan, proved using contour integration by C. T. Preece, Journal London Math. Soc., 3 (1928) p. 212. In the same issue of this journal, p. 221, G. N. Watson gave another generalization. More recently Sandham has proved that, independent of a

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{\cosh(\pi a/2) \cos(\pi a(2n+1)/2)}{\cosh(\pi(2n+1)/2)} = \frac{\pi}{8} .$$

See *Some Infinite Series*, Proc. Am. Math. Soc., 5 (1954) 430–436.

Also solved by L. Carlitz, P. R. Chernoff, P. J. de Doelder and G. W. Veltkamp, W. H. Gage, Joseph Hammer, B. K. Harrison, R. P. Kelisky, D. J. Newman, John Raleigh, Leon Steinberg, J. W. Wrench, Jr., and the proposer.

Words of Finite Length in 3 Letters

5008 [1962, 171]. *Proposed by Hans Zassenhaus, University of Notre Dame*

1. The words $W(A, B, C)$ in 3 letters A, B, C and of finite length form a semigroup with the empty word as identity element if juxtaposition is used as

the rule of multiplication: it remains a semigroup S if every word $W_1W_2W_2W_3$ with repetition of a nonempty subword W_2 is replaced by W_1W_3.

Show that the mapping $W(A, B, C) \to W(ABCBA, BCACB, CABAC)$ is an isomorphism of this semigroup onto a proper subsemigroup.

2. Find a word of shortest length in S that is not a proper subword of any other word in S.

Solution by Barbara L. Osofsky, Douglass College. The set W does not remain a semigroup when every word $W_1W_2W_2W_3$ is replaced by W_1W_3 (nor when replaced by $W_1W_2W_3$). For there is not a unique word to replace $ABCBABCBC$. If we group $(ABCB)(ABCB)(C)$ or $(ABCBA)(BC)(BC)$ we get two different words. So associativity is lacking.

As S is not a semigroup we prove only that

$$\phi: W(A, B, C) \to W(ABCBA, BCACB, CABAC)$$

is an isomorphism from W onto a proper subsemigroup of W. ϕ is defined to preserve multiplication, $A \notin W(ABCBA, BCACB, CABAC)$, and the inverse image of a word is the word formed from the letters in positions congruent to one modulo five.

2. A word of shortest length with no repetitions contained in no longer word with no repetitions is $ABACABA$. Pre- or post-multiplication by any letter creates a repetition. Any word with no repetitions must start with two distinct letters, say AB. If it starts ABC and is contained in no longer word without repetition, the block $BABC$ must appear in the word and so it contains at least seven letters. If it starts ABA, the block $CABA$ must appear and we get a minimal length word $ABACABA$.

Also solved by John Hennessey, and by Vladeta Vuckovic.

Continuously Differentiable Functions

5009 [1962, 171]. *Proposed by Albert Wilansky, Lehigh University*

Let f_n be a sequence of continuously differentiable functions on $[0, 1]$ such that $f_n \to 0$ uniformly but f_n' does not tend uniformly to 0. Show that there exists a function g which is not continuously differentiable on $[0, 1]$ and which is uniformly approximable by a finite linear combination of the f_n. (As an example, $f_n(x) = (\sin n\pi x)/n$. In this case g may be any continuous function.)

Solution by H. A. Gindler, San Diego State College. Let $C[0, 1]$ be the space of continuous functions on $[0, 1]$ with the sup norm, let X be the closed linear span of the functions f_n, and suppose that every f in X is continuously differentiable contrary to what is to be proved. Then the differentiation operator D is a closed linear transformation defined on all of the Banach space X. By the closed graph theorem D is continuous on X. Since $f_n \to 0$ uniformly we conclude that $Df_n = f_n' \to 0$ uniformly, which is contrary to hypothesis.

Also solved by James E. Potter.

Related Permutation Matrices

5010 [1962, 171]. *Proposed by Hans Schneider, University of Wisconsin*

For a permutation σ of $(1, \cdots, m)$ let $s(k)$ be the number of cycles of length k in a factorization of σ into cyclic permutations, and set $s^*(k)=\sum_{k|d} s(d)$. Let P_σ be the permutation matrix associated with σ: $P_\sigma(i, j)=1$ or 0 according as $\sigma(i)=j$ or $\sigma(i)\neq j$. For a permutation τ of $(1, \cdots, n)$ define $t(k)$, $t^*(k)$ and P_τ similarly. Show that the m by n matrix (with complex elements) of maximum rank r satisfying $P_\sigma A=AP_\tau$ has $r=\sum_k \phi(k)w^*(k)$, where ϕ is Euler's function and $w^*(k)=\min\{s^*(k), t^*(k)\}$. If $m=n$, deduce that there exists a nonsingular A if and only if $s(k)=t(k)$ for all k, and that, in this case, there exists a permutation matrix Π such that $A\Pi$ commutes with P_σ if and only if $P_\sigma A=AP_\tau$. (These results include Lemmas 1 and 2 of R. Brauer, *Connection between ordinary and modular characters of groups of finite order*, Ann. Math., 42 (1941) 926–935.)

Solution by the proposer. Since P_σ and P_τ are diagonal matrices, there exists a matrix A of rank r satisfying $P_\sigma A=AP_\tau$ if and only if P_σ and P_τ have r common characteristic roots. But the family of characteristic roots of P_σ contains the kth roots of unity exactly $s(k)$ times, and hence the $\phi(k)$ primitive kth roots of unity occur exactly $\sum_{k|d} s(d)=s^*(k)$ times; and similarly for P_τ. It follows that the number of common roots of P_σ and P_τ is $\sum_k \phi(k)w^*(k)$, and the first part of the result follows.

If $m=n$, then we may deduce from $n=\sum_k \phi(k)s^*(k)=\sum_k \phi(k)t^*(k)$ that $r=n$ if and only if $s^*(k)=t^*(k)$ for all k, and this condition is equivalent to $s(k)=t(k)$ for all k. It is well known that this implies that σ and τ are conjugate permutations: $\zeta\sigma\zeta^{-1}=\tau$ for some permutation ζ. Let $\Pi=P_\zeta$, so that $P_\tau=\Pi P_\sigma\Pi^{-1}$. Hence $P_\sigma A=AP_\tau$ if and only if $P_\sigma A=A\Pi P_\sigma\Pi^{-1}$ and the last result follows immediately.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Management and the Computer of the Future. Edited by Martin Greenberger. The M.I.T. Press, Cambridge, Mass., 1962. xxvi+340 pp. $6.00.

A series of 8 lectures and supplementary discussions given at M.I.T., addressed primarily to engineering and research management on a variety of computer topics. In general, the lectures are speculative and anecdotal.

HARLAN D. MILLS
Radio Corporation of America

Elementary Theory of Numbers. By William J. LeVeque. Addison-Wesley, Reading, Mass., 1962. vii+132 pp. $5.00.

The theory of numbers can be presented to students at virtually any level from high school to graduate school. The more advanced the level, the more advanced the mathematical techniques that can be applied to the subject. The present book is aimed at the freshman-sophomore level in college, and so is at a lower level of sophistication than Professor LeVeque's other books on number theory. Moreover, less is expected of the reader by way of mathematical maturity; for example there are explanatory discussions of mathematical induction and indirect proof that are usually omitted in more advanced books on number theory.

To give some idea of the contents we list the principal topics (the chapter headings and a little more): the Euclidean algorithm, g.c.d. and l.c.m., unique factorization, the linear Diophantine equation, congruences, residue classes, Euler's ϕ function; linear and polynomial congruences, powers of an integer modulo m, exponents and indices; continued fraction expansions of rational and irrational numbers with attention to quadratic irrationalities, approximation theorems; Gaussian integers, g.c.d., unique factorization, Gaussian primes; Diophantine equations, $x^2+y^2=z^2$, $x^4+y^4=z^4$, $x^2-dy^2=1$, $x^2-dy^2=-1$. There are plenty of problems for the student, some numerical, some theoretical. Short tables of primes and primitive roots are given for convenience.

The book provides a sound interesting approach to the theory of numbers. It is a scholarly book in that there are many observations and comments on the outer setting of the subject, to complement the careful inner analysis. A wide variety of proof techniques is presented, thus guaranteeing that the student will gain in mathematical sophistication as he proceeds.

The following special features are noteworthy: a very well-formulated seven page introduction to the whole subject; the analysis of Pell's equation by means of quadratic fields; a development showing the relation of Pell's equation to the general binary quadratic equation; an entire chapter on Gaussian integers; clever and ingenious use of notation throughout. Although it is a short volume, a great deal of material is encompassed.

The book is handsome in both mathematics and typography. As a minor exception, however, the definitions and theorems are not in italics, although they are set off by spacing. This is a deviation from the customary use of italics, as for example in LeVeque's other books by the same publisher. But this is a small matter, detracting very little from an excellent introduction to the theory of numbers.

IVAN NIVEN
University of Oregon

Elements of Abstract Algebra. By John T. Moore. Macmillan, New York, 1962. xiii+203 pp. \$6.50.

This book is designed as a text for an introductory course of a semester's duration at the junior-senior level. The order of development is logically sequential, with a discussion of sets appearing first, followed by a treatment of systems with one binary composition, and then a study of systems with two binary compositions. The principal systems covered are the integers and rational numbers, groups, rings, vector spaces, and lattices. The author does not claim originality in treatment, but he has made a selection of material that he believes to be satisfactory for a beginning one-semester course. The book is not intended to be primarily on linear algebra and matrix theory, and the treatment of these topics and some others is quite skimpy.

The reader might wonder at some of the inclusions and omissions in a text at this level. For example, a section on Noetherian rings is included, but nowhere in the book could the reviewer find mention of such an elementary concept as the characteristic of a ring.

The book is marred by what seemed to the reviewer to be an unusually large number of errors, many of which are not merely typographical. Most of these are of the type, however, that can be corrected easily by the teacher.

Those teachers who agree with the author in his choice of subject matter for an introductory course in abstract algebra might wish to consider this book for use as a text.

P. W. CARRUTH
Swarthmore College

Lectures on Game Theory. By E. S. Ventzel. Hindustan Publishing Corp., Delhi, India, 1961. 78 pp. \$4.50. Distributed by Gordon and Breach, New York.

This is a translation of a monograph on game theory written by the Russian mathematician E. S. Ventzel. The treatment is restricted to zero sum games and includes a discussion of: the minimax principle, elementary methods of solutions for 2×2 and $2\times n$ games, general methods for finite games, approximate methods for solution of certain infinite games. There are essentially no proofs and only a knowledge of elementary analysis and probability theory is required for reading this monograph. Examples occur throughout the discussion but in general are the same old examples: matching pennies, bombers flying high or low, and the like.

There is of course no shortage of elementary treatments of game theory in this country. This treatment is distinguished, however, by the clarity of the exposition and the fact that it provides a quite complete introduction to game theory in a very short monograph.

J. LAURIE SNELL
Dartmouth College

A Course in the Geometry of n Dimensions. By M. G. Kendall. Hafner, New York, 1961. viii+63 pp. $3.00.

This little book is based on a course given by Prof. Kendall to those students of statistics who need to know something of the geometry of n dimensions. It is hoped that it may also be useful to students of other disciplines. The first part, pp. 1–42, deals with the geometry of n dimensions, and the last 21 pages with applications to statistics.

A fair knowledge of matrix theory is assumed. The lectures themselves were illustrated, we are told, by diagrams of the two and three-dimensional cases, but none appears in the book. Some of the proofs are very short, and it is difficult to imagine the average student obtaining from this book the confidence to deal with situations in n dimensions which he could obtain from any good text-book on algebra. But the author does say that his book is not intended to replace Sommerville's *Introduction to the Geometry of n Dimensions.*

The lectures themselves were probably very successful, but lecture notes are not the same thing. On the other hand, the applications to statistics given in the second part of the book do show what can be done by the use of the geometry of n dimensions, and Prof. Kendall is, of course, an authority in this field.

There are a small number of misprints which would confuse some readers. On pp. 23, 24 and 25 some bold-face symbols suddenly become italic, and other italic symbols become bold-face; the bold-face symbols represent matrices, and the italic symbols represent scalars.

D. PEDOE
Purdue University

Introduction to Matrices and Vectors. By Jacob T. Schwartz. McGraw-Hill, New York, 1961. x+163 pp. $5.50.

What mathematics course to give in the last semester of secondary school is often the question; elementary matrix algebra is a plausible answer and the book under review seems to be a most suitable text for the purpose. First, the author appears to be condescending, a questionable attitude to take when one is attempting to instruct an intelligent boy of seventeen, the typical enrollee in such a course; but by the second chapter, on multiplication of matrices, and for the rest of the book, the tenor is clear, pleasant and decently adult. Could it be that, in the first chapter, the author has some doubts about the secondary school teacher of mathematics, the teacher who is so frequently the real sufferer when faced with presenting a nontraditional course? Once the author starts moving, much interesting and useful information begins to appear, although he occasionally stops short of a full proof. Polynomials in matrices, vectors as special matrices, quaternions, skew matrices and vectors in 3-space, the cap product, eigenvalues and series of matrices are some of the topics. The text is also suitable for a one quarter course at the junior college level. Misprints are few and minor.

FRANKLIN HAIMO
Washington University

Spectral Theory. By E. R. Lorch. Oxford University Press, New York, 1962. 158 pp. $5.50.

This book is a translation of lectures delivered by the author in 1953–54 at the University of Rome where he was a Fulbright Visiting Professor. It is a model of economy and clarity. The reader is first guided through the essentials of Banach space lore, linear transformation theory and the specialization to Hilbert spaces. Then to provide an approach, different from that of most texts, to the spectral theorem, the author treats the integration of vector- (and operator-) valued functions over the complex plane.

For a self-adjoint transformation A, a decomposition $A=A_1+A_2$ is introduced so that (a) A_i are restrictions of A to spanning orthogonal closed spaces $\mathbf{M}_i$; (b) A_1 has a pure point spectrum on $\mathbf{M}_1$, A_2 has a pure continuous spectrum on $\mathbf{M}_2$. The spectral decomposition of A_2 is then the hinge of the whole development. Hence A is regarded as A_2, $\mathbf{M}_2$ as Hilbert space $\mathbf{H}$. The particular form given to the spectral theorem is: For any sequence $\lambda_n<\lambda_{n+1}$, $n=0, \pm 1, \pm 2$, where $\lambda_n \to \pm\infty$ as $n\to\pm\infty$, there is a sequence of mutually orthogonal, closed linear manifolds $\mathbf{M}_n$, spanning $\mathbf{H}$, such that $\mathfrak{D}_A \subset \bigcup_n \mathbf{M}_n$, $A(\mathbf{M}_n)\subset\mathbf{M}_n$ and $\lambda_n I \leqq A \leqq \lambda_{n+1} I$ in $\mathbf{M}_n$.

The result is achieved via the integrals

$$K_{\lambda\mu}(m, n) = (2\pi i)^{-1}\int_C (\zeta-\lambda)^m(\mu-\zeta)^n R_\zeta d\zeta, \qquad \lambda<\mu,$$

where C cuts the x-axis at λ and μ, and at nonzero angles there, $R_\zeta=(A-\zeta I)^{-1}$. Furthermore C is required to be symmetric about the x-axis and smooth. These $K_{\lambda\mu}(m, n)$ satisfy certain easily verified orthogonality and addition relations and they commute with A. The desiderata then flow smoothly from these considerations.

Those accustomed to different versions of the spectral theorem and its proof will find this treatment refreshing and replete with insights not to be found in other approaches.

A concluding chapter on commutative Banach algebras whets the reader's appetite for more information. The work here is a clean exposition of Gelfand's classical papers in normed ring theory. The earlier results on nonempty spectra yield a quick proof of the Gelfand-Mazur theorem on normed fields.

The book is highly recommended as an introduction to an interesting field, although detailed and far-reaching ramifications are untouched and a great deal of motivation is ignored—no doubt to save space. The author's use of language, choice of emphasis, organization of material and the selection of fringe benefits, (e.g., the mean ergodic theorem) should find a warm reception in a considerable mathematical audience.

B. R. GELBAUM
University of Minnesota

Combinatorial Chance. By F. N. David and D. E. Barton. Hafner, New York, 1962. ix+356 pp. $10.25.

In the preface the authors indicate that they have written this book because of their interest in combinatorial chance for its own sake and not solely because of its increasing application in the field of mathematical statistics. The subject matter is a compilation of various combinatorial methods which, from time to time, have appeared in the literature; in some cases the methods have been extended by the authors.

After the usual treatment of probability-generating and moment- and cumulant-generating functions, the authors extend the discussion to factorial moments and cumulants and their use in the development of converse theorems and limiting processes for finite sets. Simple and complex matching problems and the theory of runs are discussed, starting with elementary distribution theory and ending with considerations of periodic or trend effects. Algebraic identities, necessary to convert from one set of moment coefficients to another or from moments to cumulants, are presented, and certain possible ranking criteria are summarized and their properties investigated. The authors review available techniques to aid in the computation of combinatorial extreme-value distributions. The problems discussed are in general those which do not yield explicit probability distribution functions or for which these functions and the higher moments are difficult to calculate. The detailed treatment of occupancy represents all available research information on the subject. Generalized Bernoulli numbers, approximations to differences of zero, symmetric functions and polykays are presented from the point of view of their usefulness in developing other combinatorial theory. The last chapter contains a brief discussion of possible departures from randomness and the resultant effects.

The book is clearly written, and the development of theory is carefully illustrated by many well-selected examples. A glossary of notation and a list of abbreviations are helpful. Although the subject matter is too advanced and not appropriate for the average undergraduate course, the book should prove useful to professional statisticians as a reference and to advanced graduate students as a source of information with which to broaden their knowledge of combinatorial background material and procedures.

EDWARD B. ROESSLER
University of California, Davis

BRIEF MENTION

Structural Patterns and Proportions in Vergil's Aeneid. By G. E. Duckworth. University of Michigan Press, 1962. x+268 pp., $7.50.

The author, who is Giger professor of Latin at Princeton University, gives an account of the painstaking research which has led him to the conclusion that Vergil, as well as other poets of his century, made deliberate and extensive use of the Golden Mean ratio in the structure of his poetry.

Tables of Sines and Cosines to Ten Decimal Places at Thousandths of a Degree. By Herbert E. Salzer and Norman Levine. Pergamon Press, New York, 1962. xiv+tables. $10.00.

A Physical Theory of the Living State. By Gilbert N. Ling, Blaisdell Publ. Co., New York, 1962. ix+680 pp.

The subtitle is: The Association-Induction Hypothesis with Considerations of the Mechanisms Involved in Ionic Specificity, Behavior of Proteins, Selective Accumulation of Ions and Nonelectrolytes, Cellular Electrical Potentials, Ionic Permeability and Diffusion, Excitation and Inhibition, Contractile Mechanism, Enzyme Action, Drug and Hormone Action, Antibody-Antigen Reaction, Fertilization, Chemical Embryology, Growth, Differentiation, and Cancer.

Classics in Logic. Edited by D. D. Runes. Philosophical Library, New York, 1962. xiv+818 pp. $10.00.

Sixty-three selections in Epistemology, Theory of Knowledge and Dialectics from logicians and epistemologists of many schools and epochs.

The Crescent Dictionary of Mathematics. By William Karush. Macmillan, New York, 1962. x+313 pp. $6.50.

An easy-to-understand reference tool for anybody in home or school who is studying mathematics or who wants a practical refresher in the field.

Summation of Infinitesimal Quantities. By I. Natanson. Gordon and Breach, New York, 1962. 66 pp. $4.50.

This 66-page translation of a Russian booklet published in 1953 is labeled *Vol. IX of Russian Texts in Advanced Mathematics and Physics*, although it assumes only elementary algebra and geometry and a little trigonometry. Chapter I consists of derivations of the formulas for $\sum_{k=1}^{n} k^p$, $p=1, 2, 3$. The rest of the book is made up of solutions of standard problems in water pressure, work, areas, and volumes. The method used is, in the author's (or translator's) words, "to bring the quantity to be evaluated in the form f the limit of a sum of indefinitely increasing-number of ultimately vanishing terms, or, as it is more commonly stated, in the form of indefinitely large number of infinitesimal terms." *Limit* is nowhere defined, and its properties are taken as completely evident intuitively.

The English is awkward, the spelling and punctuation are erratic, and the paper is atrocious. But the price, c'est magnifique!

Programming the IBM 1401: A Self-Instructional Programmed Manual. By J. A. Saxon and W. S. Plette. Prentice-Hall, Englewood Cliffs, N. J., 1962. xv+208 pp. $6.75.

This manual is concerned primarily with business applications of the 1401, with some attention given to scientific and mathematical uses. Self-scored tests enable the student to check his progress.

Famous Problems. By F. Klein. Chelsea Publ. Co., New York, 1962. vi+317 pp. $1.95 (paper).

This work contains reprints of F. Klein: *Famous Problems of Elementary Geometry*, W. F. Sheppard: *From Determinants to Tensor*, P. A. Macmahon: *Introduction to Combinatory Analysis*, and L. J. Mordell: *Fermat's Last Theorem.* The reason for including these four classics in one volume is purely economic: printed separately each would cost almost as much as this book.

A Table of Indices and Power Residues for all Primes and Prime Powers Below 2000. The University of Oklahoma Mathematical Tables Project, under the direction of Richard V. Andree, with an Introduction by H. S. Vandiver. W. W. Norton & Co., New York, 1962. $10.00.

An Index to Mathematical Tables. Fletcher-Miller-Rosenhead-Comrie. Addison-Wesley, Reading, Mass., 1962. xi+994 pp. 2 vol. set $42.00.

Greatly expanded second edition of a work which first appeared in 1946 and has long been out of print.

Bibliography of Nonparametric Statistics. By R. I. Savage. Harvard University Press, 1962. 284 pp.

This volume continues through April, 1961, the author's bibliography of 1953.

Mathematical Manual. By Frederick S. Merritt. McGraw-Hill, New York, 1962. xxi+378 pp. $9.50.

The range of this volume is from simple arithmetic through most of the topics of undergraduate college mathematics. Definitions, theorems, formulas and methods are given with illustrative examples and short explanations but without derivations and proofs.

Tables of the Mathematical Functions, Vol. III. By Harold T. Davis and Vera J. Fisher. Principia Press of Trinity University, San Antonio, Texas, 1962. ix+554 pp. $8.75.

Volumes I and II of this series were published in 1933 and 1935 respectively. This volume begins with an extensive introduction which discusses the arithmetical functions, the history of certain basic constants, and methods of solution of equations, both algebraic and transcendental. There are tables of special constants (some to many decimals), of reciprocals, of fractions, of certain roots and powers, and of binomial coefficients, mostly to ten decimal places (or significant figures).

Mathematics for Business and Economics. By Robert Cissell and Thomas J. Bruggeman. Houghton Mifflin, Boston, Mass. 1962. ix+229 pp. $4.75.

Sets, Relations, and Functions. By James F. Gray. Holt, Rinehart, and Winston, New York, 1962. ix+143 pp. $2.50.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dean Mina S. Rees, City University of New York, was awarded the honorary degree of Doctor of Science at Mount Holyoke College on November 8, 1962.

Professor C. T. Salkind, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Dr. Ralph G. Hoxie as President of C. W. Post College of Long Island University and the Academic Convocation honoring Sir Muhammad Zafrulla Khan, President of the General Assembly of the United Nations, on December 16, 1962.

Professor R. C. Yates, University of South Florida, represented the Association at the inauguration of Dr. William H. Kadel as First President of Florida Presbyterian College on January 18, 1963.

Missouri School of Mines: Dr. C. E. Antle, Oklahoma State University, has been appointed Associate Professor; Assistant Professor R. E. Oeffner has been promoted to Associate Professor.

New York University: Professor Marek Fisz, Columbia University, has been appointed Professor; Professor H. A. Rademacher, University of Pennsylvania, has been appointed Visiting Professor; Dr. Harold Weitzner has been promoted to Assistant Professor.

Pennsylvania State University: Assistant Professors W. L. Harkness, J. E. Kist, D. G. Johnson, W. J. Pervin, and Marian Pour-El have been promoted to Associate Professors; Associate Professor Marian Pour-El is on leave at the Institute for Advanced Study.

St. Mary's College: Brother U. Alfred, San Francisco District of Christian Brothers, has been appointed Head of the Science Department; Brother V. Dominic has been promoted to Professor.

Winthrop College: Captain T. B. Haley, USN Ret., Duke University, and Mr. B. G. Hodges, University of Oklahoma, have been appointed Assistant Professors.

Assistant Professor L. W. Anderson, University of Georgia, has been appointed Associate Professor at Pennsylvania State University.

Assistant Professor W. H. Badgley, Jr., Defiance College, has been appointed Assistant Professor at Tennessee Polytechnic Institute.

Professor Fannie W. Boyce, Wheaton College, retired in 1962 with the title of Professor Emerita.

Professor C. M. Braden, Macalester College, has been appointed Acting Chairman of the Department of Mathematics.

Mr. R. B. Brian, Johns Hopkins Applied Physics Laboratory, Silver Spring, Maryland, has been appointed Assistant Professor at San Jose State College.

Mr. G. S. Cunningham, New Hampshire State Department of Education, has been appointed Assistant Professor at the University of Maine.

Professor D. J. Dessart, State College of Education, Oneonta, New York, has been appointed Associate Professor at the University of Tennessee.

Mr. R. L. Duncan, Pennsylvania State University, has been appointed Associate Professor at Lock Haven State College.

Dr. A. L. Duquette, Jet Propulsion Laboratory, California Institute of Technology, has accepted a position with the ITT Federal Laboratories, San Fernando, California.

Dr. R. M. Durstine, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, has accepted a position as a member of the Technical Staff of the Mitre Corporation, Bedford, Massachusetts.

Mr. W. G. Firth, University of Washington, has accepted a position as Associate Research Engineer with the Jet Propulsion Laboratory, California Institute of Technology.

Assistant Professor Elinor B. Flagg, Illinois State Normal University, retired September 1962.

Mr. D. L. Flanagan, North Carolina State College, has accepted a position as Research Engineer with the Boeing Company, Seattle, Washington.

Dr. H. D. Friedman, Arcon Corporation, Lexington, Massachusetts, has accepted a position in the Applied Research Laboratory of the Sylvania Electronic Systems, Waltham, Massachusetts.

Mr. R. T. Garcia, Aeronutronic, Newport Beach, California, has accepted a position as Senior Programmer with Raytheon, Wayland, Massachusetts.

Dr. L. L. Gavurin, Brooklyn College, has been promoted to Assistant Professor.

Mr. R. R. Gero, Sperry Gyroscope Company, Great Neck, Long Island, has accepted a position as Research Specialist with the Military Development Department of the National Cash Register Company, Dayton, Ohio.

Mr. J. X. Goldschmidt, Michigan State University, has accepted a position as Scientific Analyst with the Operations Evaluation Group, Arlington, Virginia.

Professor Emeritus F. L. Griffin, Reed College, has been appointed Professor at Portland State College.

Dr. D. J. Hansen, University of Texas has been appointed Assistant Professor at North Carolina State College.

Mr. E. N. Howell, Suffolk County Community College, has been appointed Associate Professor at State University College at Oneonta, Oneonta, New York.

Associate Professor S. P. Hughart, Sacramento State College, has been promoted to Professor.

Mr. D. E. Hunter, Harvey Mudd College, has accepted a position as Mathematician with the Naval Ordnance Test Station, Pasadena, California.

Mr. C. R. Janson, Geneva Area City Schools, Geneva, Ohio, has accepted a position as Mathematician with Republic Steel Research Center, Independence, Ohio.

Dr. E. C. Johnsen, Ohio State University, has received a National Research Council Research Associateship at the National Bureau of Standards, Washington, D.C., for the year 1962–63.

Miss Beverly W. Koeppel, Seismograph Service Corporation, Tulsa, Oklahoma, has accepted a position as Instrumentation Planning Engineer with Pan American World Airways, Patrick Air Force Base, Florida.

Assistant Professor Horace Komm, Rensselaer Polytechnic Institute, has been appointed Associate Professor at Howard University.

Mr. N. J. Landry, Pentucket Regional High School, West Newbury, Massachusetts, has been appointed Assistant Professor at Northern Essex Community College.

Dr. B. R. Lane, Vanderbilt University, has been appointed Assistant Professor at Colorado State College.

Professor Sherman Lowell, Adelphi College, has been appointed Professor at Washington State University.

Mr. J. J. McMahon, Chrysler Missile, Detroit, Michigan, has accepted a position as Senior Engineer-Research with North American Aviation, Downey, California.

Dr. James Michelow, University of Washington, has accepted a position as Research Mathematician with California Research Corporation, Richmond, California.

Mr. D. E. Morrill, International Business Machines, Poughkeepsie, New York, has accepted a position as Chief of the Information Processing Center at the Woods Hole Oceanographic Institution, Woods Hole, Massachusetts.

Assistant Professor Erich Nussbaum, State University of New York at Albany has been promoted to Associate Professor.

Associate Professor R. H. Owens, University of New Hampshire, has been appointed Associate Director of the Mathematical Sciences Program of the National Science Foundation, Washington, D.C.

Mr. D. R. Pagano, Slippery Rock State College, has been appointed Assistant Professor at Clarion State College, Pennsylvania.

Mr. G. A. Reilly, Data Processing Center, San Antonio, Texas, has accepted a position as Mathematician in the Control Systems Department of Westinghouse Research and Development Laboratories, Pittsburgh, Pennsylvania.

Professor Irving Reiner, University of Illinois, has been awarded a Guggenheim Fellowship and is on sabbatical leave in Paris, France.

Assistant Professor N. J. Rothman, University of Rochester, has been appointed Assistant Professsor at the University of Illinois.

Mr. V. J. Ryba, Bell Aircraft Corporation, Niagara Falls, New York, has accepted a position as Senior Scientist with the Avco Corporation, Wilmington, Massachusetts.

Mr. H. B. Secrist, Jr., Martin Company, Denver, Colorado, has accepted a position as Senior Research Engineer at the Space and Information Systems Division of North American Aviation, Downey, California.

Professor A. K. Waltz, College of Steubenville, retired June 1962 with the title of Professor Emeritus.

Mr. Chung-Lie Wang, Rutgers, the State University, has been appointed Professor at Nova Scotia Teachers College in Truro.

Assistant Professor F. W. Weiler, College of William and Mary, has been promoted to Associate Professor.

Assistant Professor Vernon Williams, Southern University, has been promoted to Associate Professor.

Mr. J. W. Wyman, University of Kansas, has been appointed Assistant Professor at Pasadena College.

Mr. G. J. Young, Nuclear Development Corporation of America, White Plains, New York, has accepted a position as Assistant Director with the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Associate Professor A. D. Ziebur, Harpur College, has been promoted to Professor.

Assistant Professor Aaron Bakst, New York University, died on October 18, 1962. He was a member of the Association for 24 years.

Dr. R. W. Barnard, Ann Arbor, Michigan, died on March 24, 1962. He was a member of the Association for 41 years.

Mr. Luther Dawes, Jarman Junior High School, Midwest City, Oklahoma, died on October 21, 1962. He was a member of the Association for 6 years.

Mrs. Louise F. Hutchinson, Baldwin School, Bryn Mawr, Pennsylvania, died on January 29, 1962. She was a member of the Association for 6 years.

Professor F. J. Taylor, College of St. Thomas, died on November 3, 1962. He was a member of the Association for 42 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NOVEMBER MEETING OF THE NEW JERSEY SECTION

The seventh annual meeting of the New Jersey Section of the Mathematical Association of America was held at Rutgers—The State University, New Brunswick, New Jersey, on Saturday, November, 3, 1962. Dr. G. Y. Cherlin, Chairman of the Section, presided at the morning session, and Professor F. A. Varrichio presided at the afternoon session. There were 84 persons present including 57 members of the association.

The following officers were elected: Dr. S. S. Myers, Educational Testing Service, Princeton, Chairman, Dr. R. K. Brown, Signal Corps Research and Development Laboratory, Fort Monmouth, Member at large of the Executive Committee, '65, Professor F. A. Varrichio, Saint Peter's College, Secretary-Treasurer, '65, Professor J. K. Reckzeh, Jersey City State College, Associate Secretary-Treasurer, '65.

Dr. H. O. Pollack, Section Governor, reported on the August meeting of the Association at Vancouver, B. C. Reports were also given by Professor J. K. Reckzeh, Chairman of the Speaker's Bureau, and Professor I. L. Battin, retiring Secretary-Treasurer.

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LOCATION OF THE ZEROS OF INFRAPOLYNOMIALS

MORRIS MARDEN, University of Wisconsin—Milwaukee

1. Introduction. The subject of infrapolynomials originated in 1922 with a paper by Fejér, followed by a joint paper by Fekete and von Neumann. For 29 years thereafter, very little appeared on this topic. In 1951, however, Fekete published a paper in which he gave a fundamental representation for infrapolynomials. This paper was followed by a number of other papers by Fekete, Walsh, Motzkin, Nagy, Zedek, Shisha and myself. The purpose of the present paper is to survey just one aspect, namely the location of the zeros of infrapolynomials, with special reference to the results derivable from Fekete's representation.

Before discussing the matter further, let us define what is meant by an infrapolynomial. Let us denote by P_n: $\{z^n+a_1z^{n-1}+\cdots+a_n\}$ the class of all nth degree polynomials with leading coefficient one and by E a closed bounded set of points in the complex plane. If $p\in P_n$ and $q\in P_n$, we say that q is an underpolynomial of p on E [abbreviated $q\in U(p, E)$] if

$$(1.1)\qquad \begin{cases} |q(z)| < |p(z)| & \text{for } z\in E,\ p(z)\neq 0, \\ q(z)=0 & \text{for } z\in E,\ p(z)=0. \end{cases}$$

If, however, p has no underpolynomial on E [i.e., $U(p, E)=\emptyset$], then we say that p is an *infrapolynomial* on E [abbreviated $p\in I(E)$ or $I_n(E)$].

Among the best known infrapolynomials are those which minimize certain given norms. An example is the Tchebycheff polynomial $T_n(z)$ defined by the relation

$$(1.2)\qquad \max_{z\in E} |T_n(z)| = \min_{q\in P_n} \max_{z\in E} |q(z)|.$$

For instance, when E: $-1\leqq x\leqq 1$,

$$T_n(z) = 2^{-n}\{[z+(z^2-1)^{1/2}]^n + [z-(z^2-1)^{1/2}]^n\} = 2^{-n}\cos[n \operatorname{arc}\cos z].$$

Another example is the polynomial $B_n(z)$ minimizing the Bessel norm which may be defined, when E is a rectifiable arc, by the relation

$$(1.3)\qquad \int_E |B_n(z)|^k ds = \min_{q\in P_n} \int_E |q(z)|^k ds.$$

For instance, if $k=2$ and E: $-1\leqq x\leqq 1$, then B_n is essentially the nth degree Legendre polynomial. In fact, if we introduce suitable weight functions into the integrals (1.3), we obtain for $k=2$ the other classical orthogonal polynomials. More generally, any $p\in P_n$ is an infrapolynomial on E if it minimizes a "monotonically increasing norm" $\|q(z)\|$; i.e., a norm with the property

$$(1.4)\qquad \|q(z)\| < \|p(z)\| \quad \text{if} \quad q\in U(p, E),\ q(z)\not\equiv p(z).$$

Thus these extremal polynomials form a subclass of $I(E)$ and since their existence is well known, $I(E)$ is never an empty class.

2. Construction and representation. We shall consider next how we may construct and represent the infrapolynomials associated with a given closed bounded pointset E.

The simplest case is that in which $E=(z_1, z_2, \cdots, z_k)$, $1 \leqq k \leqq n$. All polynomials $p(z)=(z-z_1)(z-z_2) \cdots (z-z_k)\phi(z)$, $\phi \in P_{n-k}$, are clearly infrapolynomials since here $p(z)=0$ for all $z \in E$. As this case is trivial, we shall assume hereafter that E contains at least $n+1$ points. We have then the following:

THEOREM 2.1. FEKETE [2]. *Let $(z_0, z_1, \cdots, z_n)$ be any subset of $n+1$ distinct points in E and let λ_j be any positive constants such that $\lambda_0+\lambda_1+\cdots+\lambda_n=1$. Then $p \in I(E)$ if*

$$(2.1) \qquad p(z) = \omega(z) \sum_{j=0}^{n} \frac{\lambda_j}{z - z_j}, \qquad \omega(z) = \prod_{j=0}^{n} (z - z_j).$$

Proof. Let us suppose on the contrary that $p \notin I(E)$. Then there exists $q \in U(p, E)$. We expand p and q according to the Lagrange Interpolation Formula

$$(2.2) \qquad p(z) = \omega(z) \sum_{j=0}^{n} \frac{p(z_j)}{\omega'(z_j)(z - z_j)}, \qquad q(z) = \omega(z) \sum_{j=0}^{n} \frac{q(z_j)}{\omega'(z_j)(z - z_j)}.$$

Since $q \in P_n$, the leading coefficient is one; thus,

$$(2.3) \qquad \sum q(z_j)/\omega'(z_j) = 1.$$

But, since $q \in U(p, E)$, comparison of $p(z)$ in (2.1) and (2.2) leads to

$$\sum_{j=0}^{n} \left| \frac{q(z_j)}{\omega'(z_j)} \right| < \sum_{j=0}^{n} \left| \frac{p(z_j)}{\omega'(z_j)} \right| = \sum_{j=0}^{n} \lambda_j = 1,$$

which contradicts (2.3). Hence $U(p, E)=0$ and $p \in I(E)$, as was to be proved.

In certain cases, Theorem (2.1) has a converse which we may state as follows.

THEOREM 2.2. FEKETE [2]. *Let E be a closed bounded pointset containing at least $N+1$ points. Let $p \in I(E)$ such that $p(z) \neq 0$ for $z \in E$. Then there exist an integer m with $n \leqq m \leqq 2n$, a set of positive constants λ_j with $\lambda_0+\lambda_1+\cdots+\lambda_m=1$ and, if $m \leqq N$, a set of $m+1$ points $(z_0, z_1, \cdots, z_m) \subset E$ such that $p(z)$ is a factor of the polynomial $F(z)$:*

$$(2.4) \qquad F(z) = \Omega(z) \sum_{j=0}^{m} \frac{\lambda_j}{z - z_j}, \qquad \Omega(z) = \prod_{j=0}^{m} (z - z_j).$$

If E consists only of points on a line, we may take $N=m=n$.

In other words, the nonvanishing infrapolynomials may be finitely generated.
To establish Theorem 2.2, we shall need a number of lemmas.

LEMMA 2.1. *Let E be a closed bounded pointset. Then for a given $p \in P_n$ there exists $q \in U(p, E)$ if and only if for some $r \in P_{n-1}$ the function $w = r(z)/p(z)$ maps E onto a pointset S whose convex hull $H(S)$ does not contain the origin $w = 0$.*

Proof. If such a q exists and if $p(z) \neq 0$, then

$$\left| \frac{p(z) - q(z)}{p(z)} - 1 \right| = \left| \frac{q(z)}{p(z)} \right| < 1.$$

Thus, with $r(z) = p(z) - q(z)$, the set S and hence $H(S)$ both lie in the disk Γ: $|w - 1| < 1$, and so $w = 0 \notin H(S)$.

Conversely, if, for some $r \in P_{n-1}$, $H(S)$ does not contain $w = 0$, there exists a line L through $w = 0$ which does not intersect $H(S)$. Thus $H(S)$ lies in some disk

$$|w - \gamma| \leqq |\gamma|, \qquad \gamma \neq 0.$$

This inequality implies that $w = \gamma^{-1}[r(z)/p(z)]$ lies in the disk Γ and hence that

$$q(z) = p(z) - \gamma^{-1} r(z)$$

is an underpolynomial of p on E.

This lemma has the following counterpart in the Euclidean space of $2n$-dimensions.

LEMMA 2.2. *Let E be a closed bounded set and let $p \in P_n$ and $p(z) \neq 0$ for $z \in E$. Let Z be the corresponding $2n$-dimensional set whose points ζ are expressed in the n complex valued co-ordinates $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, where $\zeta_j = z^{n-j}/p(z)$ and $z \in E$. Then $p \in I(E)$ if and only if the origin lies in the convex hull $H(Z)$ of Z.*

Proof. Using the notation of Lemmas 2.1 and 2.2, we may write

$$w = \frac{r(z)}{p(z)} = \frac{c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n}{p(z)} = c_1\zeta_1 + c_2\zeta_2 + \cdots + c_n\zeta_n.$$

If $p \notin I(E)$, there would exist $q \in U(p, E)$ and hence by Lemma 2.1 with $c_k = c_{k1} + ic_{k2}$ and $\zeta_k = \xi_k + i\eta_k$,

$$\operatorname{Re}[r(z_j)/p(z_j)] = \sum_{k=1}^{n} (c_{k1}\xi_k - c_{k2}\eta_k) > 0$$

for all $z \in E$. Thus the points ζ for all $z \in E$ lie to one side of a hyperplane through the origin and hence $H(Z)$ does not contain the origin of $2n$-dimensional space.

We may prove the converse statement similarly.

Proof of Theorem 2.2. By Lemma 2.2, the origin is a point of $H(Z)$ if $p \in I(E)$. Hence the origin is the centroid of $m+1$ points ζ corresponding to $m+1$ points

$z_j \in E$, with $m \leqq 2n$. That is, we may find positive constants λ_j with $\lambda_0 + \cdots + \lambda_m = 1$ such that

$$\sum_{j=0}^{m} \lambda_j [z_j^{n-k}/p(z_j)] = 0 \qquad (k = 1, 2, \cdots n). \tag{2.5}$$

Writing $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, multiplying the kth equation in (2.5) by a_k for each k and adding the resulting equations, we are led to the further equation

$$\sum_{j=0}^{m} \lambda_j \frac{[p(z_j) - z_j^n]}{p(z_j)} = 0$$

which is the same as

$$\sum_{j=0}^{m} \lambda_j [z_j^n/p(z_j)] = 1. \tag{2.6}$$

The $n+1$ equations (2.5) and (2.6) may be regarded as involving the $m+1$ unknowns λ_j of which at least one is different from zero.

The matrix of the coefficients

$$\Lambda = \begin{bmatrix} \dfrac{z_0^n}{p(z_0)} & \dfrac{z_1^n}{p(z_1)} & \cdots & \dfrac{z_m^n}{p(z_m)} \\ \dfrac{z_0^{n-1}}{p(z_0)} & \dfrac{z_1^{n-1}}{p(z_1)} & \cdots & \dfrac{z_m^{n-1}}{p(z_m)} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{1}{p(z_0)} & \dfrac{1}{p(z_1)} & \cdots & \dfrac{1}{p(z_m)} \end{bmatrix}$$

has in the upper left corner a minor whose determinant has the value

$$\Lambda_{01\cdots k} = \frac{V(z_0, z_1, \cdots, z_k)}{p(z_0)p(z_1)\cdots p(z_k)},$$

where $V(z_0, z_1, \cdots, z_k)$, as the Vandermonde determinant for the distinct numbers $z_0, z_1, \cdots, z_k$, is different from zero.

If $m < n$, we may solve for the λ_j using the first m equations (2.5). Since these are homogeneous equations with nonvanishing determinant, all λ_j would be zero—a contradiction. Hence $m \geqq n$.

If $m = n$, we use the $m+1$ equations (2.5) and (2.6) and thus get the results

$$\lambda_j = (-1)^j \frac{\Lambda_{01\cdots j-1\, j+1\cdots n}}{\Lambda_{01\cdots n}} = \frac{p(z_j)}{\omega'(z_j)}, \tag{2.7}$$

where

$$\omega(z) = \prod_{j=0}^{n} (z - z_j).$$

Here therefore

$$p(z) = \omega(z) \sum_{j=0}^{n} \frac{p(z_j)}{\omega'(z_j)(z - z_j)} = \omega(z) \sum_{j=0}^{n} \frac{\lambda_j}{z - z_j}. \tag{2.8}$$

If $m > n$, we solve for $\lambda_0, \lambda_1, \cdots, \lambda_n$ in terms of $\lambda_{n+1}, \cdots, \lambda_m$, thus obtaining for $j = 0, 1, \cdots, n$

$$\begin{aligned}\lambda_j &= \overset{-1}{\Lambda}_{01\cdots n}\Big\{(-1)^j \; \Lambda_{01\cdots j-1\, j+1\cdots n} - \sum_{k=n+1}^{m} \lambda_k \Lambda_{01\cdots j-1\, k\, j+1\cdots n}\Big\} \\ &= \frac{p(z_j)}{\omega'(z_j)} - \sum_{k=n+1}^{m} \lambda_k \frac{\omega(z_k)p(z_j)}{\omega'(z_k)(z_k - z_j)p(z_k)}.\end{aligned}$$

Hence

$$\sum_{j=0}^{n} \frac{\lambda_j}{z - z_j} = \frac{p(z)}{\omega(z)} - \sum_{k=n+1}^{m} \lambda_k \frac{\omega(z_k)}{p(z_k)} \sum_{j=1}^{n+1} \frac{p(z_j)}{\omega'(z_j)(z_k - z_j)(z - z_j)}. \tag{2.9}$$

Let

$$\omega_k(z) = \omega(z)(z - z_k) \qquad \text{for } k = n + 1, \cdots, m.$$

Then

$$\omega_k'(z) = \omega'(z)(z - z_k) + \omega(z)$$

so that

$$\omega_k'(z_j) = \omega'(z_j)(z_j - z_k) \qquad (j = 0, 1, \cdots, n)$$

and

$$\omega_k'(z_k) = \omega(z_k) \qquad (k = n + 1, \cdots, m).$$

By the Lagrange Interpolation Formula

$$\frac{p(z_k)}{\omega_k'(z_k)(z - z_k)} - \sum_{j=1}^{n+1} \frac{p(z_j)}{\omega'(z_j)(z_k - z_j)(z - z_j)} = \frac{p(z)}{\omega_k(z)}.$$

The right side of (2.9) now becomes

$$\begin{aligned}\sum_{j=0}^{n} \frac{\lambda_j}{z - z_j} &= \frac{p(z)}{\omega(z)} + \sum_{k=n+1}^{m} \lambda_k \frac{\omega(z_k)}{p(z_k)} \left\{ \frac{p(z)}{\omega_k(z)} - \frac{p(z_k)}{\omega_k'(z_k)(z - z_k)} \right\} \\ &= \frac{p(z)}{\omega(z)} \left\{ 1 + \sum_{k=n+1}^{m} \frac{\lambda_k}{p(z_k)(z - z_k)} \right\} - \sum_{k=n+1}^{m} \frac{\lambda_k}{z - z_k}.\end{aligned}$$

Transposing the last sum to the left side and multiplying both sides by $\Omega(z)$, we obtain

$$\Omega(z) \sum_{j=0}^{m} \frac{\lambda_j}{z - z_j} = p(z) \left\{ \left[1 + \sum_{k=n+1}^{m} \frac{\lambda_k}{p(z_k)(z - z_k)} \right] \prod_{k=n+1}^{m} (z - z_k) \right\}.$$

This proves that $p(z)$ is a factor of $F(z)$ as required for Theorem 2.2.

Extension of Theorem 2.2 is possible to an infrapolynomial p which has as zeros the pointset K: $\zeta_1, \zeta_2, \cdots, \zeta_k$ on E, where $1 \leqq k < n$. If we set $E' = E - K$ and if we write $p(z) = p_1(z)p_2(z)$, where $p_1(z) = (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_k)$ and where $p_2 \in P_{n-k}$, then we can show that $p_2 \in I_{n-k}(E')$. For, if there exists $q_2 \in U(p_2, E')$, then $(p_1 q_2) \in U(p, E)$, a contradiction. Thus, if in Theorem 2.2 p has the zeros $\zeta_1, \zeta_2, \cdots, \zeta_k \in E$, then p has the representation

$$p(z) = (z - \zeta_1) \cdots (z - \zeta_k) p_2(z),$$

where $p_2(z)$ is a factor of a polynomial of the form (2.4).

3. Position of zeros. We shall now apply Theorem 2.2 to the problem of locating the zeros of infrapolynomials. We shall begin with

THEOREM 3.1. FEJÉR [1]. *Let E be a closed bounded pointset and let $p \in I(E)$. Then all the zeros of p lie in $H(E)$, the convex hull of E; no zero lies on the boundary $\partial H(E)$ of $H(E)$ except perhaps at a point of E on $\partial H(E)$.*

Clearly we may assume $p(z) \neq 0$ on E. If $p(\zeta) = 0$ for $\zeta \notin H(E)$, let us define τ by the relation

$$\arg(\tau - \zeta) = \sup_{z \in H(E)} \arg(z - \zeta), \bmod 2\pi. \tag{3.1}$$

Since $H(E)$ subtends at ζ an angle ϕ, $0 < \phi < \pi$,

$$0 < \arg[(\tau - \zeta)/(z_j - \zeta)] \leqq \phi < \pi, \qquad (z_j \in E). \tag{3.2}$$

As a sum of vectors in sector $0 < \arg z \leqq \phi < \pi$,

$$\sum \lambda_j [(\tau - \zeta)/(z_j - \zeta)] \neq 0.$$

That is, $F(\zeta) \neq 0$ in (2.4) and thus, $\zeta \in H(E)$ as required by Theorem 3.1.

We may give more specific information about the location of the zeros of infrapolynomials if we impose various restrictions upon E or confine the choice of polynomials p to some subclass of P_n.

We begin with the assumption that the set E is on the real axis. We have in that case

THEOREM 3.2. *Let E be a closed bounded pointset containing at least $n+1$ points and let $p \in P_n$.*
(I) *If $p \in I(E)$ but $p(z) \neq 0$ for $z \in E$, then p has only real simple zeros which are separated by points of E.*

(II) *If p has only real simple zeros which are separated by points of E, then $p\in I(E)$.*

To prove (I) we note that by Theorem 2.2

$$p(z) = \Omega(z)G(z),$$

where

$$\Omega(z) = \prod_{j=0}^{n} (z - x_j) \quad \text{and} \quad G(z) = \sum_{j=0}^{n} \frac{\lambda_j}{z - x_j},$$

where $\lambda_j > 0$ and the $x_j \in E$. We may assume that $x_0 < x_1 < \cdots < x_n$. Let us vary z from $-\infty$ to ∞ on the real axis. Then for sufficiently small $\epsilon > 0$

$$G(x_k - \epsilon) < 0, \qquad G(x_k + \epsilon) > 0, \qquad k = 0, 1, \cdots, n.$$

Furthermore, for $z \neq x_j$, $j = 0, 1, \cdots, n$,

$$G'(z) = -\sum_{j=0}^{n} \frac{\lambda_j}{(z - x_j)^2} < 0.$$

Hence G has one and only one zero between each pair x_k, x_{k+1}, $k = 0, 1, \cdots, n$. Hence (I) has been established.

To prove (II), let us set $p(z) = (z - y_1)(z - y_2) \cdots (z - y_n)$ and denote by $x_0, x_1, \cdots, x_n$ points of E separating the zeros of p. We may choose the notation so that

$$x_0 < y_1 < x_1 < \cdots < y_n < x_n. \tag{3.3}$$

By Lagrange's Interpolation Formula

$$p(z) = \Omega(z) \sum_{j=0}^{n} \frac{p(x_j)}{\Omega'(x_j)(z - x_j)} \cdot$$

Now from (3.3) (with sg$\equiv$sign)

$$\operatorname{sg} p(x_j) = \operatorname{sg}\left[(x_j - y_1)(x_j - y_2) \cdots (x_j - y_n)\right] = (-1)^j,$$
$$\operatorname{sg} \Omega'(x_j) = \operatorname{sg}\left[(x_j - x_0) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)\right] = (-1)^j.$$

Since $\lambda_j = p(x_j)/\Omega'(x_j) > 0$, $j = 0, 1, \cdots, n$, we conclude from Theorem 2.1 that $p \in I(E)$ as stated in (II).

By a different method from the above, the following more general theorem may be established.

THEOREM 3.3. WALSH-MOTZKIN [8]. *Let E be a closed bounded set containing at least $n+1$ points, let E^* denote the set of limit points of E and let $p \in P_n$. A necessary and sufficient condition that $p \in I(E)$ is that the ordered zeros $y_1, y_2, \cdots, y_\nu$ of p not on E^* separate a subset $x_0, x_1, \cdots, x_\nu$ of distinct points of E in the sense that*

$$x_0 \leqq y_1 \leqq x_1 \leqq \cdots \leqq x_{\nu-1} \leqq y_\nu \leqq x_\nu.$$

Let us next consider pointsets which are symmetric in the real axis.

THEOREM 3.4. FEKETE-VON NEUMANN [3]. *Let the pointset E be symmetric in the real axis and let J denote the circles having as diameters the pairs of conjugate imaginary points of E. If p is a real infrapolynomial on E, then any nonreal zero of p must be in at least one circle J.*

Proof. Because of the symmetry of E and $p(z)$ in the real axis, the zeros $p(z)$, not on E, will satisfy not only the equation

$$\sum_{j=0}^{m} \frac{\lambda_j}{z - z_j} = 0, \qquad (\lambda_j > 0; j = 0, 1, \cdots, m)$$

obtained from (2.4) for suitable points $z_j \in E$, but also the equation

$$\sum_{j=0}^{m} \frac{\lambda_j}{z - \bar{z}_j} = 0, \qquad \bar{z}_j = x_j - iy_j,$$

and hence the equation

$$\sum_{j=0}^{m} \lambda_j \left(\frac{1}{z - z_j} + \frac{1}{z - \bar{z}_j} \right) = 0.$$

If z_j is real, the corresponding term has the imaginary part

$$\text{(3.4)} \qquad \operatorname{Im}\left[\frac{2}{z - z_j} \right] = \frac{-2y}{|z - z_j|^2} .$$

If z_j is not real then

$$\text{(3.5)} \qquad \operatorname{Im}\left[\frac{1}{z - z_j} + \frac{1}{z - \bar{z}_j} \right] = \frac{-2y[(x - x_j)^2 + y^2 - y_j^2]}{|z - z_j|^2 |z - \bar{z}_j|^2} .$$

If the point $z = x + iy$ with $y > 0 (y < 0)$ lies in the upper (lower) halfplane outside all the circles J, then the terms of type (3.4) and (3.5) would both be negative (positive) and so such a point z could not be a zero of p.

The three preceding theorems resemble some well-known theorems on the critical points of a polynomial. Theorem 3.2 is analogous to Rolle's Theorem, whereas Theorems 3.1 and 3.4 are respectively analogous to the following two:

THEOREM. LUCAS [6]. *The critical points of a polynomial lie in the convex hull of the zeros of the polynomial.*

THEOREM. JENSEN [6]. *Each nonreal critical point of a real polynomial lies in at least one circle having as diameter a pair of conjugate imaginary zeros of the polynomial.*

The existence of these analogies comes as no surprise since the derivative of a polynomial

$$f(z) = (z - z_0)^{\lambda_0}(z - z_1)^{\lambda_1} \cdots (z - z_n)^{\lambda_n},$$

where each λ_j is a positive integer, has the form (2.1). In fact, we should suspect that analogies exist for other theorems on critical points such as the following two:

THEOREM. GRACE-HEAWOOD [6]. *If a and b, $a \neq b$, are two zeros of an nth degree polynomial, at least one critical point of the polynomial lies in the disk*

$$|z - (a + b)/2| < (1/2)|a - b| \cot (\pi/n).$$

THEOREM. MARDEN [5]. *If a disk of radius R contains k zeros of an nth degree polynomial $f(z)$ $(2 \leqq k \leqq n)$, the concentric disk of radius $R \csc [\pi/2(n-k+1)]$ contains at least $k-1$ critical points of $f(z)$.*

These two theorems are concerned with the location of just some critical points of polynomial $f(z)$ when the position of only some zeros of $f(z)$ is known. Our analogy must similarly be concerned with just some zeros of an infrapolynomial when the location of the pointset E is only partly specified. The following is such an analogy.

THEOREM 3.5. MARDEN [7]. *Let $E = E_0 + E_1$, where E_0 is a closed bounded pointset and E_1 is a set of k points $0 \leqq k \leqq n$. Let T_0 be the set comprised of all points from which E_0 subtends an angle of at least $\pi/(k+1)$. If $p \in P_n$ is a nonvanishing infrapolynomial on E, then $p(z)$ has at most k zeros outside T_0 irrespective of the location of E_1.*

Proof. If $Z_0, Z_1, \cdots, Z_k$ are any $k+1$ distinct zeros of p outside T_0, then

$$\sum_{j=0}^{m} \frac{\lambda_j}{Z_i - z_j} = 0, \qquad i = 0, 1, \cdots, k, \tag{3.6}$$

where $z_0, z_1, \cdots, z_m$ are points in E. Among the latter, let us say that only $z_m, z_{m-1}, \cdots, z_{m-s+1}$ are points of E_1 with $0 \leqq s \leqq k-1$. From the $k+1$ equations (3.6) we may theoretically eliminate $z_m, z_{m-1}, \cdots, z_{m-k+1}$ but practically this is very difficult. Instead, we use the fact that the Z_i are continuous functions of the λ_j. For a given $\epsilon > 0$ we can find a $\delta > 0$ such that for rational numbers ρ_j with $|\rho_j - \lambda_j| < \delta$, $j = 0, 1, \cdots, m$, the equation

$$\sum_{j=0}^{m} [\rho_j/(z - z_j)] = 0 \tag{3.7}$$

has roots $\zeta_0, \zeta_1, \cdots, \zeta_k$ with $|\zeta_i - Z_i| < \epsilon$ for $i = 0, 1, \cdots, k$ and hence also lie outside T_0. If N is taken as a sufficiently large integer so that each $\nu_j = \rho_j N$ is an integer, the equation

$$\sum_{j=0}^{m} [\nu_j/(z - z_j)] = 0, \tag{3.8}$$

also having roots $\zeta_0, \zeta_1, \cdots, \zeta_k$, is satisfied by the zeros of the logarithmic

derivative of the polynomial

$$f(z) = \prod_{j=0}^{m} (z - z_j)^{\nu_j}.$$

We may now apply the following.

THEOREM. MARDEN [5]. *Let* $\zeta_0, \zeta_1, \cdots, \zeta_k$ *be any* $k+1$ *distinct critical points of the polynomial*

$$f(z) = \prod_{j=0}^{m} (z - z_j)^{\nu_j}.$$

Then there exist positive constants $M_{j_0 j_1 \cdots j_k}$ *such that*

$$\sum \frac{M_{j_0 j_1 \cdots j_k}}{(\zeta_0 - z_{j_0})(\zeta_1 - z_{j_1}) \cdots (\zeta_k - z_{j_k})} = 0, \tag{3.9}$$

where $j_0, j_1, \cdots, j_k$ *take on independently each number in any given set of* $m-k$ *distinct numbers chosen from* $0, 1, 2, \cdots m$.

In particular, we may choose the latter set as $0, 1, \cdots m-k$ so that the z_j in (3.9) are those remaining after eliminating $z_m, z_{m-1}, \cdots, z_{m-k+1}$ from (3.6). All the z_j remaining in (3.9) belong to E_0. Since $\zeta_0, \zeta_1, \cdots, \zeta_k$ all lie outside T_0, at each ζ_i the convex hull $H(E)$ subtends an angle ϕ_i, $0<\phi_i<\pi/(k+1)$. If we now define τ_i by the relation

$$\arg(\tau_i - \zeta_i) = \sup_{z \in H(E)} \arg(z - \zeta_i), \text{ mod } 2\pi,$$

we find that for all $z \in E$

$$0 \leqq \arg[(\tau_i - \zeta_i)/(z - \zeta_i)] \leqq \phi_i < \pi/(k+1).$$

Hence

$$0 \leqq \arg \prod_{i=0}^{k} [(\tau_i - \zeta_i)/(z_{j_i} - \zeta_i)] < \pi.$$

That is, if multiplied by $(\zeta_0-\tau_0)(\zeta_1-\tau_1) \cdots (\zeta_k-\tau_k)$, the left side of (3.9) would be a sum of vectors each of which is drawn from the origin to a point in the upper half-plane. Such a sum could not vanish. We are thus led to a contradiction in supposing that $k+1$ zeros of the infrapolynomial p lie outside of T_0.

Two immediate corollaries to Theorem 3.5 are the following:

COROLLARY 3.5a. *If in Theorem* 3.5, *the pointset* E_0 *lies in a disk of radius* R, *at most* k *zeros of any infrapolynomial lie outside the concentric disk of radius* $R \csc[\pi/2(k+1)]$.

COROLLARY 3.5b. *If in Theorem* 3.5, *the pointset* E_0 *lies on the line segment from* a *to* b, *at most* k *zeros of any infrapolynomial lie outside the domain bounded*

by the two circles of radius L csc $[\pi/(k+1)]$ *and centers* $z=c\pm iL$ cot $[\pi/(k+1)]$ *where* $c=\frac{1}{2}(a+b)$ *and* $L=\frac{1}{2}|a-b|$.

4. Further results. In the above theorems we have discussed infrapolynomials when the pointset E is subject to various limitations. If instead, or in addition, we impose further restrictions on the class P_n of polynomials, we may again expect the zeros of the infrapolynomials to be located in more limited regions of the plane.

An example of such restrictions is to choose p and q in (1.1) from the subset $P_{n,k}\subset P_n$ comprised of all polynomials

$$P_{n,k}: \{z^n + a_1z^{n-1} + \cdots + a_kz^k + a_{k+1}z^{k+1} + \cdots + a_n\}$$

in which the coefficients $a_1, a_2, \cdots, a_k$ are the same for all $q\in P_{n,k}$ but that $a_{k+1}, \cdots, a_n$ are still arbitrary. For such infrapolynomials there is a representation due to Shisha and Walsh [10] which generalizes Fekete's representation. A number of results about the zeros of such infrapolynomials have been developed by Zedek [11], Fekete and Walsh [4], and Shisha and Walsh [10].

Furthermore, in the recent papers written by Motzkin and Walsh [8, 9] we find various other interesting results on the zeros of infrapolynomials as well as a treatment of infrapolynomials as elements in function space and from other points of view. However, these and many other results in the now very extensive literature on infrapolynomials lie outside the scope of the present brief survey, which as stated earlier has been limited to the Fekete representation (Theorem 2.2) and its consequences.

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GENERALIZED FIBONACCI SEQUENCES AND SQUARED RECTANGLES

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1. Introduction. A partition of a rectangle, R, into a finite number, n, of nonoverlapping incongruent squares (elements) is called a *squaring* of R of *order* n. The problem of squaring a rectangle first appeared in the literature as a mathematical puzzle. The difficulty in solving this puzzle concerns the semi-topological problem of how the elements fit together. Originally, squarings were obtained by empirical methods; a general approach to the problem was developed, however, by R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte [1]. This general approach bypassed the semi-topological problem by ingeniously associating the squaring of R with an electrical network (linear graph). After this fundamental paper had been published, other papers appeared, giving numerical results and a method of tabulating these numerical solutions [2], [3], [4]. Interest was focused on the development of families of squarings, each family having a particular order.

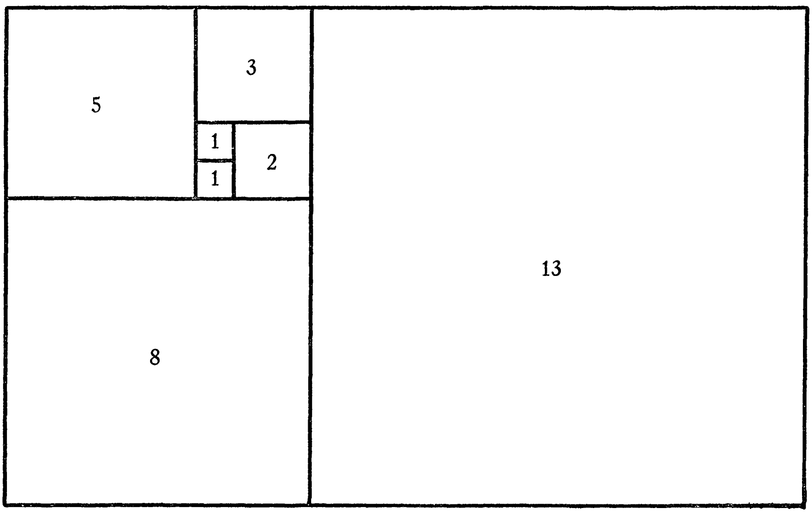

FIG. 1a. Whirling squares.

The following classifications of squarings were developed and used in the earlier work [4] for the purpose of restricting the investigation to certain subsets of the many possible squarings. A *perfect* squaring is a squaring of R consisting of nonoverlapping incongruent elements. It was proven in [1] that 9 is the least order of a perfect squaring. If a squaring is not perfect it is called *imperfect*, and the set of imperfect squarings contains two subsets. *Nontrivially imperfect* squarings contain at least one pair of congruent elements with each element of the pair not adjacent to its congruent partner. *Trivially imperfect* squarings contain two or more adjacent congruent elements. A squaring is

simple if it does not consist of two or more squared rectangles in juxtaposition or if a smaller squared rectangle is not contained within the perimeter of the squaring of R. A squaring which is not simple is *compound*. A squaring of R is *trivially compound* if it contains a square within its perimeter whose side equals one of the two unequal sides of R. If this square is removed, we obtain a squared rectangle of order $(n-1)$. Conversely, squarings of order $(n+1)$ can be formed by adding a square to one of the two unequal sides of any squared rectangle.

The purpose of this paper is to demonstrate how generalized Fibonacci numbers [5] can be used to generate squared rectangles. The application of Fibonacci numbers to the squaring of rectangles is mainly of academic interest; practical applications follow immediately, however, since the associated electrical networks can then be analyzed by inspection, using properties of the generalized sequence as well as properties of the well-known Fibonacci numbers.

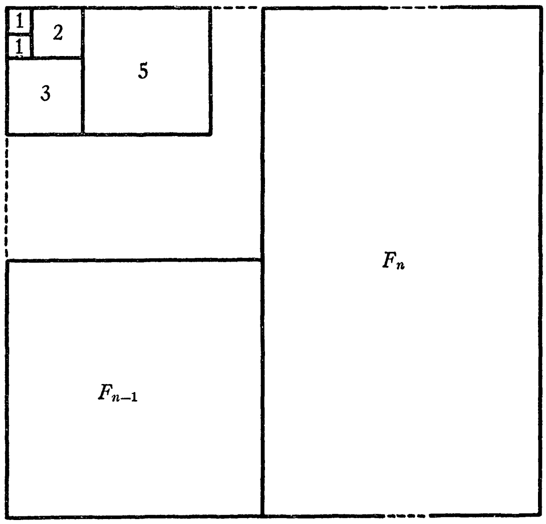

FIG. 1b. Imperfect squaring using the first n terms of $\{F_i\}_{i=1}^{\infty}$.

Consider the "whirling squares" in Figure 1a, and Figure 1b. Each of the elements has sides equal to a Fibonacci number. The linear recurrence relationship

$$F_n = F_{n-1} + F_{n-2}, \qquad (n \geqq 3) \tag{1}$$

allows elements of dimension $F_1, F_2, \cdots, F_n$ to fit together quite naturally. The resulting squarings of Figures 1a, and 1b are not very interesting since the squarings are trivially imperfect (also trivially compound). These squarings, however, stand as geometric models of the identity

$$\sum_{i=1}^{n} F_i^2 = F_nF_{n+1},$$

since the area of the rectangle equals the sum of the square areas. As the order n

is increased the squared rectangles approach closer and closer to a golden rectangle (sides in golden ratio). This occurs since

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \left(\frac{1+\sqrt{5}}{2}\right) = 1.61803398\cdots$$

2. Use of a generalized sequence. Interest in the use of a generalized Fibonacci sequence grew out of the observation that the rectangles of Figs. 1a and 1b form trivially imperfect squarings and it was conjectured that a set of perfect squarings could be constructed through the use of a generalized sequence. In [5] a generalized Fibonacci sequence was developed. The linear recurrence relationship, familiar to the well-known Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \cdots\}$ is preserved, but the first two terms are generalized, hence

$$H_n = H_{n-1} + H_{n-2} \ (n \geqq 3), \qquad H_1 = p, \qquad H_2 = p + q, \tag{2}$$

where p and q are arbitrary integers. The generalized sequence $\{H_n\}_{n=1}^{\infty}$ is:

$$\{p, p+q, 2p+q, 3p+2q, 5p+3q, 8p+5q, \cdots\}. \tag{3}$$

From (3) observe that

$$H_n = pF_n + qF_{n-1}. \tag{4}$$

The following relationships, derived from (3) and (4), will be useful: Summation of odd terms in (3):

$$\sum_{i=1}^{n} H_{2i-1} = H_{2n} - q. \tag{5}$$

Summation of even terms in (3):

$$\sum_{i=1}^{n} H_{2i} = H_{2n+1} - p. \tag{6}$$

The recurrence relationship (2) allows elements of dimension H_1, H_2, H_3, $H_4, \cdots$, to fit together in a natural configuration (Fig. 2). The problem that remains is to determine the number and size of elements required to add to the elements $\{H_n\}_{n=1}^{m}$ before a perfect squaring is achieved.

A squaring of order N can be constructed using the first m terms of (3), $\{H_1, H_2, \cdots, H_m\}$, Fig. 2. Note that $N=(m+4)$. Since $ABCD$ is a square, the following Diophantine equations must be satisfied:

$$\text{(A)} \quad \begin{cases} \text{For } N \text{ even: } & \displaystyle\sum_{i=1}^{(N-4)/2} H_{2i} \doteq \sum_{i=1}^{(N-4)/2} H_{2i-1} + (p-3q) \\ \text{For } N \text{ odd: } & \displaystyle\sum_{i=1}^{(N-5)/2} H_{2i} = \sum_{i=1}^{(N-5)/2} H_{2i-1} + (p-3q). \end{cases}$$

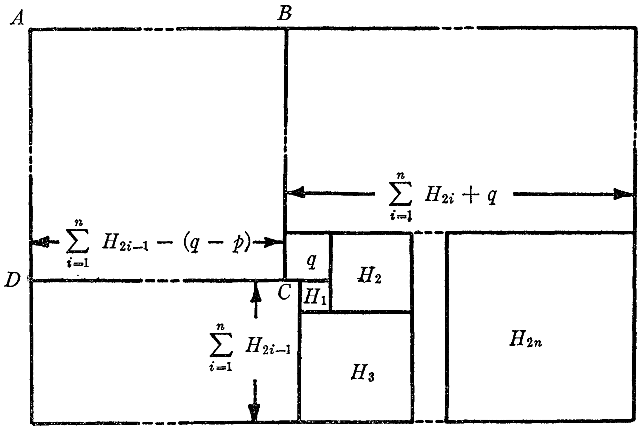

FIG. 2. Squaring a rectangle with generalized Fibonacci numbers. Order is $m+4$.

Using (4), (5) and (6) in equations (A) we have:

$$\text{(B)} \qquad \begin{cases} \text{For } N \text{ even:} & p(F_{N-5} - 2) + q(F_{N-6} + 4) = 0 \\ \text{For } N \text{ odd:} & p(F_{N-5} + 2) + q(F_{N-6} - 4) = 0. \end{cases}$$

Using equations (B) we easily obtain a perfect squaring of order 9, since $p(F_4+2)+q(F_3-4)=0$, $p(5)=q(2)$, and $p=2$, $q=5$.

Equations (B) do not prove to be fruitful in generating perfect squarings of order >9, since the coefficients of p and q are both positive, forcing $p=q=0$.

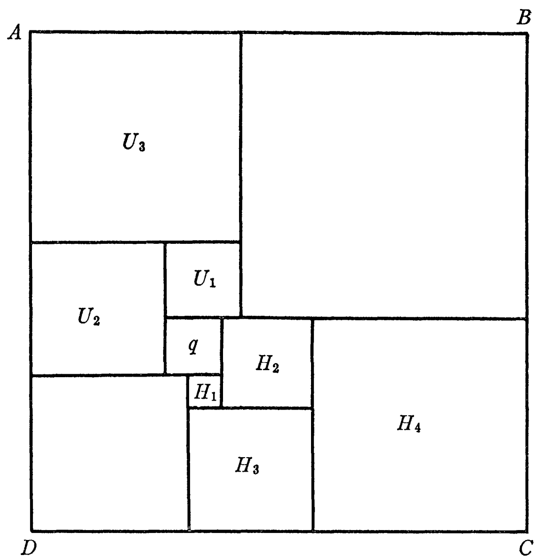

FIG. 3. Constructing a perfect squaring with two generalized Fibonacci sequences $\{H_m\}_{m=1}^{n}$, $\{U_k\}_{k=1}^{n-1}$.

3. Using two generalized sequences. Perfect squarings can be constructed using more than one generalized Fibonacci sequence. Consider the construction in Fig. 3; the elements of sequence (3) together with a new sequence $\{U_k\}_{k=1}^{\infty}$ where

$$U_k = U_{k-1} + U_{k-2} \ (k \geqq 3) \qquad U_1 = r, \qquad U_2 = r + q \tag{7}$$

form perfect squarings of order greater than 9. Note that the properties (4), (5) and (6) hold for the new sequence $\{U_k\}_{k=1}^{\infty}$ if we replace p by r. If we choose $n=k$ it can be shown that the resulting squared rectangles are trivially imperfect (also trivially compound). Concentrating on developing perfect squarings of order 9 and higher, take $n=k+1$ with $k \geqq 3$.

The Diophantine system for the case $n=k+1$: if k is even,

$$\text{(C)} \qquad \begin{aligned} \sum_{i=1}^{k/2} U_{2i-1} &= \sum_{i=1}^{k/2} H_{2i} + (q - r) \\ \sum_{i=1}^{(k/2)+1} H_{2i-1} &= \sum_{i=1}^{k/2} U_{2i} + (q - p); \end{aligned}$$

if k is odd,

$$\text{(C}'\text{)} \qquad \begin{aligned} \sum_{i=1}^{(k+1)/2} U_{2i-1} &= \sum_{i=1}^{(k+1)/2} H_{2i} + (q - r) \\ \sum_{i=1}^{(k+1)/2} H_{2i=1} &= \sum_{i=1}^{(k-1)/2} U_{2i} + (q - p). \end{aligned}$$

Substituting (4), (5) and (6) into the Diophantine equations (C) and (C′), we have:

$$[F_{k+1} + (-1)^{k+1}]p - [F_k + (-1)^k]r + [F_{k-2} + 2(-1)^k]q = 0 \tag{8}$$

$$[F_{k+2} + (-1)^{k+2}]p - [F_{k+1} + (-1)^{k+1}]r + [F_{k-1} + 2(-1)^{k+1}]q = 0. \tag{9}$$

The solution to equations (8) and (9) can be obtained instantly if we define the following vectors:

$$\begin{aligned} \text{Let } A &= (F_{k+1} + (-1)^{k+1},\ -F_k + (-1)^{k+1},\ F_{k-2} + 2(-1)^k) \\ B &= (F_{k+2} + (-1)^{k+2},\ -F_{k+1} + (-1)^{k+2},\ F_{k-1} + 2(-1)^{k+1}) \\ C &= (p, r, q). \end{aligned}$$

Equations (8) and (9) can now be written as,

$$A \cdot C = 0 \tag{8$'$}$$

$$B \cdot C = 0 \tag{9$'$}$$

and

$$C = A \times B. \tag{10}$$

Performing the cross product in (10) we have,

$$p = F_{k+3} - 1, \qquad r = F_k + 2(F_{k+3} - 1), \qquad q = F_{k+4} - 1.$$

Since the order $N=2k+4$, perfect squarings of order 10, 12, 14, 16, $\cdots$ can be constructed taking $k=3, 4, 5, 6, \cdots$. As N increases, this set of squared rectangles approaches a perfect squared square since $BC-AB=1$ in Fig. 3.

4. The electrical analog. The general theory developed in [1], concerning the squaring of rectangles and squares, is based on establishing a convenient model of the squaring of R of order n. The details of constructing this model are clearly outlined in the early literature [1] and [4]. Briefly, the relationships between the puzzle and the model are as follows: the upper and lower peripheral edges of any squared rectangle correspond to the poles of the network, each element of the squaring corresponds to a unit resistance, each horizontal line segment in the squaring corresponds to a vertex or node of the network and finally, each vertical line segment corresponds to a loop. Application of Kirchhoff's Laws to any given network yields the currents that flow through each branch. These currents correspond to the dimension of the individual elements. Since each element has unit resistance, the potential difference across the element equals the current through it. The vertical side of the squared rectangle corresponds to the potential difference between the poles of the network.

Some interesting results follow from the examination of the electrical network associated with the squaring in Fig. 1b.

The network in Fig. 4 is called a ladder-network and is an important network in communication systems. It consists of n-L sections in cascade and can be characterized by describing the attenuation (input voltage/output voltage) denoted by A, the input impedance Z_1, and the output impedance Z_0. The analysis of this network appeared in [6] which introduces the application of Fibonacci numbers to network analysis. The result obtained by applying Kirchhoff's and Ohm's Laws to ladder-networks with $n=1, 2, 3, \cdots$ is now tabulated for $R_1=xR_2$.

n	Z_0	A	Z_1
1	R_2	$(x+1)$	$(x+1)R_2$
2	$\left(\dfrac{x+1}{x+2}\right)R_2$	(x^2+3x+1)	$\left(\dfrac{x^2+3x+1}{x+2}\right)R_2$
3	$\left(\dfrac{x^2+3x+1}{x^2+4x+3}\right)R_2$	(x^3+5x^2+6x+1)	$\left(\dfrac{x^3+5x^2+6x+1}{x^2+4x+3}\right)R_2.$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The polynomials which appear in the above table possess interesting analytic as well as numerical properties and have been developed by matrix methods (the transfer functions of the electrical networks) in [7].

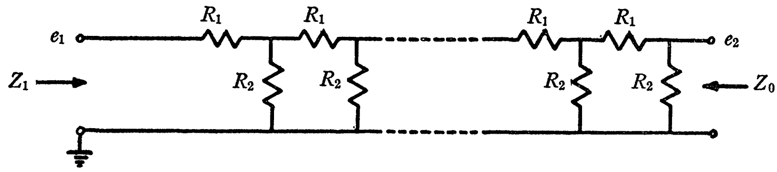

FIG. 4. Electrical model of FIG. 1b.

The Fibonacci properties of the ladder-network associated with Fig. 1b become evident if $x=1$, $R_2=1$ ohm, then

$$Z_0 = \frac{F_{2n-1}}{F_{2n}}, \qquad A = (F_{2n-1} + F_{2n}) = F_{2n+1}, \qquad Z_1 = \frac{F_{2n+1}}{F_{2n}}.$$

In other words, the network can be analyzed by inspection; as n is allowed to increase, $n=1, 2, 3, \cdots$, the value of Z_0 for n-L sections coincides with the nth term in the sequence of Fibonacci ratios $\{1/1, 2/3, 5/8, 13/21, \cdots\}$. The value for the attenuation A is given by the sum of the numerator and denominator of Z_0. The value of Z_1 is also clearly related to the expression for A and Z_0. Let the voltage across each resistor in Fig. 4 be denoted by v, the current through the resistors as i, the voltage between poles V and current between poles I. Note that $v=i=F_i$ then the expression $\sum vi = VI$ shows that the power VI is dissipated in the network as joule heat in each element. This expression is given by the Fibonacci identity

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

The networks associated with the more general squarings also possess Fibonacci properties. The squarings developed in this paper far from exhaust all the possibilities of using generalized Fibonacci numbers in the squaring of rectangles. For a further discussion, see [8].

It is the author's desire that this paper will stimulate continued work in the application of the Fibonacci sequence to physical phenomena.

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A SIMPLE NEW PROOF OF THE MEAN VALUE THEOREMS FOR CERTAIN ELLIPTIC EQUATIONS

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1. The results given here are mostly known (see Courant-Hilbert, Mathematische Physik, Vol. II, p. 249–261), but are obtained in a simple manner which does not depend on Green's functions or complex variables.

2. Consider $\nabla^2 u = 0$ in the interior of a circle of radius a. Divide the circumference in n equal arcs, on $n-1$ of which we prescribe $u=0$, while on the remaining arc $u=1$. The value of the solution at the center of the circle is denoted by $\bar{u}_0$. Now consider the n boundary value problems obtained by rotating the boundary values through an angle $\theta_k = 2\pi k/n$, $k=1, 2, \cdots, n-1$. From symmetry considerations, each problem has the same central value $\bar{u}_0$. On the other hand, superposition of the n problems yields the problem with $u=1$ over the whole circumference, whose solution is denoted by $u^{(1)}$. Clearly $u^{(1)}=1$ in the whole circle and in particular at the center $u_0^{(1)}=1$. Therefore $\bar{u}_0 = 1/n$.

Now suppose the prescribed value of u on the circumference is an arbitrary piecewise continuous function $f(\theta)$. Divide the circumference in a large number n of equal arcs; then, by superposition

$$u_0 = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\theta_k) \frac{1}{n} = \lim_{n\to\infty} \frac{1}{2\pi} \sum_{k=0}^{n-1} f(\theta_k)\{\theta_k - \theta_{k-1}\} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = f,$$

where f denotes the average value of f on the boundary.

The preceding argument can be made rigorous by appealing to the fact that u_0 is a continuous linear functional of the boundary values of $f(\theta)$.

3. Let M be a linear, elliptic, partial differential operator of the second order in k variables. We suppose that M is spherically symmetric (i.e., the form of M is the same in all spherical coordinate systems). We further assume that there is one and only one solution $u^{(1)}$ of the problem $Mu=0$, with $u=1$ on the surface of a k-sphere of radius a. The central value of this solution is denoted by $u_0^{(1)}$.

Consider the boundary value problem $Mu=0$ inside a sphere of radius a, with $u=1$ on an infinitesimal portion ds of the boundary and $u=0$ on the remainder of the boundary. From the spherical symmetry, the solution $\bar{u}$ of this problem has central value

$$\bar{u}_0 = \frac{ds}{A} u_0^{(1)},$$

where A is the total area of the boundary. Then, for any solution of $Mu=0$, we have

$$u_0 = \int f u_0^{(1)} \frac{ds}{A} = \frac{u_0^{(1)}}{A} \int f ds = u_0^{(1)} \hat{f},$$

where $\hat{f}$ is the average of f on the spherical surface. For the potential equation in *any* number of dimensions $u^{(1)}=1$, and $u_0^{(1)}=1$, and $u_0=\hat{f}$.

4. For $\nabla^2 u + k^2 u = 0$ in two dimensions,

$$u^{(1)} = \frac{J_0(k\rho)}{J_0(ka)},$$

where ρ is the distance from the center. Therefore

$$u_0^{(1)} = \frac{1}{J_0(ka)}, \qquad u_0 = \frac{1}{J_0(ka)} \hat{f}.$$

For $\nabla^2 u + k^2 u = 0$ in three dimensions

$$u^{(1)} = \frac{a}{\sin ka} \frac{\sin kr}{r}, \qquad u_0^{(1)} = \frac{ka}{\sin ka}, \qquad u_0 = \frac{ka}{\sin ka} \hat{f}.$$

5. These ideas can be extended to higher order linear elliptic operators which are spherically symmetric. For the biharmonic equation $\nabla^2\nabla^2 u=0$, in two dimensions, we consider two auxiliary problems:

(a) $\nabla^2\nabla^2 u=0$ with $u=1$ and $\partial u/\partial n=0$ on the circumference of a circle of radius a. The solution is $u^{(1)}=1$, with central value $u_0^{(1)}=1$.

(b) $\nabla^2\nabla^2 u=0$ with $u=0$ and $\partial u/\partial n=1$ on the circumference. The solution is $u^{(2)}=1/2a(\rho^2-a^2)$, with central value $u_0^{(2)}=-\frac{1}{2}a$.

This leads immediately to the mean value theorem for any function u which is biharmonic inside a circle

$$u_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta - \frac{a}{4\pi}\int_0^{2\pi} g(\theta)d\theta = \hat{f} - \frac{a}{2}\hat{g}.$$

In the above equation $f(\theta)$, $g(\theta)$ are respectively the values of u and $\partial u/\partial n$ on the circumference.

MODELS OF MANY-VALUED LOGICS

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This paper describes electrical circuits which exemplify the structure of systems of three-valued logic. These can be assembled into simple logical computers which will evaluate polynomials and decide tautologies in these three-valued systems, just as the circuits described by Puig Adam [1] and the author [2] do in two-valued logic. In the course of the discussion the relations of three-valued logic which are used are expressed in terms of sets of elements from traditional two-valued logic by a method which is general, and which enables electrical models of logics with any finite number of values to be designed in a similar way. The method also suggests models based on punched cards. Logical machines of this kind have been described by other writers [6, 7, 8], but the machines discussed here are especially simple and can be made from inexpensive, easily obtained components. For this reason they make good demonstration models for classroom instruction.

Lukasiewicz's bracket-free notation is employed for the three-valued systems for typographical simplicity. The symbols used are p, q, $\cdots$ for propositional variables; N and N' for the forms of negation introduced by Lukasiewicz and Post; and A, K, C and E for disjunction, conjunction, implication and equivalence, respectively. In two-valued logic we employ traditional notation with $+$ denoting disjunction, $\times$ (or juxtaposition) denoting conjunction, and a prime denoting negation.

The first logical system to be discussed is described by Goodstein [3]. Variables take the three values T, I and F, and there are four logical operations N', A, K and C which are defined by the tables

p	T	I	F
$N'p$	I	F	T

	Kpq	T	I	F	q
	T	T	I	F	
p	I	I	I	F	
	F	F	F	F	

	Apq	T	I	F	q
	T	T	T	T	
p	I	T	I	I	
	F	T	I	F	

	Cpq	T	I	F
	T	T	I	F
p	I	I	I	I
	F	T	T	T.

TABLE I

The aim is to construct electrical networks isomorphic with the logical relations. Each circuit consists of three stages; an input stage containing switches or relays on which the variables are coded, a series of junction boxes corresponding to the logical connectives, and a display stage which is usually three lamps, T, I, and F, one of which lights up to indicate the value of the functions. Some-

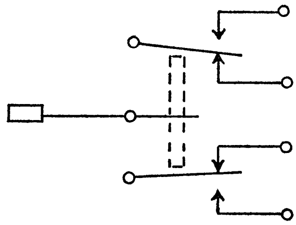

FIG. 1

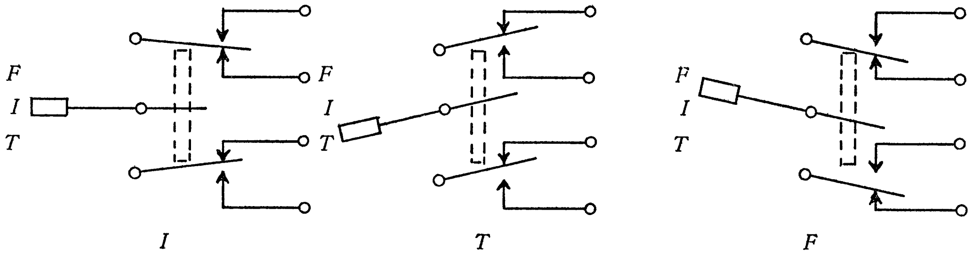

FIG. 2

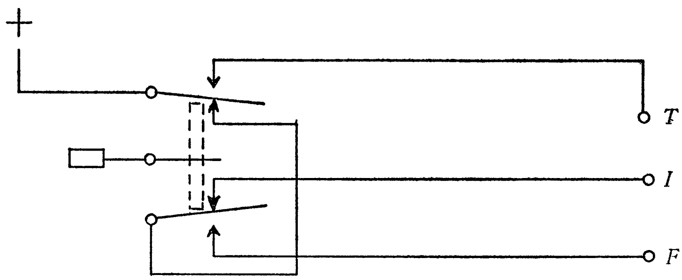

FIG. 3

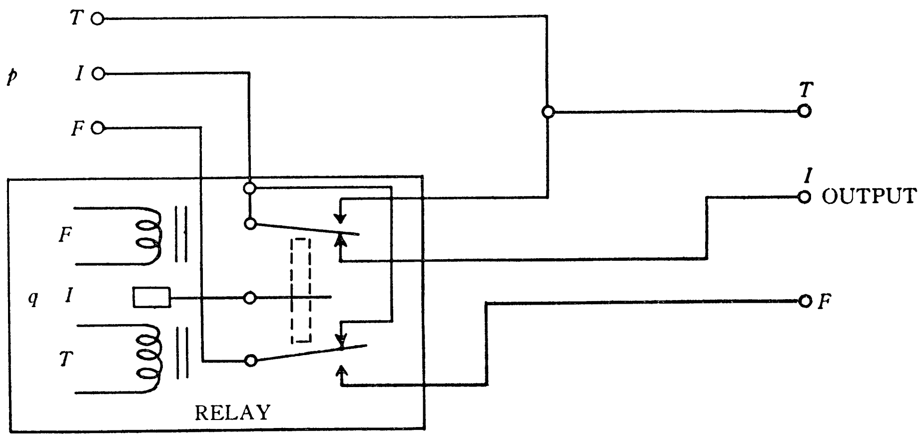

FIG. 4

times a different form of display is used, with fewer lamps which merely indicate if two functions are equal or not.

The circuit elements are standard switches and relays with three positions, rest (centre), up and down. The action of the input switches, which correspond to the variables, is as follows. Each switch has two independent banks of changeover contacts, shown in Fig. 1 in the rest position. When the lever is pressed down the upper contact is changed over and the lower contact remains at rest. When the lever is raised up the upper contact remains at rest and the lower is changed over. Thus I, T and F are coded (respectively) as in Fig. 2. It can be seen that each variable requires a pair of changeover contacts for its expression. Of course, switches contain a number of changeovers in each bank, and these are usually needed to handle the several appearances of a variable in a formula; since each appearance, in general, requires a separate set of contacts. (The upper changeover may be interpreted as "true" and the lower as "false." Both cannot occur simultaneously, but we may have neither.)

An input switch is wired as follows, in Fig. 3.

The aim of the systems described in references [1, 2] is to represent the logical relations entirely by wiring in junction boxes. If this can be done without using relays, valves, or similar devices there is an obvious gain; but it is not possible to manage some of the more complicated connectives with this type of wiring alone; so we develop a system in which the connectives A, K and C are represented by junction boxes, each of which contains one relay. The relays closely resemble the switches described above, having three positions and two input coils which switch from the rest position to the T or F positions. The output from each circuit element is on a three-line cable which feeds the next element with a definite input T, I or F, only one of which can occur at any one time. Each junction box has to take in two three-line cables and give out one, the output being the appropriate function of the two inputs.

Fig. 4 shows the wiring for an A-junction box.

For a K-junction box we have Fig. 5.

Fig. 6 shows how the C-junction box is wired.

The N'-operation is carried out by a link of cable which permutes contacts (see Fig. 7).

Combinations of the circuits in Fig. 3 with those in Figs. 4, 5 and 6 enable demonstration models of A, K and C to be built which do not require relays at all, as the relays may be replaced by other input switches. In general, however, relays are needed if the operations are performed at later stages in the circuit when building up more complicated polynomials. To evaluate a polynomial or to test a tautology it is necessary to assemble a tree-like structure, starting at the ends of the branches with the input switches and proceeding through the junction boxes to the final display stage.

The system described so far is a three-wire system with relays, and as demonstration apparatus it has the advantage that the three connectives are represented by junction boxes of a similar type; but for some purposes it might be

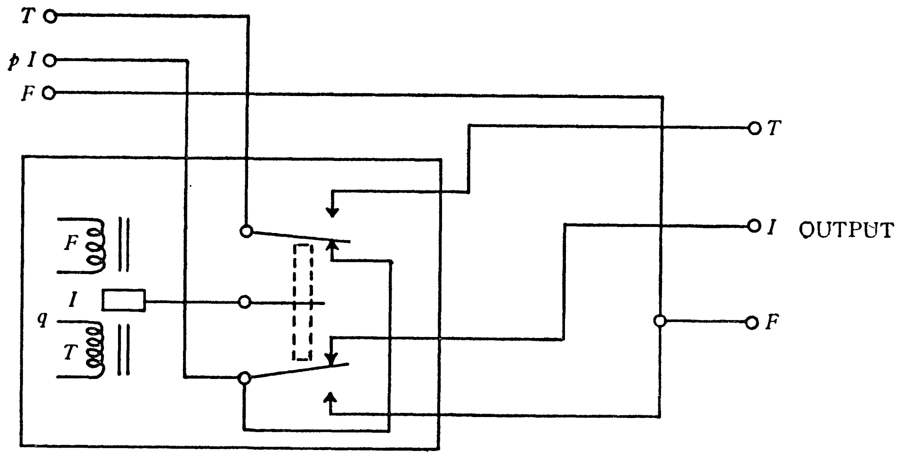

FIG. 5

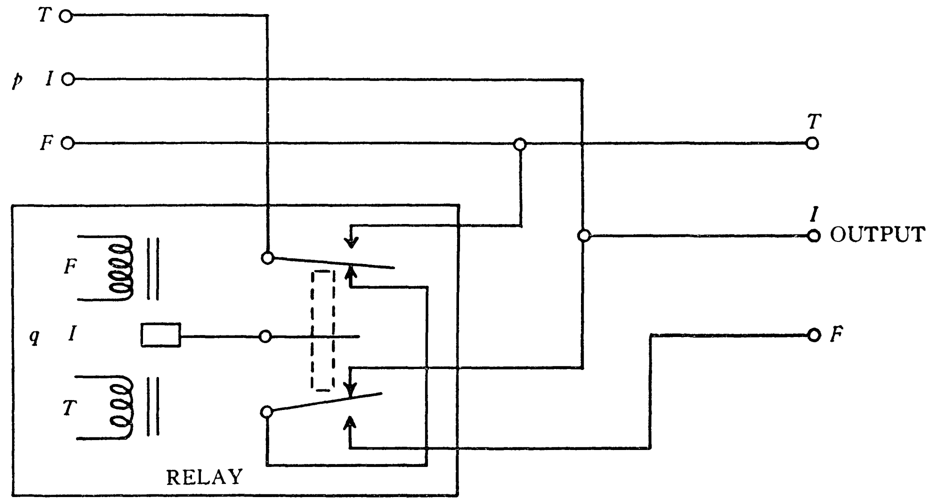

FIG. 6

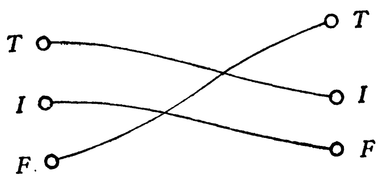

FIG. 7

preferable to economise on relays by performing the A and K operations with junction boxes of a different type which incorporate interior wiring only. It does not seem possible to make a C box in this way for reasons which arise in the later theoretical discussion.

The above system has been described as a three-line system because three wires link the separate stages of the circuit, but Figures 1 and 2 reveal that a six-line coding is also being employed, and that it is easy to translate from one code to the other. The three positions of the switches, or relays, set up relations between the six contacts of the pairs of changeovers, and these relations in turn are translated into the three-line code by the circuit of Fig. 3. The junction boxes, without relays, for A and K may be wired in a six-line code which may be constructed by defining the elements of the three-valued logic as ordered pairs of elements of traditional two-valued logic. (The use of ordered pairs is, of course, a standard method of constructing new mathematical systems from systems which are already known, and the representation of elements of an m-valued logic by ordered m-tuples was introduced in Post's original paper [5].)

Consider two-valued logic with variables $a, b, c \cdots$, an operation of negation denoted by $'$, and operations of disjunction and conjunction denoted by $+$ and $\times$ respectively. For brevity, the $\times$ sign will usually be replaced by juxtaposition. The variables take the values 0 and 1. By a method to be described later, Table I leads to the adoption of the following definitions.

DEFINITION 1. *An element of the three-valued logic is an ordered pair of elements* (a, b) *where* a *and* b *are not both* 1; *i.e.* $ab=0$.

DEFINITION 2. $A(a, b)(x, y)=(a+x, by)$.

Note: $ab=0$ and $xy=0$ so $(a+x)by=0$.

DEFINITION 3. $K(a, b)(x, y)=(ax, b+y)$.

Note: $ab=0$ and $xy=0$ so $ax(b+y)=0$.

DEFINITION 4. $C(a, b)(x, y)=(ax+b, ay)$.

Note: $ab=0$ and $xy=0$ so $(ax+b)ay=0$.

DEFINITION 5. $N'(a, b)=(b, a'b')$.

Note: $N'N'(a, b)=(a'b', a)$ and $N'N'N'(a, b)=(a, b)$.

This system is isomorphic with the system in Table I.

Definitions 2 and 3 suggest the form of six-line junction boxes for A and K. The A circuit is as shown in Fig. 8, and the K circuit is similar. These circuits enable two variables or two logical functions which are coded on the switch contacts to be combined by A or K to give an output on another six contacts in the same code, and the process can be continued so that any expression, however complicated, in variables A and K can be formed without relays, provided only that sufficient initial inputs are available. Definitions 4 and 5 show that C

and N' cannot be handled in the six-line system without relays because letters are repeated in the brackets on the right-hand sides of the definitions. Post showed [5] that all functions can be defined in terms of the set $\{N', A\}$ or $\{N', K\}$. The interdependence of the connectives in this system can be seen from the following relations. (I am indebted to Mr. C. A. Meredith for this information.) Define N (another form of negation) by $N(T, I, F) = (F, I, T)$. Then

$$\begin{aligned}
Np &= AN'N'ApN'N'pN'AN'pN'N'p\\
Np &= KN'KpN'pN'N'KN'N'pN'p\\
Kpq &= NANpNq\\
Apq &= NKNpNq\\
Cpq &= AN'N'ApN'N'pN'AN'N'pN'N'AN'N'pq\\
Apq &= CN'N'CN'N'qqCCpN'N'pq\\
Kpq &= N'N'CN'N'CqN'qN'CCppN'N'CpN'q.
\end{aligned}$$

These relations are far from obvious, and their verification with an electrical model would make a very good demonstration.

Similar models may be constructed for the system of three-valued logic described by Kleene [4]. The A, K and N connectives have the same tables as before, and the connectives C_1 and E (see Table II) are also employed.

	C_1pq	T	I	F	q
	T	T	I	F	
p	I	T	I	I	
	F	T	T	T	

	Epq	T	I	F	q
	T	T	I	F	
p	I	I	I	I	
	F	F	I	T	

	E_1pq	T	I	F	q
	T	T	F	F	
p	I	F	T	F	
	F	F	F	T	

TABLE II

In ordered pair notation we have:

DEFINITION 6. $C_1(a, b)(x, y) = (b+x, ay)$.

DEFINITION 7. $E(a, b)(x, y) = (ax+by, ay+bx)$.

DEFINITION 8. $N(a, b) = (b, a)$.

This system behaves very much like ordinary two-valued logic; for example

$$C_1pq = ANpq, \quad \text{and} \quad Epq = KC_1pqC_1qp,$$

but $C_1C_1pqC_1NqNp$ is not always true, and there are no tautologies until the relation E_1 is introduced. This connective is an operator within the system which tests whether or not two variables or two functions have the same truth value. In ordered pair notation we may define E_1 by

DEFINITION 9. $E_1(a, b)(x, y) = [(ax+a'x')(by+b'y'), ax'+a'x+b'y+by']$.

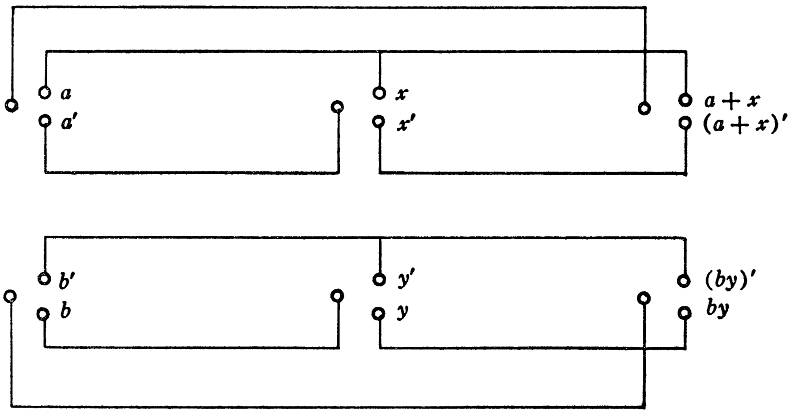

Fig. 8

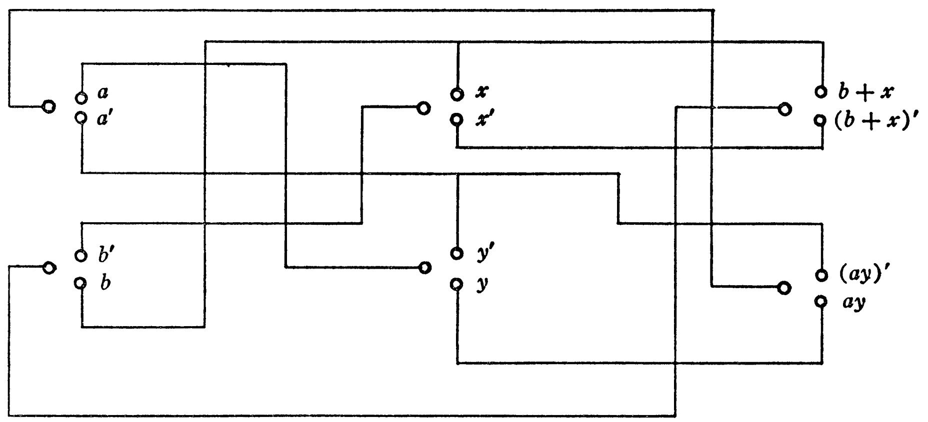

Fig. 9

The impossibility of tautologies without the E_1 relation is indicated by the absence of primes in Definitions 2, 3, 6, 7 and 8. A tautology has the value (1, 0) and the 1 in the first place cannot be built up by unprimed variables alone.

The simple form of Definition 6 shows that the operation C_1 can be carried out merely by wiring in a junction box, and no relay is needed. The wiring is shown in Fig. 9. If E is defined in terms of the other connectives all of the operations in Kleene's system can be carried out by junction box wiring, without relays, but in practice one might not wish to do so. This does not include the E_1 operation, and the difficulty of handling this is indicated by the complicated form of Definition 9. The number of appearances of each variable indicates that four changeovers on a relay are needed, and a suitable circuit on the three-line system is shown in Fig. 10.

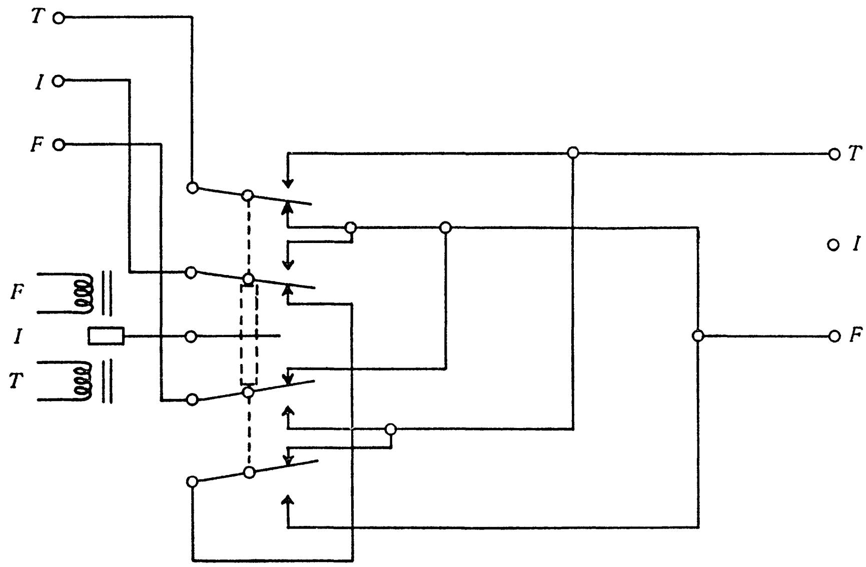

FIG. 10

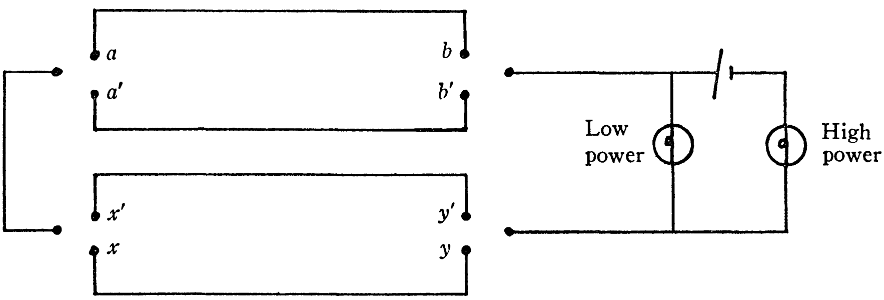

FIG. 11

This circuit is useful in all systems as it compares the values of two functions. Another method of doing so is to adapt a circuit (in [1]) which has the advantage that it requires no relays, but which may seem "unethical" as it is not a pure switching technique. It depends on the different resistances of two lamps to ensure that one of them does not light up when insufficient current passes through it. Since it is a circuit without relays it is based on the six-line system, and it may be wired as in Fig. 11. For combinations of the variables *TT*, *II*, or *FF* the low power lamp is shorted out and the voltage goes across the high power lamp, which lights. For any other combination the voltage is through the two lamps in series and the current is only sufficient to light the low power one. If it is sufficient to have a lamp which is lit when the values of the two functions are the same and which is out when they are not, the low power lamp may be

omitted. These circuits are useful only at the display stage as they do not transmit information in the code in which they receive it.

It is possible to define E_1 by

$$E_1pq = AAN'N'ACpN'N'qN'N'CpN'N'qN'N'ACN'pqN'N'CN'pqN'N'ACN'N'pN'qN'N'CN'N'pN'q,$$

and so by N' and A, K or C.

Lukasiewicz's original system differs from Kleene's in having $C_1II=T$ and $EII=T$. Definitions 6 and 7 then have to be replaced by

Definition 10. $C_2(a, b)(x, y) = (b+x+a'b'x'y', ay)$.

Definition 11. $E_2(a, b)(x, y) = (ax+by+a'b'x'y', ay+bx)$.

These complicated relations can be realised only by relays with a considerable number of contacts, and it is not worth drawing the circuits; especially as we later describe other systems which handle these connectives more simply.

Punched Card Systems. The ordered pair notation also shows how a three-valued logical computer can be designed on punched cards. The simplest type of punched card system codes two-valued variables in positions round the edge of the card with holes and slots. If the cards are in a stack and a probe is inserted through a hole and lifted, then cards with a hole in this position are raised up and the cards with a slot are left behind. It is obvious how the operations of $\times$, $+$ and $'$ can be performed by stabbing holes. Such a set of cards is a distributive lattice, and pack of 2^n cards can be used to verify all of the formulae of the sentence calculus involving up to n variables. To each polynomial there corresponds a sequence of operations, and a tautology is a sequence which removes all of the cards from the stack; so such a set of cards may be regarded as a "logical abacus" for carrying out truth table calculations.

The ordered pair notation shows how to code cards using two positions for each variable in three-valued logic, and shows the sorting procedures to follow to sort the cards into three piles T, I and F. For example, in the system in (3) we have $C(a, b)(x, y) = (ax+b, ay)$. To realise this first carry out the operation $ax+b$ and place these cards in the T pile, then perform the operation ay and place these cards in the F pile. The remaining cards have the value I. 3^n cards are needed to handle n variables, and it is an open question whether it is quicker to use a truth table or a pack of cards, but the card method can be put on existing machinery and is capable of unlimited development.

We have so far represented three-valued logics by ordered pairs over two-valued logic. This can be extended and any p-valued logic may be represented by ordered n-tuples over a two-valued logic if $p \leqq 2^n$. If $p < 2^n$ there are redundancies and it is convenient to absorb these by introducing identical relations between the components of the n-tuple. The functions required in the two-valued logic for the n-tuples to represent a connective in the p-valued logic can be computed in their disjunctive normal form from the table for the connective.

They should then be simplified as much as possible by using the identical relations. For example, when coding T, I and F by (1, 0), (0, 0) and (0, 1) the C connective of Definition 4 is treated as follows.

In the present notation its table becomes:

	C	$(1,0)$	$(0,0)$	$(0,1)$	(x, y)
	$(1,0)$	$\underline{(1,0)}$	$(0,0)$	$(0,1)$	
(a, b)	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	
	$(0,1)$	$\underline{(1,0)}$	$\underline{(1,0)}$	$\underline{(1,0)}$	

To find the value of u in $C(a, b)(x, y)=(u, v)$, examine each position in the table where $u=1$ (these are underlined) and form the corresponding disjunctive normal form,

$$u = ab'xy' + a'bxy' + a'bx'y' + a'bx'y.$$

The four terms come from the four positions of the table, and the four elements in each are unprimed or primed according to whether they take the values 1 or 0 at the particular positions in the table. In this case we have the identical relations $ab=xy=0$, and these reduce the expression for u to $u=ax+b$. In a similar way $v=1$ at only one position in the table, so there is only one term in the disjunctive normal form, $v=ab'x'y$, and using the identical relations this reduces to $v=ay$. The full normal form is not usually required in practice as it is often possible to employ short cuts.

On punched cards it is reasonable, and perhaps more obvious, to use three positions (a, b, c) to code each variable in the three-valued system. The three positions can denote T, I and F respectively, so that the cards code the values of the variables as (1, 0, 0), (0, 1, 0) or (0, 0, 1). The identical relation can here be expressed as $ab+bc+ca=0$. As an exercise in this system it may be verified that the A table becomes

	A	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	(x, y, z)
	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	
(a, b, c)	$(0,1,0)$	$(1,0,0)$	$(0,1,0)$	$(0,1,0)$	
	$(0,0,1)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	

which gives the defining relation in ordered triple notation as

$$A(a, b, c)(x, y, z) = (a + x, by + bz + cy, cz).$$

When sorting cards on this system it is only necessary to carry out the operations corresponding to two of T, I, or F since at any stage of the sorting the aim is to divide the cards into three mutually exclusive sets. In this case one would perform the simplest operations, which are $a+x$ (these cards are T) and cz (these cards are F); the remaining cards are then I. These operations are just

the same as when the two position code is used; but there are other occasions when the new code is simpler. For example, consider C_2, E_2, and E_1. We have

$$C_2(a, b, c)(x, y, z) = (x + c + by, ay + bz, az),$$

and sorting requires six operations when Definition 10 would demand eight.

$$E_2(a, b, c)(x, y, z) = (ax + by + cz, ay + cy + bx + bz, az + cx),$$

which would require ten operations (which could be reduced to eight), whereas Definition 11 needs twelve.

$$E_1(a, b, c)(x, y, z) = (ax + by + cz, 0, a(y + z) + b(x + z) + c(x + y))$$

and only six operations are needed because the I position is empty in any case. Definition 9 would need eight.

This system could be expressed electrically by using three-pole three-way switches as the basic unit; but it would not be so convenient as the system using ordered pairs.

If an electrical machine of any size is built to carry out this type of computation it is desirable to provide some form of automatic scanning, and this is not difficult. One convenient method is to use rotary switches coded to give a ternary output, and a unit of this kind, together with a set of junction boxes wired according to Kleene's system of three-valued logic has been constructed by one of my students, Mr. F. M. Harkin.

The principles developed here are sufficient to enable any finitely many-valued logic to be represented by n-tuples over two-valued logic, and hence to construct an electrical or a punched card model which will serve as a concrete representation of the system, or as a computer for it.

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CONTINUOUS SOLUTIONS OF THE FUNCTIONAL EQUATION

$f(x+y)+f(x-y)=2f(x)f(y)$

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1. It has been shown by Vaughan [4] and Rosenbaum and Segal [3] that, under very mild conditions of continuity, certain functional equations of a form similar to addition theorems for the trigonometric functions are sufficient to characterize these functions. Among the various formulae associated with the addition theorem for the trigonometric functions, the formula for the sum of two cosines, i.e.

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y,$$

is noteworthy, for it involves only the function cos. A similar functional relation, namely

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{1}$$

is shared by the hyperbolic cosine cosh, and, more generally, by the function $x\to\cosh\alpha x$, where α is any real or complex constant, and it is a plausible conjecture that every nontrivial solution of the equation (1) is of this latter form.

The equation (1) has received a certain amount of attention in the literature, and a very elegant treatment of the real case is given by Cauchy [1]. More precisely, it is proved by Cauchy that if f is a real-valued function, not identically 0, which is continuous and satisfies the equation (1) for all real x and y, then either $f(x)=\cos\alpha x$ or $f(x)=\cosh\alpha x$ for some constant α. Cauchy's result has been extended by Kaczmarz [2], who has proved that this result still holds if the condition of continuity is replaced by measurability, and his argument covers the case in which f takes complex values. Both writers, however, restrict themselves to the case of real x.

In this note we consider solutions of the equation (1) for complex values of the argument. It is no longer true that every continuous solution is of the form $f(x)=\cosh\alpha x$, and in order to obtain such a result we have to impose a condition equivalent to the differentiability of f (as a complex-valued function of a complex variable) at a single point. We prove in fact the following

THEOREM. *Suppose that there exists a complex-valued function f defined for all complex z, continuous at at least one point, and satisfying the functional equation*

$$f(z+w)+f(z-w)=2f(z)f(w) \tag{2}$$

for all complex z and w. Then either f is identically 0 *or identically* 1, *or there exist complex constants* α, β, *not both* 0, *such that for every* $z=x+iy$

$$f(z)=f(x+iy)=\cosh(\alpha x+\beta y).$$

Further, if f is differentiable at at least one point which is not a zero of $\sinh(\alpha x+\beta y)$, *then* $\beta=i\alpha$.

2. The only constant solutions of the equation (2) are the functions which are identically 0 or identically 1, for if $f(z)=k$ for all z, then $2k=2k^2$, and so $k=0$ or 1.

We suppose now that f is not identically constant (and, in particular, is not identically 0). Taking z to be a point where $f(z)\neq 0$, and setting $w=0$ in (2), we obtain immediately that

$$2f(z) = 2f(z)f(0),$$

and so $f(0)=1$. Further, f is even, for if in (2) we take $z=0$ we get

$$f(w)+f(-w) = 2f(0)f(w) = 2f(w),$$

and so $f(-w)=f(w)$.

3. We prove next that *if f is a solution of* (2) *which is continuous at at least one point, then f is continuous everywhere.* We observe first that if we take $w=z$ in (2) we obtain

$$f(2z)+1 = 2f^2(z). \tag{3}$$

Hence also, by (2) and (3),

$$\begin{aligned} f(z+h)f(z-h)-f^2(z) &= \tfrac{1}{2}\{f(2z)+f(2h)\}-f^2(z)\\ &= \tfrac{1}{2}\{2f^2(z)-1+f(2h)\}-f^2(z)\\ &= \tfrac{1}{2}\{f(2h)-1\}. \end{aligned} \tag{4}$$

From (4) we deduce that if f is continuous at any particular z, then also $f(2h)\to 1$ as $h\to 0$, so that f is continuous at 0. Hence also, for any z,

$$f(z+h)f(z-h)-f^2(z)\to 0 \text{ as } h\to 0.$$

Further, by (2) with $w=h$, we have that

$$f(z+h)+f(z-h)\to 2f(z) \text{ as } h\to 0, \tag{5}$$

and so also

$$\{f(z+h)-f(z)\}\{f(z-h)-f(z)\}\to 0 \text{ as } h\to 0. \tag{6}$$

It follows now that f is continuous at z, for if f is discontinuous at z, there exists $d>0$ and some sequence (h_n) converging to 0 such that $|f(z+h_n)-f(z)|\geqq d$. By (6), we can find $\delta>0$ such that

$$|f(z+h)-f(z)|\,|f(z-h)-f(z)| < \tfrac{1}{2}d^2,$$

whenever $|h|<\delta$, and this implies that

$$|f(z-h_n)-f(z)| < \tfrac{1}{2}d,$$

whenever $|h_n|<\delta$. But then

$$|f(z+h_n)+f(z-h_n)-2f(z)| \geqq |f(z+h_n)-f(z)| - |f(z-h_n)-f(z)| > \tfrac{1}{2}d$$

for $|h_n| < \delta$, and this contradicts (5). Hence f is continuous at z, and so is continuous everywhere.

4. In the remainder of the proof we assume without further reference that f is continuous everywhere and not identically 0 or 1.

We show now that *if f satisfies the equation* (2) *for all real z and w, there exists a complex constant α such that $f(z) = \cosh \alpha z$ for all real z.* More precisely, we show that if f satisfies the equation (2) for all real z and w, then for real z the function $z \to f(z)$ (as a complex-valued function of a real variable) has derivatives of all orders and satisfies the differential equation

$$f''(z) = f''(0)f(z). \tag{7}$$

It follows from this that

$$f(z) = \cosh \alpha z,$$

where $\alpha = \pm\sqrt{f''(0)}$ (for if $f''(0) = 0$, then f is linear, and so is identically 1, so that $f(z) = \cosh \alpha z$ with $\alpha = 0$).

The result (7) is proved in the paper of Kaczmarz, but for the sake of completeness we reproduce Kaczmarz's proof. Integrating both sides of the equation (2) with respect to w from 0 to v, where v and z are real, we obtain

$$\begin{aligned} 2f(z)\int_0^v f(w)dw &= \int_0^v f(z+w)dw + \int_0^v f(z-w)dw \\ &= \int_{-v}^v f(z+w)dw = \int_{z-v}^{z+v} f(w)dw. \end{aligned} \tag{8}$$

Since f is continuous and $f(0) = 1$, the integral on the left is nonzero for all small v, and since the function $z \to \int_{z-v}^{z+v} f(w)dw$ (for a fixed v) has a derivative at z, it follows that f (as a function of a real variable) has a derivative $f'(z)$ at z and that

$$2f'(z)\int_0^v f(w)dw = f(z+v) - f(z-v)$$

for all real z and all real v. Further, since the function $z \to f(z+v) - f(z-v)$ (for a fixed v) has a derivative at z, we see that $f''(z)$ exists for all real z, and, more generally, that f has derivatives of all orders. Finally, to prove the equation (7) we observe that, by (2),

$$\begin{aligned} f(z+h) + f(z-h) - 2f(z) &= 2f(z)\{f(h) - 1\} \\ &= f(z)\{f(h) + f(-h) - 2f(0)\}, \end{aligned}$$

and taking z and h to be real, dividing by h^2, and making $h \to 0$, we obtain immediately the required result.

5. We return now to complex values of the argument, and we suppose that f satisfies the equation (2) for all complex z and w. Writing $z=x+iy$, where x and y are real, we see from the result of Section 4 applied to the functions $x\to f(x)$ and $y\to f(iy)$ that there exist complex constants α, β such that for all real x and y

$$f(x) = \cosh \alpha x, \qquad f(iy) = \cosh \beta y. \tag{9}$$

We prove that α and β are not both 0, *and that for all complex z*

$$f(z) = \cosh (\alpha x \pm \beta y), \tag{10}$$

where the choice of the sign $\pm$ is the same for all z.

By (2) and (3), we have

$$\begin{aligned} \{f(z+w) - f(z-w)\}^2 &= \{f(z+w) + f(z-w)\}^2 - 4f(z+w)f(z-w) \\ &= 4f^2(z)f^2(w) - 2\{f(2z) + f(2w)\} \\ &= 4f^2(z)f^2(w) - 4\{f^2(z) + f^2(w) - 1\} \\ &= 4\{f^2(z) - 1\}\{f^2(w) - 1\}, \end{aligned} \tag{11}$$

and taking $z=x$, $w=iy$ in (2) and (11) and using (9) we obtain

$$\begin{aligned} f(x+iy) + f(x-iy) &= 2 \cosh \alpha x \cosh \beta y, \\ f(x+iy) - f(x-iy) &= \pm\, 2 \sinh \alpha x \sinh \beta y, \end{aligned} \tag{12}$$

and so also

$$f(x+iy) = \cosh (\alpha x \pm \beta y), \tag{13}$$

where the choice of the sign $\pm$ in (12) and (13) may depend on x and y. Since f is not identically 1, it follows immediately that α and β are not both 0. Further, if one of α, β is 0, the expression on the right of (12) is equal to 0, so that (13) holds with no ambiguity of sign.

It remains to show that if neither of α, β is 0, the choice of the sign in (12) and (13) is independent of x and y. By the continuity of f, the sign can change from $+$ to $-$ or vice versa only at a zero of $\sinh \alpha x \sinh \beta y$, but if α and β are wholly imaginary, these zeros form a rectangular grid in the plane, dividing the plane into an infinite set of rectangles, while if one of α, β is wholly imaginary, the zeros divide the plane into parallel strips. It can be shown, by considering the values of f at certain points in these rectangles or strips and using the equation (2), that the same sign holds for all the rectangles or strips, but the proof proceeds by enumeration of various cases and is rather tedious. We therefore prefer to give an alternative argument, more sophisticated, but perhaps more elegant.

We observe first that the identity (8) holds for complex z and real v provided that the integral on the right is taken along the segment joining $z-v$ to $z+v$. Hence if $h>0$ and v is real, we have

$$\begin{aligned}\frac{1}{h}\{f(z+h)-f(z)\}\int_0^v f(w)dw &= \frac{1}{h}\left\{\int_{z+h-v}^{z+h+v} f(w)dw - \int_{z-v}^{z+v} f(w)dw\right\}\\ &= \frac{1}{h}\int_{z+v}^{z+h+v} f(w)dw - \frac{1}{h}\int_{z-v}^{z+h-v} f(w)dw,\end{aligned}$$

where the integrals are taken along the appropriate segments. It follows that the limit

$$l(z) = \lim_{h\to 0+}\{f(z+h)-f(z)\}/h$$

exists and satisfies

$$l(z)\int_0^v f(w)dw = f(z+v) - f(z-v).$$

Taking $v=x$ and $z=iy$, and using the fact that f is even, we obtain

$$\begin{aligned}f(x+iy)-f(x-iy) &= l(iy)\int_0^x f(w)dw = l(iy)\int_0^x \cosh(\alpha w)dw \qquad (14)\\ &= \alpha^{-1}l(iy)\sinh\alpha x,\end{aligned}$$

and combining this with (12) we have $l(iy)=\pm\alpha\sinh\beta y$. Here, however, the sign depends only on y, so that, by (14),

$$f(x+iy)-f(x-iy) = \pm\sinh\alpha x\sinh\beta y,$$

and so also

$$f(z) = \cosh(\alpha x\pm\beta y),$$

where the choice of the sign $\pm$ is independent of x. Applying this result to the function $iz\to f(z)$, we see also that

$$f(z) = \cosh(\alpha x\pm\beta y),$$

where the choice of the sign is independent of y, and so (10) holds, with the choice of sign independent of x and y.

6. We now absorb the sign $\pm$ into the β, so that we have for all $z=x+iy$

$$f(z) = \cosh(\alpha x+\beta y).$$

It remains to prove that if f is differentiable at any point z which is not a zero

of sinh $(\alpha x+\beta y)$, then $\beta=i\alpha$. If h is real, then

$$\lim_{h\to 0}\{f(z+h)-f(z)\}/h=\frac{\partial}{\partial x}\cosh(\alpha x+\beta y)=\alpha\sinh(\alpha x+\beta y)$$

and

$$\lim_{h\to 0}\{f(z+ih)-f(z)\}/ih=-i\frac{\partial}{\partial y}\cosh(\alpha x+\beta y)=-i\beta\sinh(\alpha x+\beta y),$$

and since f is differentiable at z, the two expressions on the right are equal, whence $\beta=i\alpha$, as required.

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THE 152-nd PROOF OF THE LAW OF QUADRATIC RECIPROCITY

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In a paper devoted mainly to the proof of the quadratic reciprocity theorem for imaginary quadratic fields (using an elegant refinement of a technique of Eisenstein), Kubota [1] gives, by way of illustration of the ideas, a proof of the ordinary law of quadratic reciprocity by analytic means. Kubota's proof, which by the present author's hasty count should be about the 151-st, can be abstracted to give a simple algebraic proof in which the Jacobi symbol is factored explicitly into a product of units in a field of roots of unity. Both Kubota's proof and its algebraic skeleton, about to be presented, are in the large class of those depending on the "Gauss' lemma."

Let m be an odd positive integer, and M be a set of $\frac{1}{2}(m-1)$ integers such that the set $\{0, M, -M\}$ is a complete system of residues mod m; such a set M is usually called a "half-system" of residues mod m. If n is another odd positive integer relatively prime to m, then for any a in M, there is an a' in M such that $na\equiv\pm a'$ mod m. The Jacobi symbol (n/m) may be defined to be $+1$ if the negative sign appears an even number of times as a runs through M, and

to be -1 if it appears an odd number of times. The multiplicativity of (n/m) in n and m is readily proved, and the Gauss' lemma asserts that if n and m are primes, say p and q, then $(p/q)=1$ if p is a quadratic residue mod q, and $(p/q)=-1$ if p is a non-residue.

Let ζ be a primitive mth root of unity. Then

$$\left(\frac{n}{m}\right)=\prod_{a\in M}\frac{\zeta^{na}-\zeta^{-na}}{\zeta^{a}-\zeta^{-a}}$$

for the factors appearing in the numerator are, in their totality, the same as those in the denominator up to sign, and the sign will be positive or negative according as $na\equiv +a'$ or $na\equiv -a'$ (mod m) for some a' in M. Now let η be a primitive nth root of unity. Then, since n is odd, we have identically in x,

$$x^n - x^{-n} = \prod_{b \bmod n} (\eta^b x - \eta^{-b}x^{-1}).$$

We may therefore factor the numerators in the expression for the Jacobi symbol, and after canceling the denominators, obtain

$$\left(\frac{n}{m}\right)=\prod_{a\in M}\prod_{\substack{b \bmod n\\ b\not\equiv 0}}(\eta^b\zeta^a-\eta^{-b}\zeta^{-a}).$$

Let N be a half-system of residues mod N. Then we may write

$$\left(\frac{n}{m}\right)=\prod_{a\in M}\prod_{b\in N}(\eta^b\zeta^a-\eta^{-b}\zeta^{-a})(\eta^{-b}\zeta^a-\eta^{b}\zeta^{-a}),$$

and after multiplying out, we obtain

$$\left(\frac{n}{m}\right)=\prod_{a\in M, b\in N}[(\zeta^{2a}+\zeta^{-2a})-(\eta^{2b}+\eta^{-2b})].$$

Now in the product on the right there are $\frac{1}{2}(m-1)\cdot\frac{1}{2}(n-1)$ factors, and interchanging the roles of m and n simply changes the sign of each. Therefore,

$$\left(\frac{n}{m}\right)=(-1)^{\frac{1}{2}(m-1)\cdot\frac{1}{2}(n-1)}\left(\frac{m}{n}\right),$$

which is the law of quadratic reciprocity.

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A MAPPING OF THE PROJECTIVE n-SPACE ON THE PROJECTIVE PLANE

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Introduction. In a previous paper [3] we introduced a transformation T of the Euclidean n-space into a space P_n which lies in a Euclidean plane E. It was shown that the incidence properties of P_n were identical with those of E_n, and this fact was used to solve graphically certain problems in E_n.

In order to introduce and apply this transformation the notion of distance had to be used. It is possible, however, to introduce a more general transformation T which does not depend on the notion of distance and can be used, therefore, to transform a projective space p_n into a space P_n which lies on a projective plane π. In this note we shall introduce such a transformation and study some of its properties.

In particular, it can be shown, by arguments similar to those used in [3], that the incidence properties of the projective space p_n and the new space P_n are identical. This will lead to a new proof of the fact that the axioms of the projective plane together with the axioms of existence of k-planes are sufficient to determine all the properties of the projective space p_n.

Transformation T. Consider a real projective plane π with n concurrent lines $OX_1, OX_2, \cdots, OX_n$ lying in it, assuming $n>2$. These lines do not have to be distinct. We shall prove that it is possible to establish a one-to-one correspondence between the points of the real projective n-dimensional space p_n and the ordered sets of n points $A_1, A_2, \cdots, A_n$ lying on the lines $OX_1, OX_2, \cdots, OX_n$ respectively, having the following property:

If the point C of p_n, which transforms into the set $C_1, C_2, \cdots, C_n$, is collinear with the two points A and B, the lines C_iC_j must pass through the points $X_{i,j}$ respectively, $X_{i,j}$ being the intersection of A_iA_j and B_iB_j.

The above correspondence subject to this restriction is the transformation which is under consideration in this paper. It will be denoted by T.

To show that such transformations exist, consider the points O, A_i and B_i on the line OX_i of the plane π, where A_i and B_i are chosen arbitrarily. It is possible to construct a net of rationality on OX_i based on $O=\infty$, $A_i=1$ and $B_i=0$. Thus to each of the n lines OX_i will correspond a net of rationality which depends on A_i and B_i.

It is clear that a construction similar to the one used in [3] could now be used to find the transformation T between p_n and the plane π: Point $A(a_1, \cdots, a_n)$ of p_n will go into a set of points $A_1, A_2, \cdots, A_n$ lying on the lines $OX_1, OX_2, \cdots, OX_n$ of projective plane π respectively, such that the real number associated with A_i, from the corresponding net of rationality, is a_i. It is easy to show, by using arguments similar to those used in [3], that this

correspondence leads to a transformation of the type described above. It is also clear that this transformation is not unique and depends on the choice of the base points of the nets of rationality on OX_i.

Under the transformation T, the space p_n will transform into a new space which is denoted by P_n. The lines $OX_1, \cdots, OX_n$ are called the n axes of the space P_n.

Some definitions. The set of n points $A_1, A_2, \cdots, A_n$ lying on the n axes respectively, is called a point in the space P_n and is denoted by $P(A_n)$. A_i is called the ith coordinate of this point.

The set of $n-1$ points $X_{i,i+1}$ (the points of intersection of A_iA_{i+1} and B_iB_{i+1}) represents the line determined by the points $P(A_n)$ and $P(B_n)$ and is denoted by $L_n(X)$. The $n-1$ points $X_{i,i+1}$ are called the $n-1$ components of the line $L_n(X)$.

We shall call a line a 1-plane and a point a 0-plane and define an i-plane by induction: Consider two distinct $(i-1)$-planes, $2 \leqq i < n$, having an $(i-2)$-plane in common. The set of all $(i-1)$-planes having two distinct $(i-2)$-planes in common with the two given $(i-1)$-planes is called an i-plane.

An i-plane and a j-plane $(j \leqq i < n)$ intersect if they have a $(j-1)$-plane in common.

n-planes in P_n. Next we shall state some theorems involving some of the properties of n-planes. The proofs of these theorems are simple but in most cases tedious, and will be omitted in this note. We mention only that the proofs are based only on the axioms of the real projective plane and Desargues' theorem. It should also be noted that, since it was assumed that $n>2$, the results obtained in this note do not hold for non-Desarguian projective spaces.

THEOREM 1. *In every $(n-m)$-plane there exist at least m lines, each line having m distinct components, such that $n-m$ successive components of the line coincide and the other $m-1$ components are single and distinct. These lines are called the m axes of the $(n-m)$-plane.*

If in Theorem 1 we replace m by 1, the following theorem follows:

THEOREM 2. *In every $(n-1)$-plane, lying in P_n, there exists at least one line such that all the components of that line coincide. This line is called the axis of the $(n-1)$-plane.*

COROLLARY. *If there exist two such lines in the $(n-1)$-plane, then there exist infinitely many lines having this property and all of them are in the same plane.*

THEOREM 3. *Given the set of all the $(m-1)$-planes imbedded in a given m-plane $(m<n)$, the jth axis of each of these $(m-1)$-planes will intersect the jth axis of the given m-plane, for $j=1, 2, \cdots, m-2$.*

The next two theorems are rather important for later developments and their proofs will be given.

THEOREM 4. *Given any line t in p_n, it is possible to find a transformation T such that, under T, all the components of the image of t coincide.*

Given any two points $M(m_i)$ and $N(n_i)$ in p_n, we shall construct the previously mentioned nets of rationality in such a way that all the $n-1$ components of the image of MN will coincide with each other in π. Consider two lines m and n in π, neither of them passing through O, and find the intersection of these two lines with OX_i. Let us call the points of intersection M_i and N_i, respectively, and assign the numbers m_i and n_i (coordinates of the points M and N) to the points M_i and N_i. Now a net of rationality is determined by any 3 points of the net and hence O, M_i, and N_i determine a net of rationality on OX_i. This net of rationality will determine a transformation T such that, under this transformation, the images of all the coordinates of the points M and N are collinear (in π) and hence all the components of the image of the line MN coincide. This proves the theorem.

Theorem 4 can be used to prove the following theorem:

THEOREM 5. *Any line in P_n is either incident with a given $(n-1)$-plane of P_n or intersects the $(n-1)$-plane in a point.*

The proof follows easily from the preceding discussions. Without loss of generality, it can be assumed that all the components of the given line t coincide with each other. (We only have to find the proper transformation T.) Theorem 2 shows that in any $(n-1)$-plane, lying in P_n, there exists at least one line such that all its components coincide. Call this line t'. It is clear that two such lines always intersect and therefore the given line and the given $(n-1)$-plane will always have, at least, one point in common.

On axioms of the projective n-space. It can be shown, using arguments similar to those used in [3], that P_n is actually a model of the projective n-space p_n. Therefore Theorem 5 holds for p_n as well as P_n.

Now it is well known that the axioms of the projective plane, together with the statement of Theorem 5, are sufficient for proving all the incidence properties of the projective n-space, [1]. But since Theorem 5 was proved by using the axioms of projective plane, and only those axioms, it follows that all the incidence properties of P_n, and consequently p_n, can be proved using only the axioms of the projective plane.

This is a well-known fact, of course, and other proofs of it exist, [2].

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MATHEMATICAL NOTES

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ON A FUNCTION OF VAN DER WAERDEN

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Real function theory has always displayed a curious pathology which manifests itself in a wealth of unusual counterexamples. The subject of this note provides another such counterexample.

Let C denote the Cantor set on the closed interval $[0, 1]=I$ and m Lebesgue measure on the real line R. Then it is well known that $m(C)=0$ and that there exists a function $c: I\to R$ (the Cantor function) which is nondecreasing, "climbs" on C, is constant on every subinterval of $I-C$, and provides an example of a nonconstant continuous singular function.

One might, in a similar manner, address himself to a question of the following nature. Does there exist a continuous mapping $f: I\to R$ and a set $A\subset I$ such that f is nowhere differentiable, $f\equiv 0$ on A, and $m(A)>0$? Intuitively, one is tempted to answer no since f would be constant over the relatively substantial set A. But, as one might expect, such is not the case. In fact, given $\epsilon>0$, it is possible to construct such a function f on I with the corresponding $A\subset I$ such that $m(A)>1-\epsilon$. One such construction proceeds as follows.

Let $\epsilon>0$ be given and let C_ϵ denote a Cantor-like set on I such that $m(C_\epsilon)>1-\epsilon$. Then

$$C_\epsilon = I - \bigcup_{n=1}^{\infty} E_n,$$

where $E_n=(a_n, b_n)$ is an open interval in I. Consider van der Waerden's example [1] of an everywhere continuous nowhere differentiable function

$$W(t) = \sum_{n=0}^{\infty} \frac{\{10^n t\}}{10^n},$$

where $\{t\}$ denotes the distance from t to the nearest integer, restricted to I. Note that $W(0)=W(1)=0$. Now define f on I as follows. Set $f\equiv 0$ on C_ϵ and define f to be a copy of W on the closure $\overline{E}_n$ of E_n for each n. Thus f is a continuous nondifferentiable function on I whose set C_ϵ of zeros has measure greater than $1-\epsilon$.

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ON CERTAIN NONLINEAR RECURRING SEQUENCES

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For each integer $r=0, 1, 2, \cdots$, define the sequence $\{b_n^{(r)}\}$ by

$$(1)\qquad b_0^{(r)} = 1, \qquad b_{n+1}^{(r)} = \prod_{i=1}^{n} b_i^{(r)} + r \qquad \text{for } n \geqq 0.$$

Letting $\rho=r/2$ and $\beta_n^{(r)}=b_n^{(r)}-\rho$, there are the following equivalent formulations:

$$(2)\qquad b_0^{(r)} = 1, \qquad b_1^{(r)} = 1 + r, \qquad b_{n+1}^{(r)} = (b_n^{(r)} - \rho)^2 + (2\rho - \rho^2) \qquad \text{for } n \geqq 1,$$

and

$$(3)\qquad \beta_1^{(r)} = 1 + \rho, \qquad \beta_{n+1}^{(r)} = (\beta_n^{(r)})^2 + \rho(1 - \rho) \qquad \text{for } n \geqq 1,$$

as well as

$$(4)\qquad b_0^{(r)} = 1, \quad b_1^{(r)} = 1 + r, \quad b_{n+1}^{(r)} = (b_n^{(r)} - 1)(b_n^{(r)} - (r - 1)) + 1 \qquad \text{for } n \geqq 1.$$

Thus,

$$(5)\qquad \begin{aligned}
\{b_n^{(0)}\} &= \{1, 1, 1, 1, 1, 1, 1, \cdots\} \\
\{b_n^{(1)}\} &= \{1, 2, 3, 7, 43, 1807, 3263443, \cdots\} \\
\{b_n^{(2)}\} &= \{1, 3, 5, 17, 257, 65537, 4294967297, \cdots\} \\
\{b_n^{(3)}\} &= \{1, 4, 7, 31, 871, 756031, 571582116931, \cdots\} \\
\{b_n^{(4)}\} &= \{1, 5, 9, 49, 2209, 4870849, \cdots\}.
\end{aligned}$$

A consequence of (4) is that every term of $\{b_n^{(r)}-1\}$ is divisible by each of the preceding terms. On the other hand, it follows from (1) that for fixed r, the terms of $\{b_n^{(r)}\}$ are pairwise relatively prime, since r is relatively prime to all terms of $\{b_n^{(r)}\}$ by the recursion relation and since $b_j^{(r)}\equiv r \pmod{b_i^{(1)}}$ for all $j>i$. The fact was used by Polya, in connection with the sequence $\{b_n^{(2)}\}$ of the "Fermat numbers," to prove the existence of infinitely many primes. Clearly the same proof holds using any sequence $\{b_n^{(r)}\}$ with $r>0$.

A consequence of (3) is that the successive terms of $\{\beta_n^{(r)}\}$ are *approximately* obtained by successive squaring. More specifically, if we attempt to find a real number $\theta=\theta(r)$ such that $\theta^{2^n}\approx\beta_{n+1}$, we may proceed as follows: Let $\theta_n=\sqrt[2^n]{\beta_{n+1}}$. Then

$$\begin{aligned}
\theta_{n+1} &= \sqrt[2^{n+1}]{\beta_{n+2}} = \sqrt[2^{n+1}]{[\beta_{n+1}^2 + \rho(1-\rho)]} = \sqrt[2^n]{\beta_{n+1}}\ \sqrt[2^{n+1}]{[1 + \rho(1-\rho)/\beta_{n+1}^2]} \\
&= \theta_n(1 + \rho(1-\rho)/\beta_{n+1}^2)^{2^{-(n+1)}}
\end{aligned}$$

Assuming that the θ_n's converge to θ, we have

$$(6)\qquad \theta(r) = \beta_1 \prod_{i=1}^{\infty}\left(1+\frac{\rho(1-\rho)}{(\beta_i^{(r)})^2}\right)^{2^{-i}} = (1+\rho)\prod_{i=1}^{\infty}\left(1+\frac{\rho(1-\rho)}{(\beta_i^{(r)})^2}\right)^{2^{-i}}.$$

For $r=0$ and $r=2$, $\rho(1-\rho)=0$, leading to $\theta(0)=1$ and $\theta(2)=2$. Specifically, $b_n^{(0)}=1$ for all n, and $b_n^{(2)}=2^{2^{n-1}}+1$ for $n\geqq 1$. For all other values of r the product is nontrivial. The rapid growth of the β_i's assures the convergence of the products (6), which is further accelerated by the exponents 2^{-i}.

From (6), we derive

$$(7)\qquad \theta^{2^n}(r) = \theta_n^{2^n}(r)\prod_{i=n+1}^{\infty}\left(1+\frac{\rho(1-\rho)}{(\beta_i^{(r)})^2}\right)^{2^{n-i}} = \beta_{n+1}^{(r)}\prod_{i=n+1}^{\infty}\left(1+\frac{\rho(1-\rho)}{(\beta_i^{(r)})^2}\right)^{2^{n-i}}.$$

Thus,

$$\frac{\theta^{2^n}(r)}{\beta_{n+1}^{(r)}}$$

lies between

$$\left(1+\frac{\rho(1-\rho)}{(\beta_{n+1}^{(r)})^2}\right)^{1/2}$$

and

$$\left(1+\frac{\rho(1-\rho)}{(\beta_{n+1}^{(r)})^2}\right),$$

being closer to the former value, or approximately

$$1+\frac{\rho(1-\rho)}{2(\beta_{n+1}^{(r)})^2}.$$

Note that for $r=1$, the partial products of (6) *underestimate* $\theta(1)$, while for all $r>2$, the partial products *overestimate* $\theta(r)$, in view of the sign of $\rho(1-\rho)$. As an example,

$$(8)\qquad \begin{aligned}\theta(1) &= \frac{3}{2}\left(1+\frac{1}{3^2}\right)^{1/2}\left(1+\frac{1}{5^2}\right)^{1/4}\left(1+\frac{1}{13^2}\right)^{1/8}\left(1+\frac{1}{85^2}\right)^{1/16}\left(1+\frac{1}{3613^2}\right)^{1/32}\cdots\\ &= 1.5979102\cdots.\end{aligned}$$

The comparison of $\theta^{2^n}(1)$ with $\beta_{n+1}^{(1)}$ is as follows:

n	0	1	2	3	4	5
θ^{2^n}	1.59791	2.55331	6.51942	42.50294	1806.50006	3263442.50000
β_{n+1}	1.50000	2.50000	6.50000	42.50000	1806.50000	3263442.50000

Naturally it would be interesting to determine the algebraic or transcendental character of the real numbers $\theta(r)$ for $r=1, 3, 4, 5, 6, \cdots$.

For small values of r, the sequences $\{b_n^{(r)}\}$ are worthy of individual study. Thus, $\{b_n^{(2)}\}$ is the famous sequence of Fermat numbers. Perhaps equally interesting is $\{b_n^{(1)}\}$, which arises in a wide variety of number-theoretic situations. For example, it is conjectured that the closest approximation to 1 from below which is a sum of k reciprocal integers is given, for every value of k, by

$$\frac{1}{b_1^{(1)}}+\frac{1}{b_2^{(1)}}+\cdots+\frac{1}{b_k^{(1)}}=1-\frac{1}{b_{k+1}^{(1)}-1}.$$

FUNCTIONS WITH UNIFORM INVERSES

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If A and B are topological spaces and f is continuous on A onto B, then f is said to be an $(n, 1)$ function, n a positive integer, provided that for each $b\in B$, $f^{-1}(b)$ contains exactly n elements of A. Such functions have been studied by Gilbert [**1**], Harrold [**2, 3**], Roberts [**4**] and others but results are largely for continua, especially linear graphs, and compact 2-manifolds. What we are calling an $(n, 1)$ function is often called an exactly $(n, 1)$ function in this literature.

In this paper we will take A and B to be nondegenerate, connected subsets of the real line (including the entire real line) and show that there exists an $(n, 1)$ function $f(A)=B$ when, and only when, A and B are both open and n is odd.

LEMMA 1. *If A and B are non-degenerate connected subsets of the real line and n is a positive even integer, then there are no $(n, 1)$ functions f on A onto B.*

Proof. Suppose that f is an $(n, 1)$ function on A onto B, where A and B are connected subsets of the reals and n is even. Let $b\in B$ be any interior point of B and consider the n points $f^{-1}(b)$. Let these points be $a_1<a_2<\cdots<a_n$ and denote the intervals $[a_i, a_{i+1}]$ by I_i. For each $i=1, 2, \cdots, n-1$, $I_i\subset A$ since A is connected. Of the $n-1$ intervals I_i, at least $n/2$ of them contain relative minima of f smaller than b or at least $n/2$ of them contain relative maxima greater than b. We take the case of the guaranteed relative minima remarking that the case of the maxima is similar.

Let b_m be the absolute minimum of $f(x)$ on $[a_1, a_n]$. If more than $n/2$ of the points $f^{-1}(b_m)$ are in $[a_1, a_n]$, then there is a b' slightly greater than b_m for which $f^{-1}(b')$ must have more than n points since b' can be selected to have at least two distinct inverses near each point of $f^{-1}(b_m)$. (Since n is finite and f is continuous, the reader will have no trouble selecting epsilons, deltas, etas, etc., to formalize the arguments.)

If no more than $n/2$ of the points $f^{-1}(b_m)$ are in $[a_1, a_n]$, then there is at least one point $a\in A-[a_1, a_n]$ for which $f(a)=b_m$. Since at least $n/2$ of the intervals I_i contain relative minima of f, there is a b'' slightly smaller than b for which $f^{-1}(b'')$ has at least n points in $[a_1, a_n]$ since corresponding to each I_i wherein f

falls below b at all, there will be at least two distinct points of I_i in $f^{-1}(b'')$. Since $f(a)=b_m<b$ and $a\notin[a_1, a_n]$, $f^{-1}(b'')$ must also have a point in $[a_n, a]$, if $a>a_n$, or in $[a, a_1]$, if $a<a_1$. In any case $f^{-1}(b'')$ contains at least $n+1$ points and the lemma is proved.

We now exhibit a (3, 1) function defined from all reals to all reals. In a similar way one can construct an $(n, 1)$ function, n odd, from any open interval of reals. In the (x, y)-plane join successive points of the following doubly infinite sequence of points by line segments. $\cdots$ $(-9, 3)$, $(-8, 2)$, $(-7, 3)$, $(-6, 2)$, $(-5, 1)$, $(-4, 2)$, $(-3, 1)$, $(-2, 0)$, $(-1, 1)$, $(0, 0)$, $(1, -1)$, $(2, 0)$, $(3, -1)$, $(4, -2)$, $(5, -1)$, $(6, -2$, $(7, -3)$, $\cdots$. The function corresponding to this graph is (3, 1).

LEMMA 2. *If A and B are nondegenerate connected subsets of the real line such that B contains at least one of its endpoints, then there is no $(n, 1)$ function f on A onto B unless $n=1$.*

Proof. Assume that B contains its lower endpoint, b, and that f is $(n, 1)$ on A onto B. In view of Lemma 1 we may assume that $n>2$. $f^{-1}(b)$ consists of exactly n points of which at least $n-2$ are interior to A and correspond to absolute minima of f on A. Thus there is a b' slightly larger than b for which $f^{-1}(b')$ contains at least $2n-4$ points near these $n-2$ points that are interior points of A and absolute minima of f on A. Near the other two points of $f^{-1}(b)$, $f^{-1}(b')$ contains at least two points; hence $f^{-1}(b')$ has at least $2n-4+2=2n-2$ points. However, $2n-2>n$ if $n>2$ which is a contradiction. This completes the proof in case B contains its lower endpoint. The case where B contains its upper endpoint but not its lower endpoint is disposed of by considering the negative of f.

As is well known, the image of a closed segment under a continuous real function is a closed segment and hence, by Lemma 2, if A is a closed interval, there are no $(n, 1)$ functions f on A onto B unless $n=1$. This is also implied by the following lemma which is needed because under real continuous functions, in general, an interval with one endpoint may go onto an open interval.

LEMMA 3. *If A and B are nondegenerate connected subsets of the real line such that A contains at least one of its endpoints, there is no $(n, 1)$ function f on A onto B unless $n=1$.*

Proof. Assume that A contains its lower endpoint, a. The other possible case is treated similarly. Consider the n points of $f^{-1}f(a)$. Clearly a is the smallest of these numbers; let a' be the largest number of the collection $f^{-1}f(a)$. If a' is the last point of A, then A is the closed interval $[a, a']$ and its image, B, must be a closed interval in violation of Lemma 2. If a' is not the last point of A, then $A-[a, a']$ is not empty but either $f(x)>a$ for all $x\in A-[a, a']$ or $f(x)<a$ for all $x\in A-[a, a']$ since a' is the last point of $f^{-1}f(a)$. In the first case, f has an absolute minimum on A occurring for some number in $[a, a']$ and in the second case, f has an absolute maximum on A occurring for some number in $[a, a']$. In

either case, this implies that B contains at least one of its endpoints in violation of Lemma 2 and the proof is complete.

Clearly, the proofs given above are valid when A and B are arcs with or without one or both endpoints. We summarize these results in the following theorem.

THEOREM 1. *If A and B are nondegenerate connected subsets of arcs and if n is a positive integer, then there exist $(n, 1)$ functions f on A onto B when, and only when, A and B are open (i.e., both A and B lack both endpoints) and n is odd.*

The following theorem shows how limited the examples of $(n, 1)$ functions, n odd, must be. They are all about like the $(3, 1)$ example given above.

THEOREM 2. *Let $n>0$ be an odd integer and let f be an $(n, 1)$ function on A onto B, where A and B are open intervals of the real line. For $b\in B, f^{-1}(b)$ contains as many relative maxima for f as it contains relative minima. In particular, f can not have an absolute maximum or absolute minimum on A.*

Proof. Suppose $f^{-1}(b)$ contains M relative maxima and m relative minima with either $m>0$ or $M>0$. If $m\neq M$ we may assume, without loss of generality, that $0\leqq m<M$. There are n numbers in $f^{-1}(b)$ but there is a number, b', slightly smaller than b such that $f^{-1}(b')$ contains at least $n-m+M$ numbers since there are M maxima and $f^{-1}(b')$ will have at least two elements near each of these maxima. However, $n-m+M>n$ since $M>m$ and this is a contradiction.

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ON A THEOREM OF USPENSKY

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Let α be a real number and define S_α to be the sequence $([\alpha], [2\alpha], [3\alpha], \cdots)$, where $[\ ]$ denotes the greatest integer function. The following result has been obtained by Uspensky [1]:

THEOREM. *Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are n positive real numbers which have the property that every positive integer occurs exactly once in some one of the sequences $S_{\alpha_1}, S_{\alpha_2}, \cdots, S_{\alpha_n}$. Then $n<3$.*

The proof given by Uspensky is somewhat elaborate and based on an approximation theorem of Kronecker. It is the purpose of this note to present a direct and elementary proof that $n<3$.

Proof. Assume that $n \geqq 3$ and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n positive real numbers satisfying the hypothesis of the theorem. Certainly $\alpha_i > 1$ for all i and without loss of generality, we can assume that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Since 1 occurs in just one of the S_{α_i}, then it must occur in S_{α_1} and we have $[\alpha_1] = 1$. Thus $\alpha_1 = 1 + \delta$, where $0 < \delta < 1$.

Note that

$$\begin{aligned}[n\alpha_1] = l &\Rightarrow n\alpha_1 < l + 1 \\ &\Rightarrow (n+1)\alpha_1 = n\alpha_1 + \alpha_1 < l + 1 + 1 + \delta < l + 3 \\ &\Rightarrow [(n+1)\alpha_1] \leqq l + 2\end{aligned}$$

so that two consecutive integers cannot be missing from S_{α_1}.

Let m be the smallest positive integer which does not occur in S_{α_1}. Then m satisfies $(m-1)\delta < 1 \leqq m\delta$. By hypothesis, m must occur in some S_{α_i} and since $\alpha_2 < \alpha_3 < \cdots < \alpha_n$ then we have $[\alpha_2] = m$, i.e., $\alpha_2 = m + \epsilon$, where $0 \leqq \epsilon < 1$.

Now, let x be any integer which does not occur in S_{α_1}. Then x is of the form $[p\alpha_1] + 1$ for some p and $[(p+1)\alpha_1] - [p\alpha_1] > 1$. But

$$\begin{aligned}[(p+1)\alpha_1] - [p\alpha_1] > 1 &\Leftrightarrow [p + 1 + (p+1)\delta] - [p + p\delta] > 1 \\ &\Leftrightarrow [(p+1)\delta] - [p\delta] > 0 \\ &\Leftrightarrow p\delta < k\end{aligned}$$

and $(p+1)\delta \geqq k$ for some integer k.

Since $(p+m-1)\delta = p\delta + (m-1)\delta < k+1$ and $(p+m+1)\delta = (p+1)\delta + m\delta \geqq k+1$, there are two possibilities:

$$(1) \qquad (p+m)\delta \geqq k + 1$$

in which case the *next* integer which does not occur in S_{α_1} is

$$[(p + m - 1)\alpha_1] + 1 = p + m - 1 + k + 1 = p + k + m;$$

$$(2) \qquad (p+m)\delta < k + 1$$

in which case the *next* integer which does not occur in S_{α_1} is $[(p+m)\alpha_1] + 1 = p + m + k + 1$.

Since $x = [p\alpha_1] + 1 = p + [p\delta] + 1 = p + k$, we have shown that if x is any integer which is missing from S_{α_1} then the next integer which is missing from S_{α_1} is either $x + m$ or $x + m + 1$.

Notice that

$$\begin{aligned}[n\alpha_2] = y &\Rightarrow n\alpha_2 - 1 < y \leqq n\alpha_2 \\ &\Rightarrow (n+1)\alpha_2 = n\alpha_2 + \alpha_2 < y + 1 + m + 1 = y + m + 2\end{aligned}$$

and $(n+1)\alpha_2 = n\alpha_2 + \alpha_2 \geqq y + m$. Therefore, $[(n+1)\alpha_2] = y + m$ or $y + m + 1$.

To complete the proof, suppose that the kth integer x_k which is missing from S_{α_1} is exactly the kth term $y_k = [k\alpha_2]$ of S_{α_2}. We have just shown that $x_{k+1} = x_k + m$ or $x_k + m + 1$ and $y_{k+1} = y_k + m$ or $y_k + m + 1$. But two consecutive integers cannot

be missing from S_{α_1} and, by hypothesis, no integer can occur in both S_{α_1} and S_{α_2}. Consequently, we must have $x_{k+1}=y_{k+1}$. Since $x_1=m=y_1$, then by induction on k, we conclude that $x_n=y_n$ for all n. In other words, every positive integer occurs in either S_{α_1} or S_{α_2}. This is a contradiction to the assumption that $n\geqq 3$ and the proof is completed.

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A REMARK ON SINGULAR STURM-LIOUVILLE DIFFERENTIAL EQUATIONS

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1. In 1934, Hahn [1] showed, as an *intermediate result*, that any system of orthogonal polynomials, whose first derivatives also form an orthogonal system, satisfies a differential equation of the form

$$p(x)y'' + s(x)y' + \lambda y = 0 \tag{1}$$

for which the coefficients are real and at most $p(x)$ is quadratic, $s(x)$ is linear, and λ is constant. As a *final result* he showed by examining the singularities that (1) is equivalent, up to a linear transformation, to the Hermite, Jacobi, or Laguerre differential equations.

We propose to arrive at Hahn's *final result* by a different method; the method and the results are given in section 2. We first state some definitions and known results that we shall need in our development.

We shall consider (1) as a singular Sturm-Liouville differential equation. It has been shown in [2] that (1) can be transformed into the self-adjoint form usually given as a Sturm-Liouville differential equation by multiplying through by the factor

$$T(x) = \frac{1}{p}\exp\int\frac{s}{p}\,dx. \tag{2}$$

It is known [3, 4] that, for a certain discrete set of values of λ, there exists over an interval (a, b) a set of solutions, called eigenfunctions, of the Sturm-Liouville problem consisting of (1) and a set of appropriate boundary conditions which are to be considered in the next section. Upon considering the orthogonality of these eigenfunctions over the interval, a weight function $w(x)$ is obtained. It follows from [2] that the weight function that corresponds to (1), and is customarily assumed to be positive over (a, b), is the same as (2). That is, $T(x)=w(x)$.

2. We start with the equation of Hahn's *intermediate result* (1) which may be written as

$$(\alpha x^2 + \beta x + \gamma)y'' + (\delta x + \epsilon)y' + \lambda y = 0. \tag{3}$$

We assume that the Sturm-Liouville differential equation (1) has only real distinct singularities, and that these occur at the endpoints of the interval (a, b). The boundary conditions we apply are that the solutions and their first derivatives be continuous or bounded at these singularities, see [**3, 4, 5**].

In this paper we wish to show that the differential equation of the singular Sturm-Liouville eigenvalue problem consisting of (3) and the above boundary conditions is equivalent to the differential equations of Hermite, Jacobi, or Laguerre by relying on the fact that $w(x)$ must be positive and integrable over (a, b). That is, we prove the following theorem.

THEOREM. *Under the conditions that* (i) *the Sturm-Liouville differential equation* (3) *be real and the coefficient of y'' be at most quadratic, the coefficient of y' be at most linear, and the coefficient of y be constant, that* (ii) *equation* (3) *have real discrete singularities occurring only at the endpoints of the interval* (a, b), *and that* (iii) *the weight function be positive and integrable over* (a, b), *it follows that* (3) *is equivalent, up to a linear transformation of the independent variable, to the differential equations of Hermite, Jacobi, or Laguerre.*

Since the assumed singularities, the zeros of $p(x)$, occur only at the endpoints of a finite, semi-infinite, or infinite interval, there is no loss of generality in assuming that the intervals are $(-1, 1)$, $(0, \infty)$, or $(-\infty, \infty)$ respectively; it is readily shown that a linear transformation of the form $x = mt + c$ may be applied so that the interval with singularities at the endpoints of (a, b) may be transformed to one of these.

3. For the finite interval, in view of the facts that the weight function must be positive and that the singularities occur at ± 1, (3) must be of the form

$$(1 - x)(1 + x)y'' + (\delta x + \epsilon)y' + \lambda y = 0. \tag{4}$$

From (2) the weight function for (4) becomes

$$\begin{aligned} w(x) &= \frac{1}{(1-x)(1+x)} \exp \int \frac{\delta x + \epsilon}{(1-x)(1+x)} dx \\ &= (1-x)^{-1-(\delta+\epsilon)/2}(1+x)^{-1+(\epsilon-\delta)/2} \\ &= (1-x)^c(1+x)^d, \end{aligned} \tag{5}$$

where $c = -1-(\delta+\epsilon)/2$ and $d = -1+(\epsilon-\delta)/2$. In order for the weight function to be integrable we must have $c > -1$ and $d > -1$. If we make in (4) the substitutions suggested in (5), our differential equation becomes

$$(1 - x^2)y'' + [d - c - (c + d + 2)x]y' + \lambda y = 0, \tag{6}$$

the form usually given for Jacobi's equation.

4. For the semi-infinite interval, we have singularities at 0 and ∞. Then, since $p(x)$ is quadratic (at most), it must be of the form

$$p(x) = x(hx + k) \tag{7}$$

with $k \neq 0$, since the singularities are assumed to be discrete. If $h \neq 0$ the factor $hx+k$ implies a singularity at $x=-k/h$, but this is contrary to our assumption of singularities only at 0 and ∞. Hence, $h=0$ and

$$p(x) = x, \tag{8}$$

after we divide through by the nonzero constant k. Thus our differential equation can be written as

$$xy'' + (\delta x + \epsilon)y' + \lambda y = 0. \tag{9}$$

For considerations of orthogonality of the eigenfunctions over the interval $(0, \infty)$, the weight function becomes

$$w(x) = \frac{1}{x} \exp \int \frac{\delta x + \epsilon}{x} dx = x^{\epsilon-1}e^{\delta x}. \tag{10}$$

The integral of the weight function,

$$\int_0^\infty w(x)dx = \int_0^\infty x^{\epsilon-1}e^{\delta x}dx, \tag{11}$$

can be shown to converge for all $\epsilon > 0$ if $\delta < 0$ (see, e.g., No. 493 in [6]).

If we let $\delta x = -t$, $\epsilon = c+1$, and $\lambda = -\delta\eta$ in (9), our differential equation becomes

$$t\frac{d^2y}{dt^2} + (c + 1 - t)\frac{dy}{dt} + \eta y = 0 \tag{12}$$

with $c > -1$ and the interval of orthogonality being $(0, \infty)$. This is the usual form of Laguerre's differential equation.

5. In case the singularities of (3) are at $\pm\infty$, we observe immediately that $p(x)$ is constant. To see this we note that if $p(x) = \alpha x^2 + \beta x + \gamma$ in (3), then $p(x) = 0$ at

$$x = \frac{-\beta \pm \sqrt{(\beta^2 - 4\alpha\gamma)}}{2\alpha} \quad \text{if} \quad \alpha \neq 0. \tag{13}$$

Equation (13) gives two real or imaginary zeros, and this is contrary to our assumption relative to the singularities. Thus $\alpha = 0$; and this implies that $\beta = 0$, since otherwise $p(x) = 0$ at $x = -\gamma/\beta$. If we then assume that $\gamma = 1$ for simplicity, the weight function for the orthogonality of the solutions of

$$y'' + (\delta x + \epsilon)y' + \lambda y = 0 \tag{14}$$

must be of the form

$$w(x) = \exp \int (\delta x + \epsilon)dx = \exp\left(\frac{\delta}{2}x^2 + \epsilon x\right). \tag{15}$$

For the weight function to be integrable we may take ϵ arbitrary if $\delta<0$. The transformation

$$x = \frac{t}{\sqrt{-\delta}} - \frac{\epsilon}{\delta}$$

takes (14) into the usual form of Hermite's differential equation,

$$\frac{d^2y}{dt^2} - t\frac{dy}{dt} + \eta y = 0, \tag{16}$$

where $\eta = -\lambda/\delta$.

Thus, from considerations of the various cases of singular Sturm-Liouville differential equations with at most quadratic, linear, and constant coefficients, we arrive at the result that the corresponding equations are equivalent to the equations of Jacobi, Laguerre, or Hermite.

Mention should be made of other papers in this area by Bochner [7], Krall [8], and Stein [9] as well as of Chapter 10, and pp. 163 ff. in particular of [10].

This work was supported by AFOSR Grant No. 62-219.

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NOTES ON MAGIC SQUARES

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1. Not many general methods are known for constructing magic squares of doubly even order. We propose here a very simple method for constructing magic squares of order $n=4m$, where m is any integer. This method is illustrated by considering $n=8$, i.e. $m=2$.

We write out the integers from 1 to 64 as shown in Figure 1. The first $2n$ numbers are written going clockwise, the next $2n$ are written going anti-clockwise, and so on alternately. We then transfer the numbers lying outside the two sides of the square, without changing their configuration, to the empty cells by the side directly opposite. The resulting square, shown in Figure 2, is magic.

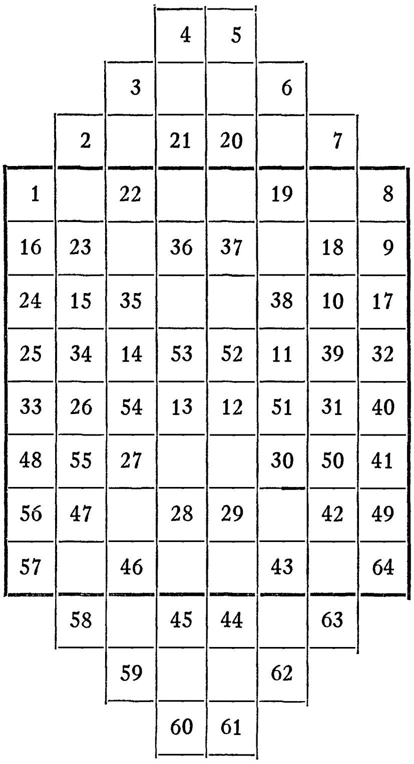

FIG. 1

1	58	22	45	44	19	63	8
16	23	59	36	37	62	18	9
24	15	35	60	61	38	10	17
25	34	14	53	52	11	39	32
33	26	54	13	12	51	31	40
48	55	27	4	5	30	50	41
56	47	3	28	29	6	42	49
57	2	46	21	20	43	7	64

FIG. 2

2. Several methods are known for constructing magic squares of odd order. One of them is due to De la Loubère [1]. Magic squares of odd order constructed by this method have the following property, which so far as is known to us, has not been noticed before. One finds that *the sum of squares of numbers in any row (column) is equal to the sum of squares of numbers in the complementary row (column).*

The above result can be proved as follows. If one constructs an odd order magic square by the method of De la Loubère using consecutive integers from $-\frac{1}{2}(n^2-1)$ to $+\frac{1}{2}(n^2-1)$, integers from $-\frac{1}{2}(n-1)$ to $+\frac{1}{2}(n-1)$ will lie along the main diagonal joining the left hand bottom corner to the right hand top corner, with zero occupying the centre of the square. It is further not difficult to

see that in this method of construction, if any integer γ lies in a particular cell, $(-\gamma)$ will lie in the complementary cell. It follows that not only is the sum of numbers the same in any two complementary rows (or columns) (magic property), but the sum of even powers of all numbers in any row (column) is equal to the sum of numbers raised to the same power and taken from the complementary row (column).

This general result is not of much interest, however, since it holds only for what may be called "the original magic square," constructed from integers $-\frac{1}{2}(n^2-1)$ to $+\frac{1}{2}(n^2-1)$. If we confine ourselves to second power only, the above result will hold for magic squares constructed from a consecutive sequence of integers starting from any value a. A magic square for this sequence is obtained by adding $a+\frac{1}{2}(n^2-1)$ to all numbers in the original magic square. For any given row (column) the sum of squares of numbers is, then, equal to

$$\sum_{\substack{\text{row}\\ \text{(column)}}} (a+\gamma)^2 = na^2 + 2a \sum_{\substack{\text{row}\\ \text{(column)}}} \gamma + \sum_{\substack{\text{row}\\ \text{(column)}}} \gamma^2.$$

Whereas $\sum\gamma$ is the same for all rows and columns, $\sum\gamma^2$ is the same for two complementary rows (columns). Hence the result.

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AN INTRODUCTION OF SPHERICAL COORDINATES: A CORRECTION

DAVID SHELUPSKY, Princeton University

It has been brought to my attention that in my paper in this MONTHLY, 69 (1962) 644–646, two errors appear on the first line of page 646. The first is the definition of $[x]$ as the greatest integer less than x rather than the greatest integer less than or equal to x.

The second error is in the last term which appears in the recurrence relation for s_n, the number of spherical coordinates which arise using the method of the paper in n-dimensional space. The relation given is correct for n odd, but for n even the term $s_{\frac{1}{2}n}s_{\frac{1}{2}n}$ should be replaced by $\frac{1}{2}s_{\frac{1}{2}n}(s_{\frac{1}{2}n}+1)$.

This term arises from the spherical coordinates which begin with the decomposition of the n-dimensional Euclidean space, into the sum of two subspaces each of dimension $\frac{1}{2}n$, for n even. The following counting argument gives us the term. The number of spherical coordinates in each of these two subspaces is $s_{\frac{1}{2}n}$. For any of these $s_{\frac{1}{2}n}$ spherical coordinates taken in the first subspace any of the same number may be taken in the second. Since we do not get distinct coordinate systems for the space when the subspaces are interchanged the total number should only be $\frac{1}{2}s_{\frac{1}{2}n}(s_{\frac{1}{2}n}+1)$ rather than $s_{\frac{1}{2}n}s_{\frac{1}{2}n}$.

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University
and A. L. SHIELDS, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, University of Michigan.

CALCULUS—A NEW LOOK

D. E. RICHMOND, Williams College

1. Introduction. It is something of an historical accident that analytic geometry was discovered before calculus, so that the differential calculus was formulated in terms of the tangent problem and the integral calculus in terms of the area problem. It is now possible to see that these formulations are geometrical applications of more fundamental ideas: (1) The idea of a local multiplier from the domain space of a function to its range space; (2) The idea of the average value of a function over an interval. These ideas simplify the understanding of calculus to a surprising extent. It is the purpose of this paper to explain and justify this statement. We shall confine ourselves, for simplicity, to functions of one variable.

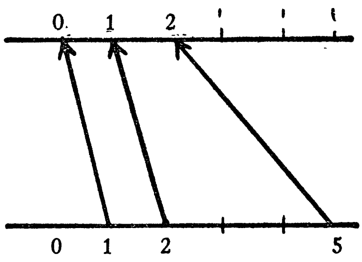

FIG. 1

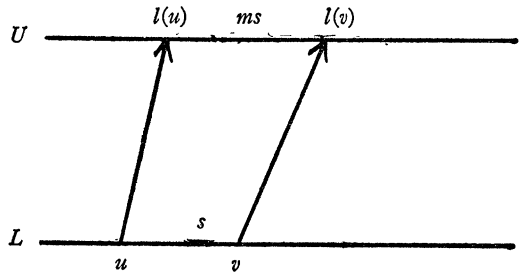

FIG. 2

2. Mappings. Constant and linear functions. Many recent elementary texts introduce functions as mappings from a set D, the *domain*, onto a set R, the *range*. For our purposes D and R shall be subsets of the real numbers. They may be represented as point-sets, usually intervals, on two parallel lines, provided with identical scales.

Thus the function

$$f: x \rightarrow \sqrt{(x-1)}$$

maps the reals $x \geqq 1$ onto the nonnegative reals (Fig. 1). This function is *increasing*, since an increase in x corresponds to an increase in $f(x)=\sqrt{(x-1)}$.

The simplest functions are the *constant functions*,

$$c: x \to c$$

(the function c maps all reals onto the constant c) and the *linear functions*

$$l: x \to mx + b,$$

(m and b constant, $m \neq 0$). Clearly l maps 0 into b. To interpret m, note that $l(v) - l(u) = m \cdot (v - u)$. Hence the step $s = v - u$ on the lower line L corresponds to the step ms on the upper line U. Thus m is the constant *multiplier* in passing from the domain space to the range space. (l increases if $m > 0$, decreases if $m < 0$).

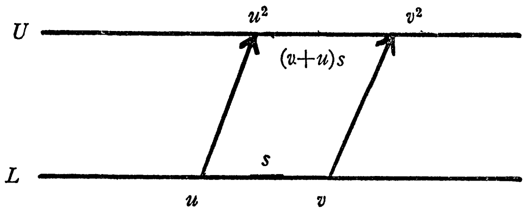

FIG. 3

3. The derivative. We turn now to functions that are not linear, but which are *locally linear* at points of their domain.

For example, if $f: x \to x^2$,

$$f(v) - f(u) = v^2 - u^2 = (v + u) \cdot (v - u) = (v + u)s.$$

The multiplier $v + u$ is no longer constant but depends on the coordinates of both end points. If we let u and v approach x, we obtain the limiting value $2x$, the *local multiplier* at x. For X near x, and x *fixed*, f is approximated by the function $l: X \to x^2 + 2x(X - x)$ (Fig. 4), which, as we observe, is *linear* in X.

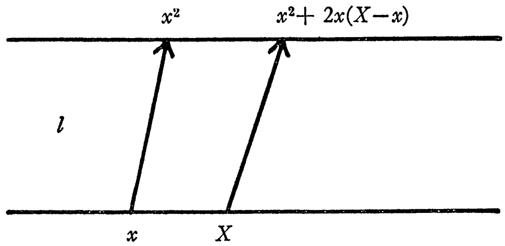

FIG. 4

Generally, if for a given function f and given x,

$$f(v) - f(u) = m(u, v) \cdot (v - u),$$

where the multiplier $m(u, v)$ approaches a limit $f'(x)$ as u and v approach x, f is

said to be *differentiable* at x and $f'(x)$, the local multiplier at x, is called the *derivative* of $f(x)$. (We assume that $u \leqq x \leqq v$ and, if necessary, that $v-u$ be small.)

It is easy to see that if f is differentiable at x, it is continuous there. The proofs of the formulas for the derivatives of sums, products, powers, and quotients of differentiable functions require no discussion here. However, the chain rule becomes particularly intuitive in terms of the mapping picture.

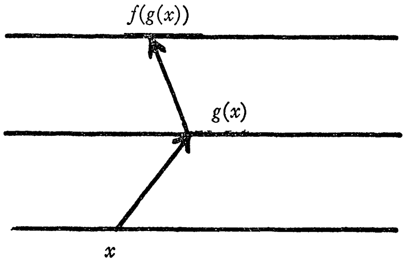

FIG. 5

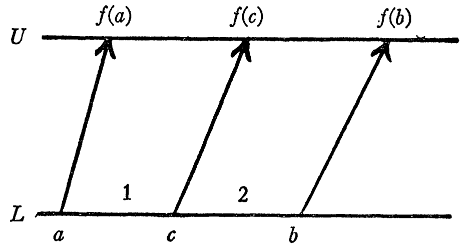

FIG. 6

Let x be mapped into $g(x)$ and $g(x)$ into $f(g(x))$ (Fig. 5). The local multiplier of g at x is $g'(x)$. At $g(x)$ the local multiplier of f is $f'(g(x))$. The over-all local multiplier at x is the product, $f'(g(x))\cdot g'(x)$. This intuitive argument is easily turned into a simple rigorous proof. The interpretation of the chain rule in terms of the graphs of $u=g(x)$, $y=f(u)$, and $y=f(g(x))$ introduces needless complications.

For a one-one function f, we may represent the inverse function by reversing the direction of the arrows. The formula for the derivative of the inverse is immediate at points x at which $f'(x)\neq 0$.

4. Averages. Let f be differentiable on the closed interval $[a, b]$, $a<b$. To obtain $f(b)-f(a)$ on U from $b-a$ on L, we multiply by $\{f(b)-f(a)\}/(b-a)$ which may be called the average multiplier over the interval $[a, b]$. In fact, we shall take this ratio to be the *mean* or *average* value of the local multiplier $f'(x)$ over $[a, b]$ and accordingly write

$$M(f') = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

This terminology is justified by the following considerations.

Let c be any point of division. The suggested averages over the subintervals are

$$M_1(f') = \frac{f(c) - f(a)}{c - a}$$

and

$$M_2(f') = \frac{f(b) - f(c)}{b - c}.$$

Since $(c-a)M_1(f')+(b-c)M_2(f')=f(b)-f(a)$,

$$\text{I.} \qquad \frac{c-a}{b-a}M_1(f') + \frac{b-c}{b-a}M_2(f') = M(f').$$

Thus each subinterval contributes to the over-all average an amount which corresponds to the fraction of the whole interval which it represents. This is surely a property which we should want an average to have.

Moreover, our suggested average has the property that as the interval over which it is taken shrinks to a point, the average approaches the value of the function f' at that point. In fact, if we replace a by u and b by v

$$\text{II.} \qquad M(f') \text{ over } [u, v] \to f'(x)$$

as $u \to x$ and $v \to x$.

The average $M(f')$ given by (1) is the only average of f' which has the properties I and II. For let $A(f')$ be a second average with these two properties. Adjust the notation, if necessary, so that $[a, b]$ denotes a closed subinterval of the original interval for which $A(f')$ and $M(f')$ have distinct values. Assume for definiteness that $A(f') > M(f')$ (the other case is handled in a similar fashion), so that

$$A(f') = M(f') + d \qquad (d > 0).$$

Bisect $[a, b]$ and average over each half. Then either

$$(2) \qquad A_1(f') \geqq M_1(f') + d \text{ or } A_2(f') \geqq M_2(f') + d.$$

Otherwise by (I)

$$\begin{aligned} A(f') &= \tfrac{1}{2}[A_1(f') + A_2(f')] < \tfrac{1}{2}[M_1(f') + M_2(f')] + d \\ &= M(f') + d = A(f'), \end{aligned}$$

a manifest contradiction.

By choosing a subinterval for which (2) holds and continuing the bisecting process, we obtain a sequence of intervals which shrink to a point x on $[a, b]$. Since for each of these intervals the A average exceeds the M average by at least d, condition II is contradicted.

It is apparent that the use of (1) enables us to compute the averages of many functions with great ease. Thus to find the mean value of f': $x \to x^2$ over the interval $[1, 3]$, we set $x^2 = f'(x)$ and $x^3/3 = f(x)$. Then

$$M(f') = \frac{f(3) - f(1)}{3 - 1} = \frac{9 - \frac{1}{3}}{2} = \frac{13}{3}.$$

(The uniqueness proof shows that another choice of "antiderivative" for $f(x)$ can give no different answer.)

We may also use (1) to obtain the average of f' over an *open* interval (a, b), provided that f is continuous on the *closed* interval $[a, b]$. This result is obtained if we define $M(f')$ over (a, b) to be the limit of $M(f')$ over $[u, v]$ as $u \to a$ and $v \to b$.

For example, if $f(x) = 2\sqrt{x}$, $f'(x) = 1/\sqrt{x}$ and $M(f')$ over $(0, b)$ is equal to $(2\sqrt{b} - 0)/(b - 0) = 2/\sqrt{b}$.

Thus, to average a given function g over $[a, b]$ or (a, b), it suffices to discover a function f, continuous on $[a, b]$, such that $f'(x) = g(x)$ on (a, b) and then apply (1).

It is easy to extend this method of averaging to functions for which such antidifferentiation may be performed piecewise. It is sufficient to select antiderivatives on the subintervals so that the resultant function f is continuous at each function. This device is useful in the case of step-functions (Section 6).

It would be inappropriate in this place to discuss the well-known techniques of antidifferentiation.

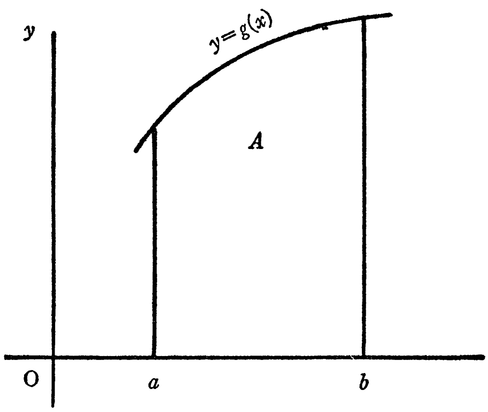

FIG. 7

5. Applications of averages. Area, volume, work. The standard applications of integration to the determination of areas, volumes, work, and so on, easily reduce to the problem of finding the average of an appropriate function. This obviates the necessity of introducing "elements" of a sum in the customary way.

For example, if $g(x) > 0$ on (a, b), we define the area A bounded by the graph of $y = g(x)$, the x-axis and the ordinates at a and b, to be the average value of the ordinate times the length of the base $(b - a)$. Hence, if $f'(x) = g(x)$,

$$A = (b - a)M(g) = f(b) - f(a).$$

The modifications to be made if $g(x) < 0$ on (a, b) or if $g(x)$ changes sign are immediate and intuitively clear.

Similarly, the area between $y=g_1(x)$, $y=g_2(x)$, $(g_1(x)<g_2(x))$, $x=a$ and $x=b$, is defined to be $(b-a)M(g_2-g_1)$ in a very natural manner.

In polar coordinates, the area bounded by $r=g(\theta)$, $\theta=\alpha_1$ and $\theta=\alpha_2$ is defined to be

$$(\alpha_2 - \alpha_1)M\left(\frac{g^2}{2}\right)$$

as is immediately suggested by the equivalent circular sector.

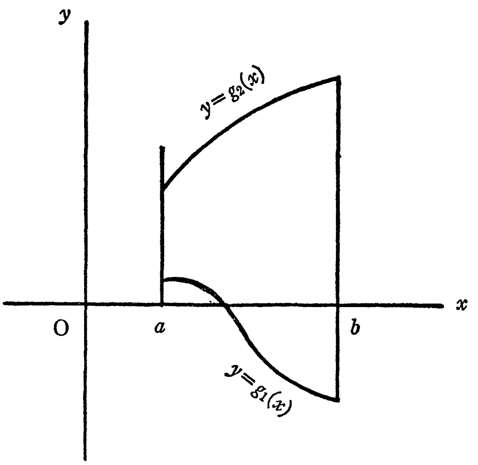

FIG. 8

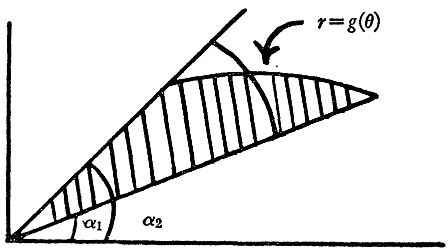

FIG. 9

We turn next to *volumes*. Let $A(x)$ be the area of the cross section obtained by cutting a solid by a plane perpendicular to the x-axis at abscissa x. Then V, the volume between the planes $x=a$ and $x=b$, may be defined to be the average cross-sectional area, $M(A)$, times the altitude $(b-a)$. For a solid of revolution we obtain

$$(b - a)M(\pi g^2) = \pi[f(b) - f(a)],$$

where $f'(x)=g^2(x)$.

The volumes customarily obtained by the use of cylindrical shells may be found by averaging the areas of cylindrical surfaces. If this is not sufficiently intuitive, the correct formula is easily derived from the previously discussed case by integration by parts. This involves no appeal to the theorems of Duhamel or Bliss and no sloppy neglect of pieces.

The reader will easily be able to see how work and pressure problems, centroids and the like can be handled in a similar way. He might, however, wonder how arc lengths fit into this scheme.

Let $x=f(t)$ and $y=g(t)$, $a\leqq t\leqq b$, be parametric equations of a plane curve where f and g are functions with continuous derivatives.

Imagine a particle moving along the curve with horizontal and vertical components of velocity given by $f'(t)$ and $g'(t)$. Then at time t the speed is

$$v = \sqrt{(f'(t)^2 + g'(t)^2)}.$$

It is then natural to define the arc length to be $(b-a)M(v)$, where the average is taken over the time interval $[a, b]$. The specialization to the case $f(t)=t$ gives the familiar result.

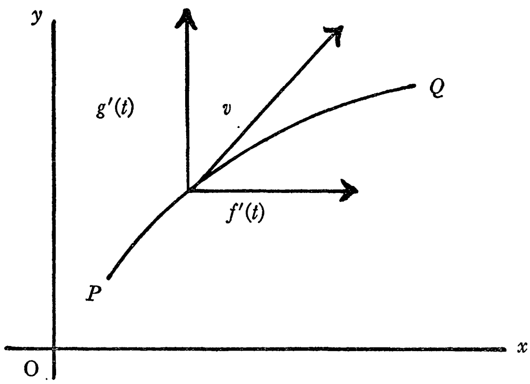

FIG. 10

These examples show that by focusing on the idea of average, we obtain a unifying concept which simplifies the application of calculus to geometry and the natural sciences. We shall not insist upon the obvious importance of this concept for probability and statistics.

To some readers this formulation will no doubt seem subversive because it short-circuits the important idea of the limit of a sum. The answer is twofold. First, if it is possible to make calculus a less formidable subject than it has been and currently is, we shall make this powerful tool accessible to a large number of people who could profitably use it. Secondly, this approach can be made every bit as rigorous as the conventional one. We have proved the *uniqueness* of the averages obtained. We have not, of course, proved the *existence* of an average for an arbitrary continuous g. This is not too difficult to do (see Section 6). But quite apart from such a proof, the student rightly feels that if he can actually find an answer (using antidifferentiation) in a given case, there is no *need* to prove existence.

6. Approximations. If we are unable to antidifferentiate a given $g(x)$ which we wish to average over an interval, it is natural to substitute an approximation, $h(x)$, which we *can* antidifferentiate.

If we can choose an $h(x)$ so that

$$-\epsilon \leqq g(x) - h(x) \leqq \epsilon$$

throughout $[a, b]$, then

$$-\epsilon \leqq M(g) - M(h) \leqq \epsilon. \tag{3}$$

This result rests upon the following theorems.

1. *If* $f(x) \geqq 0$ on $[a, b]$, *then* $M(f) \geqq 0$.

The assumption that $M(f) < 0$ leads to a contradiction by repeated bisection as in Section 4.

2. *M is a linear functional. That is, if* $M(g)$ *and* $M(h)$ *exist over* $[a, b]$, *and* C_1 *and* C_2 *are constants, then*

$$M(C_1g + C_2h) = C_1M(g) + C_2M(h).$$

This follows from the uniqueness theorem, since the suggested average, $C_1M(g) + C_2M(h)$, satisfies I and II.

3. $M(c) = c$, *where c is the function which has the value c over* $[a, b]$ *or* (a, b).

Since $\epsilon + h(x) - g(x) \geqq 0$ on $[a, b]$, then $M(\epsilon + h - g) \geqq 0$ by 1. Also $M(\epsilon) + M(h) - M(g) \geqq 0$ by 2 (twice), and $M(g) - M(h) \leqq \epsilon$, since $M(\epsilon) = \epsilon$ by 3.

The other half of the inequality follows similarly.

Now, if we can choose ϵ in (3) arbitrarily small, we can approximate the desired $M(g)$ as closely as we please.

We consider two especially important choices of $h(x)$.

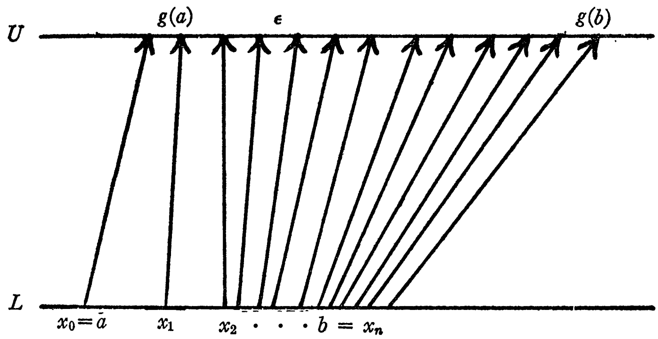

FIG. 11

a) *Step Functions.* Let g, a continuous function, increase from $g(a)$ to $g(b)$ as x increases from a to b. Subdivide this interval on U into n equal parts of length $\epsilon = (g(b) - g(a))/n$. Let the points of subdivision correspond to the x-values, $x_1, x_2, \cdots, x_{n-1}$ and let $x_0 = a$, $x_n = b$. On each half-open interval $[x_k, x_{k+1})$ replace $g(x)$ by its minimum value $g(x_k)$ on $[x_k, x_{k+1}]$. Let $h(x)$ be the "step function" so constructed. On each subinterval $[x_k, x_{k+1}]$, the average of h is

$g(x_k)$. By repeated use of I, the average $M(h)$ over $[a, b]$ is seen to be

$$M(h) = \frac{1}{b-a} \sum_0^{n-1} g(x_k)(x_{k+1} - x_k).$$

By replacing $g(x)$ on each subinterval by its *maximum* value, $g(x_{k+1})$, we obtain an approximation $h_1(x)$ from above, whose average is

$$M(h_1) = \frac{1}{b-a} \sum_0^{n-1} g(x_{k+1})(x_{k+1} - x_k).$$

It is easy to show that $M(h_1) - M(h) = \epsilon$.

The required $M(g)$ must lie between $M(h)$ and $M(h_1)$. Thus $M(g)$ is the one and only one number which corresponds to the intersection of all intervals $[M(h), M(h_1)]$. The argument is easily extended to establish the existence of $M(g)$ if g is continuous and consists of a succession of a finite number of increasing and decreasing "pieces," and indeed, if g is an *arbitrary* continuous function. However, it will be a long time before the student will encounter functions for which this degree of generality is necessary.

The use of upper and lower sums to evaluate $M(g)$ with any desired accuracy gives concrete meaning to the usual existence proofs. Also, of course, the conventional notation for the definite integral, $\int_a^b g(x)dx$, is easily motivated by these considerations.

b) *Polynomials.* Let a given $g(x)$ have successive derivatives $g'(x)$, $g''(x), \cdots, g^{(n+1)}(x)$ with

$$-K \leqq g^{(n+1)}(x) \leqq K \text{ on } [a, b]. \tag{4}$$

If we antidifferentiate each term of (4) from a to x, $n+1$ times, we obtain

$$\frac{-K(x-a)^{n+1}}{(n+1)!} \leqq g(x) - h(x) \leqq \frac{K(x-a)^{n+1}}{(n+1)!},$$

where

$$h(x) = g(a) + g'(a)(x-a) + \cdots + \frac{g^{(n)}(a)(x-a)^n}{n!}.$$

For sufficiently large n, $h(x)$ gives an arbitrarily good approximation to $g(x)$ over the whole interval, provided that (replacing K by K_{n+1})

$$K_{n+1} \frac{(b-a)^{n+1}}{(n+1)!} \to 0$$

as n becomes infinite. The generalization to a Taylor series about an interior point is obvious.

It will be seen that this treatment of series fits naturally into the general pattern of thought which we have been developing.

A SIMPLE WAY OF DIFFERENTIATING TRIGONOMETRIC FUNCTIONS AND THEIR INVERSES IN AN ELEMENTARY CALCULUS COURSE

H. A. THURSTON, University of British Columbia

We assume that the student knows (i) how to differentiate inverse functions, and (ii) that areas under graphs can be obtained by integration. The converse of (ii) is also true; if the area is known, it can be used to find an integral. This is

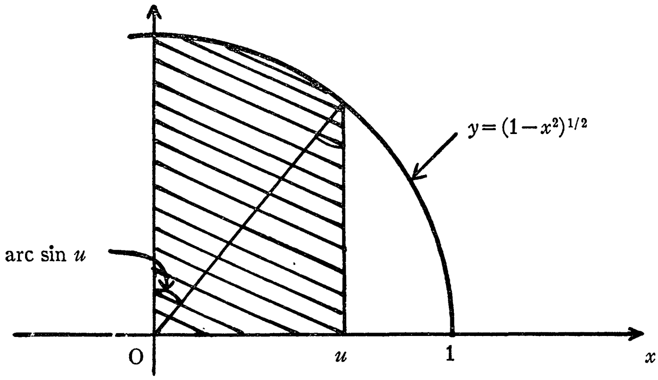

FIG. 1

the quickest way of finding $\int_0^u (1-x^2)^{1/2}dx$. From the diagram it follows that this integral is

$$\tfrac{1}{2} \arcsin u + \tfrac{1}{2}u\cdot(1 - u^2)^{1/2}.$$

From the fundamental theorem we now have

$$D_x[\arcsin x + x\cdot(1 - x^2)^{1/2}] = 2(1 - x^2)^{1/2}$$

which quickly yields the usual formula for the derivative of arc sin. Because sin is an inverse of arc sin, its derivative on the interval $[-\pi/2, \pi/2]$ is soon found, and the relation $\sin (x+\pi) = -\sin x$ gives the derivative in general. The rest of the trigonometrical results follow as usual, and finally the limit as $h \to 0$ of $h^{-1}\cdot\sin h$ can be found (if wanted) by L'Hôpital's rule.

VENN DIAGRAMS FOR MORE THAN FOUR CLASSES

DAVID W. HENDERSON, Swarthmore College

1. Introduction. Venn diagrams are used in many textbooks to illustrate relationships in logic and the algebra of classes (for example, [4], pp. 69–83; [2], pp. 235–236 and [5]). It seems to be commonly thought that Venn diagrams for more than four classes are either impossible or else employ disconnected, multiply connected, or otherwise complicated regions ([4], p. 80; [5], p. 201). The purpose of this note is to present two simple Venn diagrams of five classes and then discuss the general case.

2. Venn diagrams for five classes. It can be seen that all possible combinations of five classes are represented in each of the two diagrams in Fig. 1. The diagram on the left consists of five irregular and congruent pentagons, and can be drawn using congruent quadrilaterals. The diagram on the right consists of five congruent quadrilaterals.

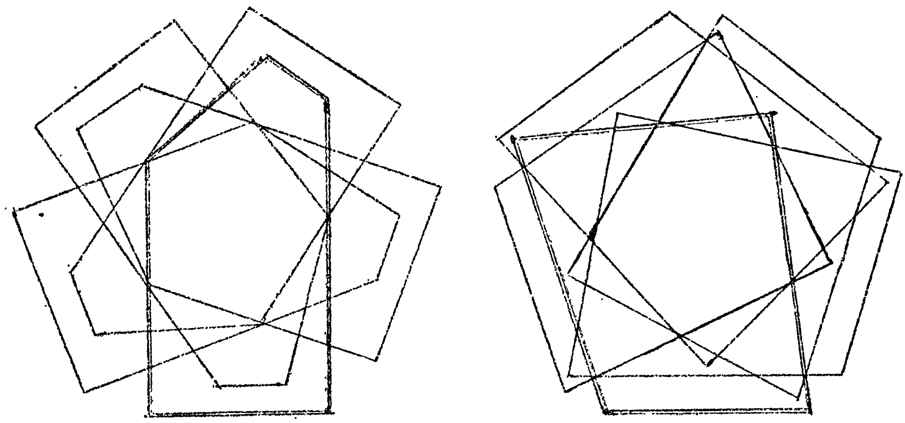

FIG. 1

3. Venn diagrams for more than five classes. There are in the literature two procedures for constructing Venn diagrams of n classes [1]; [6]; [7, p. 118]. In these procedures, regions representing different classes are quite dissimilar, some of them in each diagram taking on complicated shapes resembling coiled snakes [6] or combs [1].

We are led by the success with five classes to consider a special type of Venn diagram, which I shall call symmetric. A symmetric Venn diagram of n classes is one in which the regions representing the different classes are all congruent, the diagram being made by rotating a given region about a point through successive angles of $360°/n$. The regions must be chosen so that the resulting diagram will represent all possible combinations of n classes, as is required of a Venn diagram. The familiar diagram for three classes and the ones for five classes in Fig. 1 are seen to be symmetric Venn diagrams. Since a symmetric Venn diagram of n classes must be symmetric with respect to a rotation of $360°/n$ the number $C(n, p)$ of combinations of n things taken p at a time must be divisible by n or equal to one for each $p \leqq n$. $C(n, p)$, $p < n$, is divisible by n if n is prime; but if $n = rs$ (r and s, integers greater than one, with r prime), then $C(n, r)$ can readily be seen to be not divisible by n [3, p. 64]. Thus for a Venn diagram to be symmetric, it is necessary that it be a Venn diagram for a prime number of classes. A symmetric Venn diagram for seven classes has been found, the regions being irregular hexagons, but the existence of symmetric Venn diagrams for primes greater than seven has not been demonstrated.

The author wishes to acknowledge the suggestions and encouragement of O. Aberth, P. Carruth, and D. Rosen. The author is now at the University of Wisconsin.

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ALTERNATE CLASSROOM PROOF THAT $\sin t/t \to 1$ AS $t \to 0$

GARY PERRY, Trinity College, Hartford, Connecticut

Hoffman (This MONTHLY, 67 (1960) 671–672) offers a method of proving $\lim_{t\to 0} (\sin t)/t = 1$ which avoids the use of area. The following proof also avoids the use of area and makes use of several important concepts usually encountered in a first year analytic geometry and calculus course.

THEOREM. $\lim_{t\to 0} (\sin t)/t = 1$.

Proof. We assume for simplicity that $t > 0$. Let $P(x, y)$ be a point in the first quadrant on $x = (1-y^2)^{1/2}$, and let t denote the length of arc along the curve from the x-axis to P. The arc length formula yields

$$t = \int_0^y \frac{du}{(1-u^2)^{1/2}}, \qquad (0 \leqq y < 1).$$

Since $\sin t = y$,

$$\frac{t}{\sin t} = \frac{1}{y}\int_0^y \frac{du}{(1-u^2)^{1/2}}, \qquad (0 < y < 1).$$

Clearly $y \to 0^+$ as $t \to 0^+$, and we write

$$\lim_{t\to 0^+} \frac{t}{\sin t} = \lim_{y\to 0^+} \frac{1}{y}\int_0^y \frac{du}{(1-u^2)^{1/2}}.$$

If F is the function defined by

$$F(y) = \int_0^y \frac{dw}{(1-w^2)^{1/2}}, \qquad (0 \leqq y < 1),$$

then $F(0)=0$, so that

$$\lim_{y\to 0^+}\frac{1}{y}\int_0^y \frac{du}{(1-u^2)^{1/2}} = \lim_{y\to 0^+}\frac{F(y)-F(0)}{y} = F'(0)$$

by the definition of the right-hand derivative. Finally, by the fundamental theorem of calculus,

$$F'(y) = \frac{1}{(1-y^2)^{1/2}}, \qquad (0 \leqq y < 1).$$

Then $F'(0)=1$ and the theorem is proved.

A GENERALIZATION WHICH IS EASIER TO PROVE THAN A SPECIAL CASE

R. A. ROSENBAUM, Wesleyan University

A generalization of a theorem sometimes gets closer "to the heart of the matter" than the theorem itself, by stripping away nonessentials and exposing the significant relationships. Here is an elementary example.

Problem 384 in the *Mathematics Magazine* (September, 1959) proposes that the reader prove

THEOREM 1. *The set of nonsingular $n\times n$ matrices such that the sum of the elements of each row of each matrix equals* 1 *forms a group under multiplication.*

The significant hypothesis is that the row-sum be constant—not necessarily 1. Clearly, the most difficult part of the proof involves showing that each matrix of the set has an inverse in the set, and a student might be tempted to use some results on adjoints or similar material related to inverses. But this is not really "the way to look at it." The following theorem has a simple and straightforward proof.

THEOREM 2. *Over an arbitrary field, let $A=[a_{ij}]$ be an $m\times n$ matrix, $B=[b_{ij}]$ be an $n\times p$ matrix, and $C=A\cdot B$.*

(α) *If $\sum_{j=1}^n a_{ij}=a$, for each i, and $\sum_{j=1}^p b_{ij}=b$, for each i, then $\sum_{j=1}^p c_{ij}=a\cdot b$, for each i; and*

(β) *If $\sum_{j=1}^p b_{ij}=b\neq 0$, for each i, and $\sum_{j=1}^p c_{ij}=c$, for each i, then $\sum_{j=1}^n a_{ij}=c/b$, for each i.*

COROLLARY. *The set of all nonsingular $n\times n$ matrices over a field such that, for any one matrix, the sum of the elements of each row is constant (but perhaps not the same constant for different matrices) forms a group under multiplication.*

The group of the Corollary contains that of Theorem 1.

tem of equivalency—proficiency examinations for the large number of college students doing college level work by independent study and in television courses. The examinations are also intended to serve persons who can demonstrate abilities developed through study in adult education courses, courses at industrial plants, and other courses outside regular college curriculums. College graduates may also be helped to meet applicable requirements for teacher certification by passing such examinations.

News Release, New York State Department of Education

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
Collaborating Editor: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1581. *Proposed by Erwin Just, Bronx Community College*

If m_1, m_2, m_3 are nonzero slopes of the sides of an equilateral triangle, prove that:

(a) $$m_1m_2 + m_2m_3 + m_3m_1 = -3,$$

(b) $$\sum_{i=1}^{3} m_i \sum_{i=1}^{3} 1/m_i = 9.$$

E 1582. *Proposed by Harley Flanders, Purdue University*

Prove that

$$\iint x^6 d\sigma = 15 \iint x^2y^2z^2 d\sigma,$$

where the integrations are taken over the unit sphere centered at the origin with respect to the area element $d\sigma$.

E 1583. *Proposed by C. D. Zimmerman, Southern Missionary College*

Is $\sum_{x=0}^{m-1} m^x/x! \geqq e^{m-1}$ for all positive integral m?

E 1584. *Proposed by D. J. Newman, Yeshiva University*

What are all the sizes of rugs which will fit on a given square floor? In other words, what is the condition on a and b which insures that a rectangle with sides a and b can be contained wholly within the unit square?

E 1585. *Proposed by Irving Newman, Monroe, North Carolina*

What is the maximum number of knights which can be placed on a chessboard in such a way that no knight attacks any other?

E 1586. *Proposed by J. F. Darling, Woodstown, New Jersey*

In a triangle with circumradius R reflect the circumcenter in the sides. From these points and the vertices describe circles of radius R. Letter the outer intersections of consecutive pairs of these six circles A, B, C, D, E, F, and the corresponding inner intersections A', B', C', D', E', F'. Prove that triangles ACE and DFB, and likewise triangles $A'C'E'$ and $D'F'B'$, are equilateral, have equal and parallel corresponding sides, and are in perspective from the nine-point center of the given triangle.

E 1587. *Proposed by T. C. Brown, Washington University*

Let S be the semigroup of words in two generators a, b subject to the relation $w^3=w$ for all w in S. Show that

$$(aba^2b)^2 = (ab^2ab)^2 = (ab^2a^2b)^2 = (ab)^2.$$

E 1588. *Proposed by M. S. Klamkin, The State University of New York at Buffalo*

An ellipse has the property that the sum of the moments of inertia of its area about two orthogonal tangents is constant. Does this property characterize the ellipse?

E 1589. *Proposed by W. J. Blundon, Memorial University of Newfoundland*

Find necessary and sufficient conditions for a triangle of inradius r, circumradius R, and semiperimeter s to be isosceles.

E 1590. *Proposed by James Nearing and J. L. Pietenpol, Columbia University*

Sum, for all x and y, the series

$$\sum_{n=0}^{\infty} P_n(x)y^n/n!,$$

where $P_n(x)$ is the nth order Legendre polynomial.

SOLUTIONS

Construction of an Isosceles Triangle

E 1526 [1962, 667]. *Proposed by Ned Harrell, Melo-Atherton High School, Atherton, California*

Construct an isosceles triangle given the base and the bisector of one of the base angles.

I. *Solution by Walter Bluger, Ottawa, Ontario.* Let the triangle be ABC, with base AB, and let AD be the bisector of base angle A. Produce BA to M so that $BM=BD$. Set $AD=t$, $AB=c$, $BM=BD=p$. Then, from similar triangles MBD and ADM, $t^2=p(c+p)$. From a point P on a circle of diameter c, draw a tangent $PQ=t$. The diametral line of the circle through Q cuts the circle in S and S' such that $QS=p$ and $QS'=c+p$. The desired triangle can now be constructed.

II. *Solution by Jane W. Di Paola, Long Beach, New York.* Using the notation of Solution I, let BE be the bisector of angle B. Applying Ptolemy's Theorem to the cyclic quadrilateral $ABDE$, we find $p^2+cp=t^2$, or $(p+c/2)^2=t^2+c^2/4$, whence $p+c/2$ is the hypotenuse of a right triangle with legs t and $c/2$. The desired triangle is now easily constructed.

III. *Solution by Anders Bager, Mjørring, Denmark.* Using the notation of Solution I, let $2x$ denote the base angle A. Then, applying the law of sines to triangle ABD, $c/t=\sin 3x/\sin 2x=(4\cos^2 x-1)/2\cos x$, whence $\cos x$, and then the desired triangle, are constructible.

Also solved by R. A. Brace, Norman Brenner, Florent Cartuyvels, J. Caspino, D. I. A. Cohen, Jane Evans, Roy Feinman, D. Fink and H. Glaser (jointly), Stephen Fisk, Michael Goldberg, L. D. Goldstone, S. H. Greene, Charles Haas, Kit Hanes, B. A. Hausmann, R. T. Hood, J. A. H. Hunter, A. R. Hyde, Erwin Just and Norman Schaumberger (jointly), G. W. Kessler, R. B. Killgrove, Frank Kocher, Forrest McMains, C. F. Marion, D. C. B. Marsh, J. R. Naylor, F. D. Parker, Stanton Philipp, Frederick Renvyle, L. A. Ringenberg, Bernard Rosner, H. D. Ruderman, Perry Scheinok, R. J. Schoenberg, D. R. Simpson, H. P. Smith, P. D. Statigos, B. R. Toskey, Simon Vatriquant, Andy Vince, Charles Wexler, J. E. Wilkins, Jr., Dale Woods, K. P. Yanosko, and the proposer.

Editorial Note. An easy analysis shows that we must have $2/3<t/c<\sqrt{2}$.

Number of Solutions to an Equation

E 1527 [1962, 667]. *Proposed by R. G. Buschman, Oregon State University*

Let $N(a)$ be the number of solutions to the equation $[x]=ax$, where a and x are real and $[x]$ denotes the greatest integer not exceeding x. Find a simple formula for $N(a)$.

Solution by E. H. Umberger, Pennsylvania State University. By considering simultaneously the graphs of the line $y=ax$ and the step function $y=[x]$, we find that $N(a)$ is given by:

$$
\begin{aligned}
N(a) &= 1, \text{ if } a < 0;\\
&= \infty \text{ (uncountable), if } a = 0;\\
&= n, \text{ if } (n-1)/n < a \leqq n/(n+1), n = 1, 2, 3, \cdots;\\
&= \infty \text{ (countable), if } a = 1;\\
&= n+1, \text{ if } (n+1)/n \leqq a < n/(n-1), n = 1, 2, 3, \cdots.
\end{aligned}
$$

Solution of the inequalities for n yields the formula

$$N_1(a) = -[-|a/(1-a)|],$$

which coincides with $N(a)$ where $N(a)$ is defined; moreover, $N_1(a)\to\infty$ as $a\to 1$. However, $N_1(0)=0$, a "defect" which may be corrected if we divide by $-[-a^2/(1+a^2)]$, obtaining

$$N(a) = [-|a/(1-a)|]/[-a^2/(1+a^2)].$$

Also solved by J. C. Abad, Blossom Backal, Nyles Barnert, Joseph Beer, Norman Brenner, D. L. Burke, Leonard Carlitz, D. I. A. Cohen, D. E. Daykin, S. J. Einhorn, Jane Evans, Paul Feder, D. Fink and H. Glasser (jointly), Stephen Fisk, Dee Fuller, Michael Gemignani, Ralph Greenberg, S. H. Greene, Emil Grosswald, R. T. Hood, R. A. Jacobson, M. S. Klamkin, Joel Kugelmass, D. C. B. Marsh, J. C. Nichols, Katherine E. O'Brien, C. S. Ogilvy, Stanton Philipp, J. L. Pietenpol, David Rothman, S. J. Ryan, Jean-Pierre Samson, Perry Scheinok, D. L. Silverman, E. L. Spitznagel, Jr., Harlan Stevens, B. R. Toskey, Dennis Travis, G. W. Walker, L. J. Warren, Charles Wexler, J. E. Wilkins, Jr., Dale Woods, John Xenakis, and J. E. Yeager.

Many of these other solutions were either only partially correct or failed to obtain as simple a formula.

Integral Distances

E 1528 [1962, 667]. *Proposed by L. A. Zalcman, Southwest High School, Kansas City, Missouri*

The 1958 Putnam Competition put forth the following problem: "Given an infinite number of points in a plane, prove that if all the distances determined between them are integers, then the points are all in a straight line." What is the greatest number of points that can be arranged such that the distance between each pair is integral, and yet the points do not all lie on the same straight line?

I. *Solution by D. L. Silverman, Beverly Hills, California.* There is no maximum since for any N we can obtain N primitive Pythagorean triples (a_i, b_i, c_i). Setting $P=\prod_{i=1}^N a_i$, the points $(b_iP/a_i, 0)$, $i=1, 2, \cdots, N$, together with the point $(0, P)$, form a noncollinear set of $N+1$ integrally spaced points.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* There is no maximum, even if we insist that no *three* of the points be collinear. This is evidenced by the vertices of a rational cyclic polygon of n sides $(n>2)$ constructed by Euler's method (see L. E. Dickson, *History of the Theory of Numbers*; v. II, p. 221, Chelsea 1952 edition) and magnified to an integral scale.

Also solved by Anders Bager, J. W. Baldwin, Joseph Beer, D. E. Daykin, J. W. Ellis, James Gold, Michael Goldberg, Joel Goldman, R. L. Graham, R. T. Hood, Erwin Just and Norman Schaumberger (jointly), Frank Kocher, Sidney Kravitz, Leo Moser, C. S. Ogilvy, Stanton Philipp, David Rothman, H. D. Ruderman, S. J. Ryan, Perry Scheinok, R. E. Shafer, G. W. Walker, J. E. Wilkins, Jr., and the proposer.

Editorial Note. The problem of finding arbitrarily many noncollinear coplanar points such that all distances are integral has been considered by a number of authors; see, e.g., N. H. Anning and Paul Erdös, Integral distances, Bull. Amer. Math. Soc., 51 (1945) 598–600 and 996, E. Trost, Elem. Math., 6 (1951), and W. Sierpiński, Elem. Math., 14 (1959); also see H. Hadwiger and H. Dubrunner, Kombinatorische Geometrie in der Ebene, Instil. de Math., University Geneva, 1959. A still unsolved allied problem that has recently received attention is to determine if one can have a set of points dense in the plane and such that all distances are rational. In this direction Mordell has shown that one can find quadrilaterals whose vertices are arbitrarily close to those of any given quadrilateral and such that all six distances are rational. The straight line and the circle are the only curves on which it is known how to find a dense set of points with all distances rational. Goldberg suggested the allied problem of finding a set of five points in 3-space, no three collinear and no four coplanar, whose ten distances are integral. One such example is the set of points A, B, C, D, E where $AC=AD=AE=1625$, $BC=BD=BE=1768$, $AB=1287$, $CD=1200$, $DE=2880$, $EC=2448$. A metric space in which all distances are integral has been called a *holometric* space.

Order of Smallest Noncommutative Ring

E 1529 [1962, 667]. *Proposed by C. R. MacCluer and G. A. Wilson, Ohio State University*

What is the order of the smallest noncommutative ring?

Solution by J. L. Pietenpol, Columbia University. If the additive group is cyclic, the ring will be commutative. The smallest noncyclic group is the Klein four-group. With this as the additive group and with the multiplication rules $x0=xa=0$, $xb=xc=x$ ($x=0, a, b, c$), we have a noncommutative ring of order four.

Also solved by Anders Bager, D. M. Bloom, Joel Brawley, Jr., D. E. Daykin, J. W. Ellis, Alvin Hausner, G. A. Heuer, J. E. Homer, Jr., T. G. McLaughlin, D. C. B. Marsh, Stanton Philipp, H. D. Ruderman, Art Steger, B. R. Toskey, Dennis Travis, University of Nebraska algebra class, and the proposers.

The University of Nebraska algebra class showed how one can construct a noncommutative ring of any composite order N. Heuer asked if there is a finite noncommutative ring with identity, and, if there is, what is the order of the smallest one.

Placing the First Digit Last

E 1530 [1962, 667]. *Proposed by Sidney Penner, Illinois Institute of Technology*

What is the smallest positive integer that can be increased by 50% by moving the digit on the extreme left to the extreme right?

Solution by J. W. Ellis, Louisiana State University in New Orleans. The desired integer N can be written as $10^n x+y$, where x and y are positive integers and $x\leqq 9$; after the change, it becomes $10y+x$, whence $20y+2x=3\cdot 10^n x+3y$,

or $17y=(3\cdot 10^n-2)x$. Since $(17, x)=1$, the smallest solution will have $x=1$ and or $17y=3\cdot 10^n-2$; the smallest n for which this is possible is 15, and the resulting integer is $N=1{,}176{,}470{,}588{,}235{,}294$.

Also solved by J. C. Abad, Ronald Alter, R. H. Anglin, Anders Bager, J. J. Bailey and J. B. Bohac (jointly), J. W. Baldwin, Nyles Barnet, Joseph Beer, C. Berndtson, M. T. L. Bizley, Walter Bluger, Robert Bowen, Norman Brenner, Brother L. Raphael, R. A. Bruce, W. E. Buker, Leonard Carlitz, J. R. Caspar, D. I. A. Cohen, E. L. and Irma B. Cohen (jointly), R. J. Cormier, Frank Dapkus, D. E. Daykin, Jane Evans, Virginia Felder, Stephen Fisk, B. S. Garbow, L. J. Giegerich, Jr., Michael Goldberg, L. D. Goldstone, Ralph Greenberg, S. H. Greene, Cornelius Groenewoud, B. A. Hausmann, C. V. Heuer, G. A. Heuer, D. L. Hobde and H. W. Westfall (jointly), R. T. Hood, J. A. H. Hunter, A. R. Hyde, R. A. Jacobson, J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), Donald Kanzaki, R. A. Katz, M. S. Klamkin, R. R. Korfhage, Sidney Kravitz, Robert La Fara, Harry Langman, Lawrence Lessner, D. C. B. Marsh, Jean-Paul Martin and Jean-Pierre Samson (jointly), R. W. Means, H. V. Monks, E. F. Myers, Sam Newman, C. S. Ogilvy, C. C. Oursler, J. L. Pabrinkis, Stanton Philipp, J. L. Pietenpol, Gerald Porter, L. A. Ringenberg, J. S. Romanow, Philip Rose, H. D. Ruderman, S. J. Ryan, C. M. Sandwick, Sr., Perry Scheinok, R. J. Schoenberg, S. L. Segal, D. L. Silverman, D. R. Simpson, H. P. Smith, Harlan Stevens, R. E. Strock, R. L. Syverson, G. C. Thompson, B. R. Toskey, Dennis Travis, Simon Vatriquant, R. N. Vawter, Andy Vince, G. W. Walker, Ronald Watson, John Weissman, Charles Wexler, K. S. Williams, B. B. Winter, A. L. Wollman, J. S. W. Wong, John Xenakis, J. E. Yeager, David Zeitlin, and the proposer.

Several solvers pointed out that this problem appears, with solution, on p. 11 of Dan Pedoe, The Gentle Art of Mathematics, Macmillan, 1960, it having appeared earlier, proposed by J. Bronowski, in *New Statesman and Nation*, Dec. 24, 1949. The problem is also considered in D. E. Littlewood, A digit problem, *Math. Gazette*, 39 (1955), Note 2494, p. 58. Zeitlin observed that the problem is a special case of Problem 4012, Jan., 1943. Ogilvy called attention to W. B. Chadwick, On placing the last digit first, this MONTHLY, 48 (1941), p. 251. Newman remarked that the smallest number that is increased by 50% by moving the digit on the extreme right to the extreme left is 285,714. Bizley said a more difficult problem would be to determine all rational numbers q/p such that an integer can be found which will increase in the ratio $p:q$ when the digit on the extreme left is moved to the extreme right.

Two Hexahedra

E 1531 [1962, 667]. *Proposed by C. W. Trigg, Los Angeles City College*

There are two distinct hexahedra with equal edges. Each has congruent regular polygons as faces. Show that one can be placed inside the other.

Solution by R. J. Cormier, Northern Illinois University. Consider the unit cube K centered at the origin with faces parallel to the coordinate axes. Place the two main vertices of the second hexahedron H at (a, a, a) and $(-a, -a, -a)$, where $a=\sqrt{2}/3$, and the other three vertices at $(b, -b, 0)$, $(-b, 0, b)$, $(0, b, -b)$, where $b=\sqrt{6}/6$. All the vertices of H lie strictly inside K. The faces of H are equilateral triangles of side 1.

Also solved by J. W. Baldwin, D. I. A. Cohen, D. E. Daykin, Michael Goldberg, Ned Harrell, Harry Langman, D. C. B. Marsh, Perry Scheinok, and the proposer.

A Differential Inequality

E 1532 [1962, 668]. *Proposed by J. B. Love, Eastern Baptist College*

Let $P(x)$ be a polynomial of degree n with only real roots and real coefficients. Show that $(n-1)[P'(x)]^2-nP(x)P''(x)\geqq 0$.

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* It is easily seen that $(n-1)P'^2-nPP''=0$ for $n=1$. Define $Q(x)=(ax+b)P(x)$, where a and b are real, so that $Q(x)$ is of degree $n+1$ with real roots and real coefficients, and suppose $(n-1)P'^2-nPP''\geqq 0$. Then

$$nQ'^2-(n+1)Q''Q=[(n+1)/n][(n-1)P'^2-nP''P](ax+b)^2 + (1/n)[naP-(ax+b)P']^2\geqq 0.$$

The result now follows by mathematical induction.

II. *Solution by Stanton Philipp, Seal Beach, Calif.* Let $x_1, \cdots, x_n$ be the roots of $P(x)=0$. If $x=x_i$, the theorem is trivial. If $x\neq x_i$,

$$P'/P=\sum_{i=1}^{n} 1/(x-x_i).$$

Differentiating, we obtain

$$(PP''-P'^2)/P^2=-\sum_{i=1}^{n}[1/(x-x_i)]^2.$$

By the Cauchy-Schwarz inequality,

$$\left[\sum_{i=1}^{n} 1/(x-x_i)\right]^2\leqq n\sum_{i=1}^{n}[1/(x-x_i)]^2.$$

Therefore

$$P'^2-PP''=P^2\sum_{i=1}^{n}[1/(x-x_i)]^2\geqq (P^2/n)\left[\sum_{i=1}^{n} 1/(x-x_i)\right]^2=P'^2/n,$$

or

$$(n-1)P'^2-nPP''\geqq 0.$$

Also solved by A. N. Aheart, Joseph Beer, Robert Bowen and A. C. Branch (jointly), Norman Brenner, J. L. Brown, Jr., D. E. Daykin, Ragnar Dybvik, Ralph Greenberg, S. H. Greene, Kit Hanes, J. C. Hickman, Erwin Just and Norman Schaumberger (jointly), M. S. Klamkin, Harry Langman, D. C. B. Marsh, F. D. Parker, J. L. Pietenpol, David Rothman, H. D. Ruderman, Perry Scheinok, Andy Vince, J. E. Wilkins, Jr., David Zeitlin, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Collaborating Editors: L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; and A. WILANSKY, Lehigh University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5092. *Proposed by A. Oppenheim, University of Malaya, Kuala Lumpur*

Suppose that $A_iB_iC_i$ $(i=1,\ 2)$ are triangles with sides a_i, b_i, c_i, area Δ_i, and altitudes p_i, q_i, r_i. Define numbers a_3, b_3, c_3 by the equations $a_3=(a_1^2+a_2^2)^{1/2}$, etc. Show that

(i) a_3, b_3, c_3 are sides of a triangle;

(ii) $p_3^2 \geqq p_1^2+p_2^2$, equality occurring if and only if the original two triangles are similar;

(iii) $\Delta_3 \geqq \Delta_1+\Delta_2$, with equality as in (ii);

(iv) $\Delta_3^2 \geqq 4\Delta_1\Delta_2$, with equality if and only if the triangles are congruent.

5093. *Proposed by S. M. Shah, University of Kansas*

Let $0<\lambda_n \leqq \lambda_{n+1}$, $n=1,\ 2,\ \cdots$; let $\phi(x)$ be positive and nondecreasing for $x \geqq \lambda_1$; and suppose that $\int_{\lambda_1}^{\infty} dt/t\phi(t) < \infty$. Prove that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{\phi(\lambda_n)}\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\right) < \infty.$$

5094. *Proposed by Albert Wilansky, Lehigh University*

Find two reversible matrices whose product is not reversible. ("Reversible" is defined by Banach, p. 90.)

5095. *Proposed by Gregory J. Lodge, Rensselaer Polytechnic Institute*

Prove that

$$\zeta(n) = \frac{(-1)^n}{(n-1)!}\,\Gamma_n(k) + S_n(i),$$

where n and k are positive integers $\neq 1$, $\zeta(n)$ is the Riemann zeta function, $\Gamma_n(x)$ is the nth poly-gamma function defined by $\Gamma_n(x)=d^n \ln \Gamma(x)/dx^n$, and $S_n(i)=\sum_{i=1}^{k-1} i^{-n}$.

5096. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime. Is it possible to find a set of p integers $a_1, \cdots, a_p$ such that $\prod_{j=1}^{p}(x+a_j)\equiv x^p+1 \pmod{p^2}$?

Is it possible to find a set of p^2 integers $a_1, \cdots, a_{p^2}$ such that $\prod_{j=1}^{p^2}(x+a_j)\equiv x^{p^2}+1 \pmod{p^2}$?

5097. *Proposed by Melvin Hausner, New York University*

Let an $n\times n$ matrix have positive entries along the main diagonal, and negative entries elsewhere. Assume that it is normalized so that the sum of each column is 1. Prove that its determinant is greater than 1.

5098. *Proposed by Richard M. Dudley, University of California, Berkeley*

Let $\exp_1(z)=\exp(z)$ and $\exp_{n+1}(z)=\exp(\exp_n(z))$, $n=1, 2, \cdots$. Show that there exists a nonconstant entire function f of one complex variable such that for each positive integer n there exists an entire function f_n such that $f=\exp_n(f_n(z))$.

5099. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Let k be an even number and $J_n(x)$ be the Bessel's function of the first species. Find the sum $\sum_{n=0}^{\infty}(2n+k)J_n(x)J_{n+k}(x)$.

5100. *Proposed by Seth Warner, Duke University*

To within isomorphism, find the number of rings there are whose additive group is cyclic of order m.

SOLUTIONS

A Set of Continuous Functions

5011 [1962, 172]. *Proposed by Julius Lieblein, David Taylor Model Basin, Washington, D. C.*

For a fixed $k>0$, find all continuous functions, $f(t)$, such that $I=\int_x^{kx}f(t)dt$ is independent of x, $x>0$.

Solution by J. Ernest Wilkins, Jr., General Dynamics Corp., San Diego, Cal. Since $f(t)$ is continuous, $0=dI/dx=kf(kx)-f(x)$. If $x=e^{\xi}$, $k=e^K$, and $G(\xi)=xf(x)$, we see that $G(\xi+K)=G(\xi)$, so that $G(\xi)$ is a periodic function with period $K=\ln k$. Conversely, if $f(x)=x^{-1}G(\ln x)$ for some function $G(x)$ periodic with period $K=\ln k$, then

$$I=\int_x^{kx}t^{-1}G(\ln t)dt=\int_{\xi}^{\xi+K}G(\eta)d\eta=\int_0^K G(\eta)d\eta$$

is indeed independent of x.

Also solved by P. R. Chernoff, H. E. Chrestenson, M. S. Demos, W. G. Dotson, Jr., P. G. Heyda, J. Koekoek, W. S. Lawton, R. D. Luce, Frank Owens, J. J. Roseman, W. C. Waterhouse, and the proposer.

Editorial Note. The functional equation $kf(kx)-f(x)=0$ has been treated extensively. See van den Berg, Nieuw Archief voor Wiskunde (3) III, (1955) 79–88; Koenigs, Ann. Sci. École Norm. Sup., 1 (1884), Supplement, S1–S4, also same Journal, 2 (1885) 385–404; Milkman, Proc. Amer. Math. Soc., 1 (1950) 505–509.

Ideals in a Domain of Integrity

5012 [1962, 172]. *Proposed by M. F. Ruchte and B. T. Sims, Iowa State University*

Recall that a domain of integrity is an associative ring which contains no proper divisors of zero, and that a division ring is a domain of integrity whose nonzero elements form a multiplicative group. Prove or disprove the following converse of a theorem of Ore:

Let S be a domain of integrity with identity 1. If S is imbedded in a division ring F, the intersection of every two nonzero right (left) ideals of S is nonzero.

Solution by G. M. Bergman, University of California, Berkeley. The proposed statement is not a theorem. Take a field $G(X)$ of rational functions in one indeterminate X over a field G, and consider the algebra obtained by adjoining an element Y satisfying $YX=X^2Y$. Multiplication satisfies the distributive law, and the rule

$$r(X)Y^i\cdot s(X)Y^j = r(X)s(X^{2^i})Y^{i+j}$$

is easily verified. Then every element of S can be written uniquely in the form $\sum_{i=0}^{\infty} r_i(X)Y^i$, where $r_i(X)\in G(X)$ and is zero for all but finitely many values of i.

Let the *degree* of a nonzero element be the least i such that Y^i has a nonzero coefficient. Then the product of an expression of degree i and one of degree j is of degree $i+j$, and, in particular, nonzero; so S is a domain of integrity.

Under left multiplication by members of $G(X)$, S forms a vector space over this field. Clearly, all left ideals are subspaces. The set D_i of elements of degree $\leq i$ (and 0) also forms a subspace, of dimension $i+1$.

Let u and v be elements of degree i and j, respectively. The elements u, $Yu, \cdots, Y^ju$ are linearly independent and lie in the vector space $Su\cap D_{i+j}$. (Su designates the left ideal generated by u.) Hence the latter is of dimension at least $j+1$. Similarly, $Sv\cap D_{i+j}$ is of dimension at least $i+1$. Since the sum of their dimensions is greater than that of the whole space D_{i+j}, they must have nonzero intersection. Hence $Su\cap Sv\neq\{0\}$, and it follows immediately that any two nonzero left ideals have nonzero intersection. Then, by the direct theorem of Ore, S can be imbedded in a division ring.

On the other hand, we can display two right ideals that have zero intersection: the right ideals generated by Y and XY, respectively. For if $\sum r_i(X)Y^i$ is a right multiple of Y, all the r_i will be even functions of X (i.e., they will belong to $G(X^2)$), while if it is a right multiple of XY, they will all be odd (i.e., $\in XG(X^2)$).

Hence S can be imbedded in a division ring, yet there are right ideals with zero intersection.

Also solved by P. R. Chernoff.

Additive Isomorphism

5013 [1962, 236]. *Proposed by L. E. Ward, Jr., U. S. Ordnance Test Station, China Lake, California*

Let R denote the set of real numbers, Z the set of complex numbers. Are R and Z isomorphic when considered as additive groups?

Solution by J. W. Ellis, Louisiana State University at New Orleans. Let H be a Hamel basis for the reals R over the rationals F. Since H is infinite, $2\,\mathrm{card}(H)=\mathrm{card}(H)$, so that there are complementary sets H_1 and H_2 in H and $1-1$ onto functions f_1: $H \to H_1$ and f_2: $H \to H_2$.

Now given x and y real, from the basis property we may write uniquely $x=\sum_{i=1}^{n} a_i h_i$ and $y=\sum_{j=1}^{m} b_j k_j$, with each a_i, $b_j \in F$ and each h_i, $k_j \in H$. We then define

$$g(x+iy) = \sum_{i=1}^{n} a_i f_1(h_i) + \sum_{j=1}^{m} b_j f_2(k_j).$$

It is now a simple matter to verify that g is an isomorphism of the additive complex numbers onto the additive reals.

Also solved by K. A. Baker, W. G. Brown, Robert Bowen, J. G. Ceder, P. R. Chernoff, P. M. Cohn, S. E. Dickson, Seymour Ditor, D. M. Friedlen, Alvin Hausner, H. Kestelman, D. U. C. Kranston, Robert Kruse, W. A. LaBach, Andrzej Mąkowski, Joan Moore, Barbara L. Osofsky, N. T. Peck, Gerald Porter, P. L. Renz, N. R. Riesenberg, G. S. Rogers, J. E. Shockley and F. W. Weiler, Larry Smith, B. R. Toskey, Dennis Travis, Sherwood Washburn, W. C. Waterhouse, A. Ziv, and the proposer.

Editorial Note. The mapping outlined above is plainly not continuous. (In fact there is no way of presenting explicitly the elements of a Hamel basis. See G. Hamel, Math. Ann., 60 (1905) pp. 459–462.) Overlooking this, a number of our contributors denied the possibility of the isomorphism since, for all rational r, if $1 \leftrightarrow u+iv$, isomorphism requires $r \leftrightarrow ru+irv$.

Polygon Imbedded in a Lattice

5014 [1962, 236]. *Proposed by M. S. Klamkin, State University of New York at Buffalo.*

It is well known that an equilateral triangle cannot be imbedded in a square lattice. It can be imbedded, however, in a cubic lattice. Can this be extended, i.e., can any regular polygon be imbedded in a cubic lattice of high enough dimension?

Solution by H. E. Chrestenson, Reed College. Suppose that a regular n-gon is imbedded in a lattice. If s and d are the lengths of side and shortest diagonal, respectively, the law of cosines gives

$$d^2 = 2s^2 - 2s^2 \cos(\pi - 2\pi/n) = 2s^2 + s^2(2\cos 2\pi/n).$$

Since s^2 and d^2 are integers, 2 cos $2\pi/n$ must be rational. A theorem of D. H. Lehmer (see I. Niven, *Irrational Numbers*, Carus Monograph No. 11, p. 37) states that 2 cos $2\pi/n$ is an algebraic integer of degree $\phi(n)/2$. Thus $\phi(n)$ must be 2, whence n is 3, 4 or 6. To imbed a regular hexagon, let the origin and A and B be lattice point vertices of an equilateral triangle (e.g. A: (4, 1, 1) and B: (1, 4, 1).) By expanding the triangle by a factor of 3 and reflecting in the centroid we see that the origin, $2A-B$, $3A$, $2A+2B$, $3B$, and $2B-A$ are vertices of a regular hexagon. Thus a regular n-gon can be imbedded in a lattice if and only if $n=3$, 4 or 6, and in these cases a cubic lattice suffices.

Also solved by K. A. Baker, P. R. Chernoff, Michael Goldberg, Andrej Makowski, John McGuire, and D. L. Silverman.

A Determinant Related to $f(x)$

5015 [1962, 236]. *Proposed by Basil Gordon and E. G. Strauss, University of California, Los Angeles*

Suppose that $f(x)$ is continuous on $[a, b]$ and n times differentiable on (a, b), and that $a=x_0<x_1< \cdots <x_n=b$. Show that

$$\begin{vmatrix} 1 & 1 & \cdots 1 \\ x_0 & x_1 & \cdots x_n \\ \cdot & \cdot & \cdot \\ x_0^{n-1} & x_1^{n-1} & \cdots x_n^{n-1} \\ f(x_0) & f(x_1) & \cdots f(x_n) \end{vmatrix} = \frac{1}{n!} f^{(n)}(\xi) \prod_{i>j} (x_i - x_j),$$

where $a<\xi<b$.

Solution by M. V. Subbarao, University of Missouri. Let A be the given determinant and consider

$$(1) \qquad F(x) = \begin{vmatrix} 1 & 1 & 1 & \cdots 1 \\ x & x_0 & x_1 & \cdots x_n \\ \cdot & \cdot & \cdot & \cdot \\ x^{n-1} & x_0^{n-1} & x_1^{n-1} & \cdots x_n^{n-1} \\ x^n & x_0^n & x_1^n & \cdots x_n^n \\ f(x) & f(x_0) & f(x_1) & \cdots f(x_n) \end{vmatrix}$$

It is clear that $F(x_0)=F(x_1)= \cdots =F(x_n)=0$. Hence by repeated application of Rolle's theorem, $F^{(n)}(\xi)=0$ for $a<\xi<b$, where $F^{(n)}(\xi)$ is the determinant (1) with its first column replaced by $0, 0, \cdots, 0, n!, f^{(n)}(\xi)$. By expansion, $F^{(n)}(\xi) =(-1)^n n!A-f^{(n)}(\xi)\cdot V$, where V is the familiar Vandermonde determinant whose value is $\prod_{i>j}(x_i-x_j)$. The desired result follows immediately.

Also solved by W. G. Brown, L. Carlitz, Jyoti Chaudhuri, P. R. Chernoff, D. V. Clingan, C. R. De Prima, A. B. Farnell, Walter Gautschi, Joseph Hamer, J. R. Hatcher, J. C. Hickman,

A. P. Hillman and P. B. Manchester, J. Horvâth, W. J. Kammerer, C. Kassimatis, R. G. Kayel, H. Kestleman, T. J. Lee, J. B. Linder, J. P. Line, James Nearing, J. C. C. Nitsche, D. Pedoe, J. E. Potter, B. E. Rhoades, Paul Schaefer, Berthold Schweitzer, L. G. Settle, John Todd, J. F. Traub, J. H. van Lint, Robert Vermes, J. H. Wahab, W. J. Walbesser, Sherwood Washburn, W. C. Waterhouse, J. S. W. Wong, J. E. Yeager, David Zeitlin, and the proposers.

Editorial Note. Essentially this result is found in several places in the literature. Readers supplied the following references: Nörlund, *Vorlesungen über Differenzenrechnung*, p. 13: L. M. Milne-Thomson, *The Calculus of Finite Differences*, pp. 10–12; Polya u. Szegö, *Aufgaben und Lehrsätze*, v. II, p. 54, problems 95–97; and a more general result in Bourbaki, *Fonc. Var. Reélle*, Ch. I, Section 3, ex. 11.

A Sequence of Even Polynomials

5016 [1962, 237]. *Proposed by J. S. Frame, Michigan State University*

An infinite sequence of even polynomials $P_n(x)$ is defined by the differential equation $P_n''(x) = -P_{n-1}(x)$, together with the boundary conditions $P_0(x)=1$, $P_n(1)=P_n(-1)=0$ for $n>0$. Prove that

$$\int_{-1}^{1} P_{n-1}(x)P_{m-n}(x)dx = 4^m(4^m - 1)\cdot 2B_m/(2m)!,$$

where B_m denotes the mth Bernoulli number, given by the expansion

$$1 - \frac{t}{2}\cot\frac{t}{2} = \sum B_m t^{2m}/(2m)!.$$

Solution by J. H. van Lint, Technical University, Eindhoven, Netherlands. We define $\phi(x, t) = \sum_{n=0}^{\infty} P_n(x)t^{2n}$. From the defining properties of $P_n(x)$ we immediately find $\phi(x, t) = \cos tx/\cos t$. If we call the integral to be calculated $I_m = I_{n-1,m-n}$ we find by integrating by parts

$$I_m = I_{n-1,m-n} = I_{n-2,m-n+1} = \cdots = \frac{1}{m}\sum_{\nu=0}^{m-1} I_{\nu,m-\nu-1}.$$

Hence

$$\begin{aligned}\left(\sum_{m=1}^{\infty} I_m t^{2m}\right)' &= 2\sum_{m=1}^{\infty} t^{2m-1}\sum_{\nu=0}^{m-1}\int_{-1}^{1} P_\nu(x)P_{m-\nu-1}(x)dx \\ &= 2t\int_{-1}^{1}\phi^2(x, t)dx = \frac{2t}{\cos^2 t} + 2\tan t \\ &= (2t\tan t)'.\end{aligned}$$

Hence I_m is the coefficient of t^{2m} in the power series expansion of

$$2t\tan t = 2[(1 - 2t\cot 2t) - (1 - t\cot t)],$$

i.e., $I_m = 2\cdot 4^m(4^m - 1)B_m/(2m)!$

Also solved by R. Ainsworth, L. Carlitz, P. R. Chernoff, Edward Delahanty and David Cutler, L. M. Kaplan, Margaret M. LaSalle, J. B. Linder, A. M. Peiser, Jean M. Quoniam, and the proposer.

Algebraic Functions

5017 [1962, 237]. *Proposed by Leonard Carlitz, Duke University*

Let $w=f(z)$ be an analytic function of the complex variable $z=x+iy$ that is an algebraic function of the two real variables x, y. Show that w is an algebraic function of z.

I. *Solution by Harley Flanders, Purdue University.* Let $w^n+r_1w^{n-1}+\cdots+r_n=0$ be the irreducible equation for w, where the r_i are rational functions in x, y and hence rational functions in z, $\bar{z}$. We differentiate, using the analyticity of w in the form $\partial w/\partial\bar{z}=0$:

$$\frac{\partial r_1}{\partial \bar{z}} w^{n-1} + \cdots + \frac{\partial r_n}{\partial \bar{z}} = 0.$$

Since this algebraic equation for w has lower degree than the irreducible one, we must have $\partial r_1/\partial\bar{z}=\cdots=\partial r_n/\partial\bar{z}=0$, i.e., $r_1, \cdots, r_n$ are all analytic, hence are rational functions of z. This means that w is an algebraic function.

The same result is valid for a function of several complex variables.

II. *Solution by A. P. Hillman and P. B. Manchester, University of Santa Clara.* The hypothesis tells us that w satisfies a polynomial equation $P(x, y, w)=0$. Substituting $z-iy$ for x, we have $P(z-iy, y, w)=Q(z, y, w)=0$. Taking partial derivatives of $P(x, y, u+iv)=0$ and using the Cauchy-Riemann equations we have

$$P_x = -(u_x + iv_x)P_w, \qquad P_y = -(u_y + iv_y)P_w = -i(u_x + iv_x)P_w = iP_x.$$

But this and $Q_y=-iP_x+P_y$ imply $Q_y=0$ as desired.

Also solved by G. M. Bergman, P. R. Chernoff, Jim Shilleto, J. H. van Lint, and the proposer.

Covering a Rectangle with Two Smaller Similar Rectangles

5018 [1962, 237]. *Proposed by D. J. Newman, Yeshiva University*

Show that a rectangle can be covered by two smaller rectangles similar to itself if and only if it is not a square.

Solution by Robert Breusch, Amherst College. Let a and $b(a\leqq b)$ be the sides of the given rectangle R.

1) If $a=b$, and if any two smaller squares would cover R, each of them would have to cover two adjacent corners (Figure 1). Also, $EFGH$, of side length $x<a$, would have to cover one side, as AB, and at least half of the sides $AD+BC$.

But it covers only

$$\begin{aligned} AU + BV &= (x - a\cos\alpha)/\sin\alpha + (x - a\sin\alpha)/\cos\alpha \\ &< a(1 - \cos\alpha)/\sin\alpha + (1 - \sin\alpha)/\cos\alpha < a, \end{aligned}$$

where α is the angle between corresponding sides of the two squares. Thus a square cannot be covered.

2) If $a<b<2a$, then $k<1$ exists such that $b/2 \leqq ka < a \leqq kb < b$. Therefore there is a covering as indicated in Figure 2 with $x=ka$, $y=kb$, such that $b>y\geqq a$, $a>x\geqq b/2$ as required.

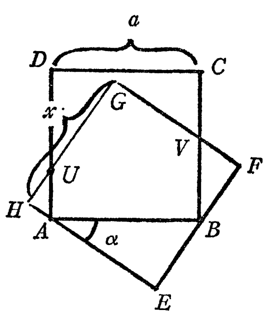

FIG. 1

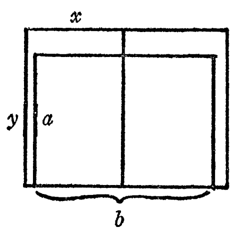

FIG. 2

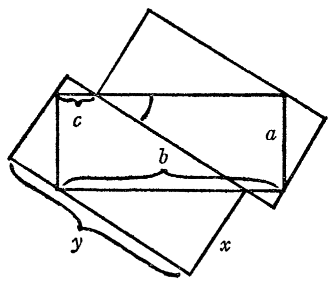

FIG. 3

3) If $b\geqq 2a$ (in fact, if $b>a\sqrt{2}$) a covering as indicated in Figure 3 is possible, with $\tan\alpha = a/(b-2c)$, and thus $c=\frac{1}{2}(b-a\cot\alpha)$. Then let

$$x = a\cos\alpha + c\sin\alpha = \tfrac{1}{2}a\cos\alpha + \tfrac{1}{2}b\sin\alpha,$$

$$y = a\sin\alpha + (b - c)\cos\alpha = a\sin\alpha + \tfrac{1}{2}b\cos\alpha + \tfrac{1}{2}a\cos^2\alpha/\sin\alpha.$$

If α is now chosen such that $x/y = a/b$, then

$$\sin\alpha = a/\sqrt{(b^2 - a^2)}, \qquad \cos\alpha = \sqrt{(b^2 - 2a^2)}/\sqrt{(b^2 - a^2)}.$$

Also, with this choice of α, $x = a(\sqrt{(b^2-2a^2)}+b)/2\sqrt{(b^2-a^2)} < a$.

Also solved by Donald Barnes and Robert Spira and Robert Bowery, Walter Bluger, C. F. Evans, Michael Goldberg, B. R. and C. E. Henry, R. G. Kayel, John McGuire, and the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

NOTE: Although the MONTHLY does not normally review high school materials, the widespread interest of the collegiate mathematical community in the improvement of the curriculum of the secondary schools makes it seem desirable to print the following thoughtful statements. It should be emphasized that they are not presented as thesis and rebuttal, but rather as essays which may clarify viewpoints on the appropriate development of tenth-grade geometry. It is not anticipated that these columns can usefully be devoted to extensive or continued discussions of this sort.

SAMPLING A MATHEMATICAL SAMPLE TEXT

ALEXANDER WITTENBERG, Laval University, Quebec

Until recently, the work of the various groups concerned with reforming the teaching of mathematics in schools proceeded amidst an idyllic lack of published critical comment [1]. It was, therefore, somewhat of a sensation when a group of 65 mathematicians, some of them most distinguished, publicly stated fundamental misgivings about current efforts, and warned of the tragedy that would occur if the curriculum reform should be misdirected and the golden opportunity for reform wasted [2]. Their statement, however, mostly confined itself to rather lofty areas of disagreement, such as the relative educational merits of modern *vs.* traditional mathematics. It might thus seem to imply a tacit acknowledgement that, at least, the reform groups were doing sound work in implementing those goals that they had set themselves [3].

Are they really? In this paper I shall consider in detail the treatment of one fundamental topic of elementary geometry, namely area of plane figures, as given in the revised edition of the Geometry text developed by the School Mathematics Study Group for use in the tenth grade.

I hope to show that the facts of the case inevitably lead to this conclusion: the text is unsatisfactory on at least three counts: 1) as an elementary introduction to geometry and geometrical thinking; 2) as an introduction to logically precise mathematical thinking (one of the SMSG's avowed aims); 3) as an introduction to the modern conception of a deductive system (another SMSG aim). By *unsatisfactory* is meant a) that the text is apt to mislead the student, and b) that the text is definitely poorer than it easily could be at the same level of difficulty. The following critical remarks are by no means exhaustive.

The text takes up the study of area in Chapter 11, Areas of polygonal regions (hereafter abbreviated as p.r.). It first discusses the notion of a p.r.; essentially, it is a plane region built up from nonoverlapping triangles. Then, without mentioning why these regions are singled out, the text immediately proceeds to introduce measure of area for p.r. by means of the following four postulates and one definition:

"*Postulate* 17. To every p.r. there corresponds a unique positive number.

Definition. The *area* of a p.r. is the number assigned to it by Postulate 17.

Postulate 18. If two triangles are congruent, then the triangular regions have the same area.

Postulate 19. Suppose that the region R is the union of two regions R_1 and R_2. Suppose R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2.

Postulate 20. The area of a rectangle is the product of the length of its base and the length of its altitude."

Let us first look at these postulates, and at the use made of them in the text, from the standpoint of an elementary introduction to geometry.

After a student has got over his initial bewilderment at meeting four postulates, the first of which talks about p.r., the second about triangles, and the fourth about rectangles, he will soon notice that the whole matter boils down to this: He is handed the formula for the rectangle ready-made, and told without further ado that this is the right formula to use. That this formula is dignified by the name postulate will make little difference to him.

In fact, using this right formula is all the text proceeds to do. It first derives from it the formula for the area of a triangle, $\frac{1}{2}bh$. Since there are three possible choices of base b and altitude h in this formula, the text states: "The three formulas $\frac{1}{2}b_1h_1$, $\frac{1}{2}b_2h_2$ and $\frac{1}{2}b_3h_3$ must give the same answer, because all three of them give the right answer for the area of the triangle." Then the text derives the formula for the area of a parallelogram, bh. A little later, it proves that two triangles with equal altitudes and equal bases have equal areas thus: "The proof of this is clear because the formula $A=\frac{1}{2}bh$ gives the same answer in each case."

Thus, "having the same area" never takes on any other meaning than this: the formula happens to yield the same value.

But this blind manipulating of formulas is as misleading as it is unsatisfactory. For p.r., "having the same area" can easily be given a meaning that is at once intuitive and precise: it simply means that one region can be cut up into pieces with which we can build up the other. This, in particular, is the simple geometric content of the formulas and proofs mentioned above: the first expresses the fact that a triangle has the same area as a rectangle with the same base and half its altitude. The formula for the parallelogram tells us that the latter has the same area as a rectangle with the same base and the same altitude. As to the surprising fact that two triangles with the same base and altitude have the same area, even if one looks fat and small and the other long and thin, it is not a blind caprice of algebra. It follows from the fact that both are equivalent to the same rectangle. All these facts are easily verified by simple and direct geometrical reasoning, without any reference to measurement of areas. Once a student has a firm grasp of this simple situation, it may be interesting for him to learn that things are not so simple in space.

The text, in effect, hides all this from the student. Worse, it suggests to the student that there is no simple geometric content to the theory of area. Indeed,

it states at the beginning of Chapter 11: "In this chapter we will study the areas of p.r. and learn how to compute them. The 16 postulates that we have introduced so far would enable us to do this, but the treatment would be extremely difficult and quite unsuitable for a beginning geometry course like this one."

This is at once misleading and pedagogically objectionable. In effect, the student is given some empty formal deductions as a substitute for genuine geometrical thinking [**4**].

This judgment is confirmed by the further developments in the text. The following chapter takes up Similarity. The whole theory of similarity rests upon a basic proportionality theorem. This is proved in the text by means of that same formula for the area of the rectangle. Thus, this postulated formula is the only ultimate reason given the student for believing a simple fact like this one: If in a triangle a parallel to one side bisects one of the other sides, it will also bisect the third. Indeed, the student is not even allowed to see, as a simple geometric fact, that doubling the side of a square results in a square with area four times larger: "Given a square of side a, and a square of side $2a$, it is easy to see that the area of the second square is 4 times the area of the first. (This is because $(2a)^2=4a^2$.)"

Needless to say, no mention is made of the exciting significance the theory of similar areas takes on outside mathematics, particularly in engineering and biology [**5**].

As is to be expected, genuine geometrical thinking is even more definitely avoided in those cases where it would raise a real challenge. Thus, Postulate 20 really aims at leaving the student unaware of the one essential difficulty in the definition of measure of area for rectangles, the case where a side of the rectangle is incommensurable with the unit of length. Later, the area of the circle is derived by means of a vague discussion of limits which omits the simple geometrical reasoning that really stands on the threshold to the modern theory of limits: if the area of a circle is approximated by an inscribed regular polygon, and the number of sides of the polygon is doubled, then the residual error is cut by more than one half.

Yet the same text is sophisticated enough to prove theorems like these:

THEOREM 2-2. *Of any three different points on the same line, one is between the other two.*

THEOREM 2-3. *Of three different points on the same line, only one is between the other two.*

THEOREM 2-5. *Every segment has exactly one mid-point.*

This odd contrast strikes the reader again and again: the text is very sophisticated about trivial matters. But where some genuine sophistication of geometrical thought would be in order, as in a careful look at the limiting processes involved in defining the measure of areas for arbitrary rectangles or for circles, the text simply glosses over the difficulty. I shall call this feature the *pseudo-sophistication* of the text.

What about the text as an introduction to precise mathematical thinking? Obviously, a tenth-grade text cannot be logically complete, and this text does not claim to be. What one must expect, however, is that a text consistently stick to one level of logical refinement. If it treats some matters with great sophistication, and others only crudely, without pointing out the difference to the student, the latter may well be confused about the very nature of logically precise thinking. This is particularly true if, at the same time, the text constantly proclaims its devotion to exalted aims of logical and linguistic stringency, and if it fails to point out even difficulties the nature of which the student might grasp quite well.

The text states, as an explicit postulate, that two congruent triangles have the same area. But it makes no mention of the essential theorem to which this postulate points: *Two congruent p.r. have the same area.* Of course, this theorem is intuitively obvious. But not more so than the postulate.

In its discussion of the area of a circle, the text very carefully points out that it is impossible to *prove* that the area can be obtained as a limit of areas of inscribed polygons: "This can never be proved, since we have not yet given any definition of the area of a circle." The text therefore proceeds to *define* the area of the circle (as limit of areas of inscribed regular polygons).

But the same text had discussed the problem "to construct a square whose area is exactly the same as that of a given circle (squaring the circle)" in an earlier chapter, thus before it had defined what the area of the circle is. Moreover, the discussion of the area of a circle is followed by problems requiring the student to find the area of various other curvilinear figures, such as rings and the so-called moons of Hippocrates (*Lunulae Hippocratis*). But the circle is the only curvilinear figure the area of which is defined in the text. And no postulate states that, if a circle is cut into two parts, then its area is the sum of the areas of those two parts.

Of course, this is intuitive! But surely no more so than the fact stated in Postulate 19! If one has to be so careful about the latter, why may one be so careless when it comes to circles?

In passing, note the lack of inner coherence: the three subjects of squaring the circle, squaring the *lunulae Hippocratis*, and measuring the area of the circle, really belong together, historically as well as materially. If we were able to square the circle, the area of the circle could be measured in the same way as, say, that of a triangle. And the hope of achieving this rested historically upon the discovery of one curvilinear figure that could indeed be squared: *the lunulae.* The text conveys no awareness of these important and interesting relationships.

From the standpoint of logical thinking, there is yet another oddity to be pointed out. Why are the areas of p.r. treated by means of postulates, while those of circles are defined? Why the difference? In fact, the difference is misleading. The same difficulty arises in both cases: the measurement of the area of the circle involves limits. But so does that of the rectangle in the incom-

mensurable case. If it is necessary to define what is meant by the area of a circle, so it is for the rectangle. If the latter can be postulated, so can the former.

The muddle becomes even more pronounced if we proceed to the treatment of volumes. If it is necessary to define what is meant by the area of a circle, surely the same is true for the volume of a sphere. But this time there is no talk of definitions. Neither is there an equivalent to Postulate 17. It would be a little difficult, it is true, to define the precise analogue in space of a p.r., and the text notes this. But the same text uses the idea of a solid as an undefined intuitive notion in one of its postulates for volume (Postulate 22). Why may it dispense with a postulate stating that a solid has a volume, when it could not dispense with Postulate 17?

I noted above the pseudo-sophistication of the text. This is quite apparent in these examples: the text does not choose one level of logical rigor, in order to carry out a consistent and coherent development at this level. Instead, it throws a hodge-podge of isolated items culled from very advanced contexts into a development that basically is less sophisticated than it might be at tenth-grade level. The result is a text that proves THEOREM 4-2: *Every angle is congruent to itself*, while requiring the student to compute areas that have never been defined.

In fact, the text's whole approach to the theory of areas and volumes, if viewed with a mathematician's eyes, is a somewhat unfortunate mixture of two advanced mathematical approaches, to neither of which it does justice. The one is "measure theory," which proves as a theorem that there essentially is one and only one way to attribute a number as "measure of area" to reasonable regions in the plane [6]. The other consists in defining measure of area directly, a natural sequence being to define it first for triangles, then for p.r. since these are sums of triangles [7], and finally for curvilinear areas as limits of p.r. The text gives no hint of the kind of sophistication required by such approaches. At their level of sophistication, for instance, it is essential to *show* that if a p.r. is cut up into triangles in two different ways, the areas of the triangles will add up to the same sum in both cases. Without the careful elaboration required by approaches like these, the text's treatment of area, despite its high sounding terminology, really boils down to a rather crude formal approach, at once less to the point, and mathematically less challenging, than would be possible at the tenth-grade level.

What about the text as an introduction to the idea of a "deductive system"? After one has noticed how the text fails to live up to simpler claims, one approaches it with some skepticism concerning this last question.

What does it mean to present geometry as a "deductive system"? At the very highest level of refinement that may, with some stretch of the imagination, be within the grasp of a tenth-grader, it means to give a development which states a number of simple fundamental properties of the geometrical entities as postulates (without proof), and then proceeds to derive each other property from these as a theorem. This in fact seems to be the predominant position

adopted in the text. At least this is how the Teacher's Commentary comments on the area postulates: "Since we are introducing a block of postulates concerning area, this may be a good time to remind your students of the significance and purpose of postulates. They are precise formulations of the basic intuitive judgments suggested by experience, from which we derive more complex principles by deductive reasoning."

But we must ask another question: Why should anybody bother to develop geometry as a deductive system? Why, in fact, did the Greeks strive to do so? From the above standpoint, the answer is perfectly clear. It is not in order to get a system in which certain statements are deduced from certain other statements. It is rather because in this way, and only in this way, can it be seen that complex geometrical statements like the Pythagorean theorem have to be true, and can it be rationally understood why they are true. The distinction between postulate and theorem, while not entirely clear-cut, is essential from this standpoint. It is an achievement to make something complex intelligible in terms of something simple. The converse would hardly be a worthwhile achievement; for instance, proving "through two points there is one and only one straight line" from a set of postulates including the Pythagorean theorem!

This essential distinction, however, is blurred in the text. This is already apparent in Postulate 20. But it is most striking in connection with the last postulate: "We haven't proved it (the principle involved here); we have merely been explaining why it is reasonable. Let us therefore (*sic*) state it in the form of a postulate: Postulate 22. (Cavalieri's Principle) Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the two intersections have equal areas, then the two solids have the same volume."

This really is a theorem of the calculus and, incidentally, one that gave rise to passionate controversies before it could be proved [8]. By stating it as a postulate, the text confuses the meaning of a deductive system. Of course, there could be no objection to the text asserting a theorem without proof, if it were described as such. But the phrase "theorem assumed without proof" is the one phrase that does not occur in the text. This creates a façade of logical perfection.

One may entertain some doubts as to whether the SMSG itself is quite clear about the nature and role of a deductive system. Its latest text, *Geometry with coordinates*, which may be taken to embody its most mature views, suggests in its introductory chapter "Introduction to Formal Geometry" that the Greeks introduced the deductive method in order to bring order and organization to a subject that previously had contained only isolated experimental facts like this one: a triangle with sides 3, 4, 5 will have a right angle. Bringing order and organization, however, may be a bureaucrat's dream; it certainly was not the Greeks'. Their tremendous achievement was to show that it is possible to *understand* why that fact is so. And it is precisely this motivation that may stir a modern youngster. As to the subsequent discussion in that chapter, it con-

trasts deductive reasoning with the inductive reasoning which draws general conclusions from individual instances. But for mathematics, this is quite a misleading antithesis. The genuine contrast is to heuristic or intuitive reasoning. This is already perfectly clear in the works of Archimedes [9].

The SMSG Geometry text has 640 pages, 8 appendices, 22 postulates, and 197 theorems and corollaries. Our findings might be summed up thus: There is much less in this enormous text than meets the eye. Its superficially impressive modern attire does not withstand close analysis, be it on pedagogical, on strictly mathematical, or on philosophical grounds. The text looks sophisticated because it avoids the simple and straightforward; yet there is no compensating gain, and sophistication is conspicuously absent where it really would matter.

For the student, and sometimes, I am afraid, for the teacher as well, this approach amounts to an education in pseudo-sophistication. He is not taught to think simply, yet carefully, about some challenging elementary mathematics. He is taught to be pedantic about trivial things. "Problem 11-1, 6a: Is the following statement true or false: A triangle is a polygonal region. Answer: False. A triangle is not a region at all, but is a figure consisting of segments." Worse still, he is taught to manipulate words like postulate or mathematical system as if he understood them, without being aware of their true implications. This is not the way to start an education in science. Pseudo-sophistication is the death, not the beginning, of science [10].

Considering the scope of the SMSG's enterprise, these are surprising findings, to say the least. It would take another paper to explore all their implications. Apparently, something must be wrong with the SMSG's whole approach to its appointed task. Here, I shall only point out one fact in this connection.

While the SMSG publicity somehow creates the impression that its texts are the brain-children of dozens of outstanding mathematicians and teachers, the actual authors of any one text remain anonymous. Moreover, nobody acknowledges editorial responsibility for any of the texts. Thus, not one mathematician has staked his professional standing on the quality or soundness of any of the SMSG texts. This is an unfortunate departure from the way in which most valuable books, texts or otherwise, have been written so far during the history of mankind [11].

References

1. A lone dissenter has been Prof. Morris Kline of N.Y.U. His paper *Math. Teaching Reforms Assailed as Peril to U.S. Scientific Progress* significantly enough did not appear in any of the major journals concerned with mathematical education, but in N.Y.U. Alumni News, October 1961.

2. *On the mathematics curriculum of the high school,* this MONTHLY and Math. Teacher, both March 1962. French translation in Bull. de l'Ass. des Prof. de Math. de l'Ens. Public, Paris, June 1962; German translation in Math. Naturwiss. Unterricht, October 1962.

3. This impression is apparent in an analysis by one thoughtful outsider, written, it is true, before the statement of the 65: Benjamin DeMott, *The Math. Wars,* The Amer. Scholar, Spring 1962.

4. See the brilliant discussion *The Area of the Parallelogram* in Max Wertheimer, *Productive Thinking,* Harper. It is very instructive to confront the SMSG approach with Wertheimer's analysis.

5. See, e.g., the chapter *On magnitude* in D'Arcy Thompson's *On Growth and Form*, abridged ed., Cambridge Univ. Press, 1961.

6. A very clear treatment of this approach is contained in one of the SMSG's own publications: A translation of a Russian textbook for teachers by B. V. Kutuzov, issued as *Studies in Mathematics, Vol. IV, Geometry*. A brilliant discussion of the nature and motivation of the modern theories of area and volume has been given in a talk to Swiss teachers by Hadwiger, *Der Inhaltsbegriff, seine Begründung und Wandlung in älterer und neuerer Zeit*, Mitt. Naturf. Ges. Bern. N.F., Vol. 11, 1954. This beautiful lecture, at once elementary and profound, would well warrant an English translation.

7. This is done in Ch. IV of Hilbert's classical *Foundations of Geometry*. One of the necessary prerequisites for this approach is to prove that, in a triangle, the three possible base times altitude products are equal. The SMSG postulates really amount to assuming this without proof. But this really is a theorem about similar triangles so that a more natural development along the SMSG's own lines would be: a) assume the proportionality theorem without proof; b) prove the above-mentioned theorem; c) define the area of a triangle as $\frac{1}{2}$ bh, and then define the area of a p.r. as a sum of areas of triangles.

8. See Otto Toeplitz, *Die Entwicklung der Infinitesimalrechnung*, Springer, 1949, p. 55 ff. Toeplitz also points out, which SMSG does not, the danger of the kind of reasoning embodied in Cavalieri's principle. A similarly reasonable principle can be used for an easy proof that doubling the radius of a circle also doubles (instead of: quadruples) its area. By inducing the student to accept postulates like these as reasonable, the text in effect throws him back to the level of mathematical thinking that prevailed in modern times before the dawn of the 19th century.

9. Witness the contrast between his Method and his rigorous proofs. (Archimedes, *Works*, edited by Heath, Dover.)

10. There are important underlying issues here. They are explored at length in A. Wittenberg, *Bildung und Mathematik*, Ernst Klett Verlag, Stuttgart, 1963. The teaching of plane area is one of the examples discussed in detail in this book.

11. I wish to thank my colleague Dr. F. Goodspeed for his very stimulating critical comments on an earlier draft of this paper. He does not share any responsibility for the views expressed herein, however.

SOME REFLECTIONS ON THE TEACHING OF AREA AND VOLUME

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1. Introduction. This article is, in a sense, a response to the preceding article by Professor Alexander Wittenberg. It is not, however, a debater's response. There are reasons for this. In the first place, the polemical tone is a handicap in exposition; and the expository problems involved here are difficult enough at best. In the second place, the rhetoric in Professor Wittenberg's article is in an important way quite incongruous with the substance of his views. The main burden of his criticism is that the School Mathematics Study Group's Geometry book was less thorough and less ambitious than it should have been, in presenting the substance of the geometric theory and in explaining its underlying logic. If these criticisms apply to the SMSG book, then surely they apply with double or triple force to the other tenth-grade books now in common use. Thus, in effect, Professor Wittenberg is accusing us of timidity; and the accusation has a certain charm, due in part to its novelty: our book has more commonly been regarded as recklessly ambitious. With scattered exceptions, it includes all

topics recommended by the College Board Commission, plus more; it furnishes many proofs, notably in solid geometry, that the Commission proposed to omit; and in general its level of rigor and exactitude is considerably higher than the Commission suggested. Moreover, polemical exchanges often tend merely to perpetuate intellectual misunderstandings.

For these reasons among others, I propose to explain in a positive sense the ideas which in my own opinion underlie the treatment of area and volume in the SMSG book. I say *in my own opinion* because I have no right to try to speak for anyone else.

2. The choice of a theory. In designing a presentation of area-theory for young students, we first need to decide which theory of area is going to be presented. There are two main alternatives.

(A) *The synthetic theory.* This was the theory implicit in Euclid. For the sake of brevity and clarity, we describe it in modern terms, but rather informally.

A polygonal region is a plane set which can be cut up into a finite number of triangular regions, intersecting one another only in edges and vertices. If R_1 and R_2 can be cut up in this way, and a one-to-one correspondence can be set up between the triangular regions in R_1 and those in R_2, in such a way that corresponding triangular regions are congruent, then R_1 and R_2 are *equi-areal*, or have *the same area*. Note that the latter phrase is grammatically peculiar, because so far in the theory there is no such thing as the area of either of them. If R_1 and R_2 are equi-areal, then we write $R_1 \equiv R_2$. If R_1 is equi-areal with a proper subset of R_2, then we write $R_1 < R_2$.

Obviously the relation $\equiv$ is reflexive and symmetric. In fact, it is also transitive, but the proof is nontrivial; it depends on the fact that every two triangulations of the same polygonal region have a common subdivision. It is possible to show that the relations $\equiv$ and $<$ are incompatible. That is, we never have both $R_1 \equiv R_2$ and $R_1 < R_2$. But the proof is not easy.

Using the theory of Eudoxus, we can now explain what is meant by a proportionality $R_1 : R_2 :: S_1 : S_2$, where R_1 and R_2 are regions and S_1 and S_2 are segments. The proportionality means that for every pair of natural numbers p, q, the following two statements are equivalent:

(i) If U is the union of p disjoint congruent copies of R_1, and V is the union of q disjoint congruent copies of R_2, then $U < V$.

(ii) If W is the union of p congruent copies of S_1, end to end, and X is the union of q congruent copies of S_2, end to end, then W is a shorter segment than X.

This is what the proportionality $R_1 : R_2 :: S_1 : S_2$ meant to Euclid; and it is in these terms that we must interpret Proposition I of Book VI of the *Elements*, which asserts that if two triangles have the same altitude, then they are to each other as their bases. In the synthetic theory, this theorem is not a mere equation between fractions. The easiest way to see the intuitive content of the Eudoxian scheme is to suppose that a real-valued area-function α is given, with

the usual properties, and that another function β is given, measuring lengths of segments. The proportionality ought then to mean that

$$\frac{\alpha R_2}{\alpha R_1} = \frac{\beta S_2}{\beta S_1}.$$

And conditions (i) and (ii) of Eudoxus' definition take the forms:

$$\text{(i}'\text{)} \quad p\alpha R_1 < q\alpha R_2, \qquad \text{(ii}'\text{)} \quad p\beta S_1 < q\beta S_2.$$

These are equivalent to

$$\text{(i}''\text{)} \quad \frac{p}{q} < \frac{\alpha R_2}{\alpha R_1}, \qquad \text{(ii}''\text{)} \quad \frac{p}{q} < \frac{\beta S_2}{\beta S_1}.$$

Thus, when we say that (i) and (ii) are equivalent, this means that the rational numbers less than $\alpha R_2/\alpha R_1$ are precisely the rational numbers less than $\beta S_2/\beta S_1$. With a little reflection, the reader can convince himself that in effect Euclid was using Dedekind cuts in the rationals as a *working definition* of real numbers. Anyone looking for geometric simplicity had better look elsewhere.

We present at length and explicitly this sample of Euclidean area-theory, because while the theory is widely alluded to, it seems to be understood only by specialists and by people who happen to have become intrigued by it. The language of this theory survives in high-school books, as a habit and as a pseudo-Euclidean gesture; but the theory itself (in America, at least) has been abandoned. I believe that there are good reasons for this.

In the first place, the theory is sophisticated, and is logically very difficult, with or without modern exactitude. The subtleties begin with Definition 4 of Book V of the *Elements;* they increase with Definition 5; and they continue thereafter.

In the second place, the theory does not lend itself to the design of suitable problem and exercise material. This point can hardly be overemphasized. One of the worst weaknesses in the recent tradition of mathematics teaching is the gulf between the problem work, which the student *does*, and the so-called "theory," of which he is merely a spectator. In many courses, this gulf is wide. (See almost any calculus book.) But in geometry, it is easy to avoid: if the theory is appropriately formulated, then the student can and does go further along the same path in proving originals. If we allude to a theory which we cannot really explain or use, and state definitions which are too subtle for the student to use in dealing with the concepts that they purport to describe, then we have thrown away a crucial virtue of geometry; and we have fallen, I fear, into pseudo-sophistication.

Third, and fatally, the synthetic concept of area (or, more precisely, *equal-area*) is not the concept that is going to be important to the student in the following few years. In all sorts of applications, in the sciences and elsewhere, lengths, areas and volumes are measured with real numbers. The same is true

in analytic geometry and calculus. In the context of modern mathematics and science, the ideas of equal-length and equal-area (not to mention the Eudoxian idea of proportionality) simply do not deserve to be regarded as central concepts.

B. Area considered as a real-valued function. These, I believe, are sufficient reasons for presenting area as a real-valued function. In doing this, we must decide what we are going to assume without proof, and what we are going to prove. To derive the formulas that everybody uses, and solve the problems that all students are supposed to solve, we surely need the following:

(1) To every polygonal region there corresponds a unique positive number.

This is merely an informal way to saying that an area-function is part of the mathematical structure to be studied. The number is called the *area* of the region.

(2) If two triangles are congruent, then the triangular regions have the same area.

(3) If two polygonal regions intersect only in edges and vertices, then the area of their union is the sum of their areas.

Logically speaking, of course, we can get by without using any postulates at all. On the basis of the Parallel Postulate, it is possible to define an area-function satisfying (2), (3) and also (4) below. But the proof is so difficult (see Chapter 14 of [2]) that to present it in the tenth grade would be quite out of the question. Moreover, students seem to accept and use (1), (2) and (3) with no trouble. At any rate, everybody uses them, either explicitly or tacitly.

If these statements hold, for a given area-function, then they continue to hold when the function is multiplied by a constant. We therefore need to fix the unit of area; and here there are several possibilities:

(4) The area of a square region of edge 1 is equal to 1.

(4′) The area of a square region of edge a is equal to a^2.

(4″) The area of a rectangular region is the product of its base and its altitude.

Using (1), (2) and (3), we can show that each of these conditions implies the next. To show that (4′) implies (4″), we decompose a square region of edge $a+b$ into two square regions and two rectangular regions, as in the figure. The two rectangular regions have the same area A, by (2) and (3). By (3) and (4′) we have

$$(a + b)^2 = a^2 + b^2 + A + A,$$

so that

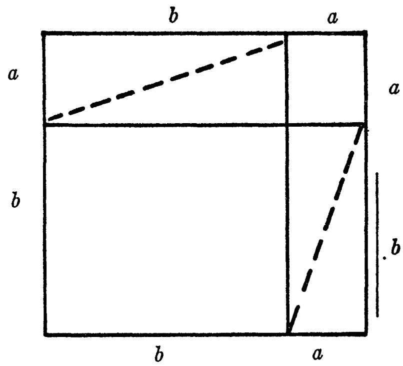
$$A = ab,$$

which was to be proved.

But a sophisticated proof is required to deduce (4′) from (4). (See Chapter 13 of [2].) It seems, therefore, that one should take (4′) as a postulate, and

prove (4″) as a theorem. In the SMSG book we used (4″) as a postulate, probably because we had never heard of its simple proof. I still believe that for the purposes of the tenth grade, (4) is quite unusable.

Obviously, if students are going to get numerical answers to area problems, then we must, at some stage, introduce area as a real-valued function. And granted that we are going to do this eventually, we should do it soon, so that the full force of the theory can be used for the things that it is good for. It leads to an easy proof of the Pythagorean Theorem, and also to an easy proof of the proportionality theorem on which the theory of similarity depends. The latter theorem asserts that a cross-bar parallel to the base of a triangle cuts the other two sides into proportional segments. Our proof is like Euclid's. I believe that it was economies of this kind that enabled our book to be both fairly rapid and fairly exact.



3. The role of deductive reasoning. The idea of a proof means a number of things, to various people at various stages of their development. To a logician, a proof is a formal process based on rules of inference stated in advance and in full. Working mathematicians (unless they happen also to be logicians) hardly ever use proofs in this sense. They have, however, an idea which one may call that of a conclusive proof, adequate to guarantee that a theorem is true, independent of the endorsement of any higher authority.

Realistically speaking, I think we should admit that in the tenth grade, proof means neither of these things. I remember being told by R. L. Moore that when he was trying to decide whether a man was ready for a doctorate, one of the things he asked himself was whether the man really knew a proof when he saw one. If Moore's criterion is significant in the third graduate year, then it is overoptimistic in the tenth grade.

There are, to be sure, some topics whose underlying logic is so simple that it can be presented virtually in full to young students. This is true, for example, of the rudiments of the theory of congruence for triangles. To blur or truncate the logic of this topic would be a gratuitous confusion, like saying that the number

of feet in a yard is a little less than π. Much the same thing is true of the theory of areas, for polygonal regions. The required assumptions are few and simple; and the standard theorems really can be deduced from them, in ways that young students can understand. Topics like this offer important opportunities: they enable us to present models of exact reasoning which can later be imitated, in more difficult contexts, when the student is more mature. Most of the topics in elementary geometry can be treated in this way, if we formulate the concepts with care, and avoid alluding to subtleties and complications which we don't seriously propose to teach.

If we present mathematics in a coherent fashion, however, and teach the things that most people think we should, we are faced with the fact that some very elementary topics have an underlying logic which is incapable of being explained at an elementary level. Thus everybody talks about π; but not until years later does anybody talk about the completeness of the real number system in the sense of Dedekind, which is needed in order to guarantee that such a number as π exists.

The word *intuition* has been so much abused lately that one hesitates to use it at all. Often it seems to be merely a euphemism for *rote.* I have heard one man speak of an intuitive treatment of solid mensuration, in which the formulas would be given without derivations.

In a more significant sense than this, I believe that the treatment of areas of circles (in the tenth grade) ought to be intuitive. By this I mean that one should explain some, but not all, of the underlying logic of the situation. The mathematical theory, in outline, might look something like this:

DEFINITIONS. *The inner measure of a bounded plane set is the supremum of the areas of the polygonal regions that lie in it. Its outer measure is the infimum of the areas of the polygonal regions that contain it. If these are the same, then the set is measurable, and their common value is called its measure, or its area. A curve G is rectifiable if the lengths of its inscribed broken lines form a bounded set W. If G is rectifiable, then the supremum of W is called the length of G.*

THEOREM 1. *Every circular region is measurable.*

THEOREM 2. *Every circle is rectifiable.*

DEFINITION. *The length of a circle is called its circumference.*

THEOREM 3. *The ratio of circumference to diameter is the same for all circles.*

DEFINITION. *π is the ratio described in Theorem 3.*

THEOREM 4. *The area of a circular region of radius r is πr^2.*

We might, of course, proceed to get numerical estimates of how short the sides of an inscribed polygon need to be, to give certain degrees of accuracy in the approximations that are involved. This is digressive, however, unless we propose to compute π.

One trouble with this material is that the proofs are too hard. But a prior trouble is that the concepts are too sophisticated. Students think that they

know what is meant by the circumference and the area of a circle. It seems unwise to raise questions about the existence of these things, because their existence is more plausible at this level than that of the suprema and infima with which we would answer our own questions. Definitions should, indeed, be furnished, but they should be simple; their appropriateness should be apparent; and the whole idea of introducing formal definitions should be motivated by the idea that the definitions make the concepts easier to work with. In other words, *we need the definitions in order to perform the calculations.*

It is in this spirit that the SMSG geometry treats areas of circles. We point out that the area of a circular region ought to be the limit of the areas of the inscribed regular polygons. We then take this statement as a *definition* of the area of the region, pointing out that previously we had no definition. Similarly, we define the circumference C of the circle to be the limit of the perimeters of the inscribed regular polygons. We then find that $A=\frac{1}{2}rC$, and since $C=2\pi r$ it follows that $A=\pi r^2$.

It appears to me that this treatment presents the crux of the matter. Obviously it would be better to teach more of the underlying theory if we can. But in the places where the SMSG book has been used, it has replaced books which taught less.

Much the same principles apply, in a stronger form, to the theory of volume. A mathematically adequate definition of measurability and volume comes quite late. Cut-and-paste methods cannot be used to replace limiting processes, even for polyhedra, because the 3-dimensional analogue of Bolyai's Theorem is false: it was shown by Max Dehn [1] that if two polyhedra in space have the same volume, it does not follow that they are equivalent by finite decomposition. Therefore some sort of limiting process must be used. But calculus does not help either. The point is that if we try to define volume as an integral, then we have the problem of showing that the integral is invariant under rotation of axes; and as far as I know, the only way to solve this problem is to set up a measure-function, independently of integration, and then show that the values of this function are given by integrals. On this basis one can finally prove Cavalieri's Theorem.

Here again we examine a complex situation and look for a crucial aspect of it which is capable of being taught, and which when taught makes the topic intuitively intelligible. And we find such an aspect: this is the transition from Cavalieri's Principle to the elementary volume formulas. We therefore introduce the Principle by an informal discussion of approximations of solids by models made of thin cards. Having made it plausible, we take it as a postulate and proceed to use it to the hilt. Our use of the word *postulate*, by the way, was not meant to be either honorific or evasive. It refers merely to a statement which is used without being proved. It is not at all uncommon for statements to be taken as postulates when their proofs, as theorems in alternative treatments, are long and difficult. See, for example, *Foundations of Algebraic Topology*, by S. Eilenberg and N. E. Steenrod.

4. Conclusion. Thus the treatment of area and volume in the SMSG geometry was based on two main ideas.

First, we decided that area and volume should be regarded as real-valued functions. This is the prevailing practice, for two good reasons: it is a concept that young students can learn and use, and it is the concept that is needed, in applications and in the further study of mathematics.

Second, we decided that the level of logical rigor should be determined by the logic of each particular situation; in each case, we would explain approximately as much as we thought we could explain intelligibly. It would be idle to hope or to claim that we managed to carry out this policy without making misjudgments in matters of detail. In re-examining the book, I have found a number of these, though not an unreasonably large number. Nevertheless, I believe that our philosophy was a valid one. As Wittgenstein has tersely put it, *wovon man nicht sprechen kann, darüber muss man schweigen.*

References

1. Max Dehn, Über raumgleiche Polyeder, Gött. Nachr. 1900, pp. 345–354.
2. Edwin Moise, Elementary Geometry from an Advanced Standpoint, Addison-Wesley, Reading, Massachusetts, 1963.
3. Ludwig Wittgenstein, Tractatus Logico-philosophicus, New York, 1922.

Linear Algebra and Geometry. By Nicolaas H. Kuiper. North-Holland Publishing Company, Amsterdam, 1962. 284 pp. $8.25.

The unification of linear algebra and geometry, as done so admirably in the pioneering work by Schreier and Sperner, is gaining favor once again. Such unification has several distinct advantages for textbooks on linear algebra: The abstract concepts take on a "concrete" meaning for the beginner, the geometry which might be called linear and quadratic geometry is neatly and powerfully handled by linear algebraic methods, and then the advantages to geometry can be turned back toward motivating more of the study of algebra. The book under review takes cognizance of these advantages, but falls somewhat short of the mark as a textbook candidate for a standard course at the junior-senior level in the United States.

On the positive side, it covers a good deal of linear algebra, treating the standard material on vector spaces, linear transformations, matrices, quadratic and bilinear forms, etc., in a most elegant manner, with a view toward generalization and application in more advanced courses. The only possible complaint (not the reviewer's) might be that it is a bit light on computation with matrices. The geometry begins with affine space and treats Euclidean and unitary spaces after inner products are introduced. Projective geometry is given a substantial chapter, which includes the proofs of the major theorems and a classification of quadrics and collineations over the complex and the real fields. Hyperbolic and elliptic planes, as derived from the real projective plane, are treated all too briefly in a 10-page chapter. The admirable feature of the geometry is that the

author has written it with a view toward modern differential geometry. Accordingly, his treatment is coordinate-free, and he observes that coordinates themselves can be regarded as functions. His definition of affine space and of a topological space are designed to prepare the reader for the definition of a manifold.

The shortcomings of the work are pedagogical. It attempts to treat far too much material in its 254 pages of text. The result is sometimes a very superficial treatment that leaves the reader with too little. Occasionally, the definitions given are unnatural and very difficult. The student-reader will probably stumble over the definition of affine space, and he will certainly be floored by the definition of a topological space. These would have been better placed in climaxing exercises.

This is a book for the instructor who doesn't want everything said for him. If he chooses to teach a one-semester course from Kuiper's book, he will have a lot left to say. Some chapters will have to be deleted and others will have to be enriched, especially with problems of a routine nature to develop the student's skill and feel for the subject. It will probably be hard work for the instructor, harder for his students, but certainly a pleasure for all.

SEYMOUR SCHUSTER
Carleton College

BRIEF MENTION

Elements of Algebra. By J. Houston Banks and F. Lynwood Wren. Allyn and Bacon, Boston, 1962. xi+514 pp. $6.50.

Russian-English Mathematical Dictionary. By L. M. Milne-Thomson. University of Wisconsin Press, Madison, 1962. xiv+191 pp. $5.00.

Linearna Algebra Analiticka Geometrija Polinomi. By Dr. D. S. Mitrinovic and Dr. D. Mihailovic. Katedra Za Matematiku (Belgrade) 1962. x+534 pp.

Zbornik Mathematickih Problema. By Dr. Dragoslav S. Mitrinovic. Katedra Za Matematiku (Belgrade). xvi+502 pp.

Grundzüge der Ausgleichungsrechnung (nach der Methode der kleinsten Quadrate nebst Anwendung in der Geodäsie). By Walter Grossmann. Springer-Verlag, Berlin, 1961. xii+345 pp.

Theory of Ground Water Movement. By P. Ya. Polubarinova-Kochina. Princeton University Press, Princeton, 1962. xix+613 pp. $10.00.

This translation of a Russian book published in 1952 gives a thorough mathematical treatment of ground water flow and should be of interest to specialists in this field.

Managerial Economics. By E. E. Nemmers. Wiley, New York, 1962. xv+498 pp. $8.50.

Statistics: An Intuitive Approach. By G. H. Weinberg and J. A. Schumaker. Wadsworth, Belmont, California, 1962. xii+338 pp.

Introduction To Calculus With Analytic Geometry. By Richard V. Andree. McGraw-Hill, New York, 1962. xi+360 pp. $6.95.

The Growth of Basic Mathematical and Scientific Concepts in Children. By K. Lovell. Philosophical Library, 1961. 154 pages.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Arizona: Professor Gordon Pall, Illinois Institute of Technology, has been appointed Professor; Dr. D. P. Squier, California Research Corporation, La Habra, California, has been appointed Assistant Professor; Associate Professor D. L. Webb has been promoted to Professor.

University of California, Los Angeles: Assistant Professor E. W. Cheney, Iowa State University, has been appointed Assistant Professor; Associate Professors L. J. Paige and J. D. Swift have been promoted to Professors.

Georgia Institute of Technology: Associate Director and Associate Professor B. M. Drucker has been promoted to Director and Professor; Associate Professor R. H. Kasriel has been promoted to Professor; Assistant Professor J. P. Line has been promoted to Associate Professor.

University of Omaha: Mr. Benjamin Stern has been promoted to Assistant Professor; Professor J. M. Earl retired with the title of Professor Emeritus.

San Jose State College: Dr. R. E. Barlow, General Telephone and Electronics Laboratories, Menlo Park, California, has been appointed Assistant Professor; Associate Professors L. I. Holder, G. C. Preston and L. H. Lange have been promoted to Professors; Professor L. H. Lange has been appointed Head of the Department of Mathematics; Assistant Professors Leonard Feldman, I. D. Ruggles and D. E. Thoro have been promoted to Associate Professors; Professor A. R. Lovaglia has returned from a year's leave of absence at Antioch College.

Mr. R. W. Beals, Jr., Rand Corporation, Santa Monica, California, has accepted a position as Staff Programmer with International Business Machines, Yorktown Heights, New York.

Dr. R. P. C. Caldwell, University of Illinois, has been appointed Assistant Professor at the University of Rhode Island.

Mr. R. L. Cooley, Purdue University, has been appointed Assistant Professor at Wabash College.

Mr. N. E. Cuff, Martin Company, Orlando, Florida, has accepted a position as Mathematician with the Vitro Corporation of America, Silver Spring, Maryland.

Associate Professor P. H. Doyle, Michigan State University, has been appointed Professor at Virginia Polytechnic Institute.

Professor E. I. Glover, Agricultural and Mechanical College of Texas, has been appointed Professor and Head of the Department of Mathematics at Florida Agricultural and Mechanical University.

Professor Michael Golomb, Purdue University, is on leave at the Institute for Applied Mathematics, Eidgenössische Technische Hochschule, Zürich, Switzerland.

Mrs. Vivian J. Heinzelman, Douglas Aircraft, Santa Monica, California, has accepted a position as Associate Systems Engineer with International Business Machines, Los Angeles, California.

Associate Professor R. J. Koch, Louisiana State University, has been promoted to Professor.

Assistant Professor John McCarthy, Massachusetts Institute of Technology, has been appointed Professor of Computer Science at Stanford University.

Mr. D. M. Marrian, Rutgers, The State University, has accepted a position as Associate Mathematician in the Applied Physics Laboratory of Johns Hopkins University.

Mr. Laval Mathieu, University of Montreal, has accepted a position as Actuarial Assistant with the Standard Life Assurance Company, Montreal, Canada.

Mr. P. W. Mielke, Jr., University of Arizona, has been appointed Senior Statistician in the Biostatistics Department of the University of Minnesota.

Mr. W. R. Neal, Daystrom, LaJolla, California, has accepted a position as Senior Analyst with the Systems Programming Corporation, Santa Ana, California.

Mr. M. L. Poage, South High School, Denver, Colorado, has accepted a position as Associate Editor of Mathematics with Science Research Associates, Chicago, Illinois.

Assistant Professor Albert Schild, Temple University, has been promoted to Professor.

Assistant Professor Michael Skalsky, Southern Illinois University, has been promoted to Associate Professor.

Mr. R. O. Spruiell, New Mexico State University, has accepted a position as Computer Analyst with Douglas Aircraft Company, Santa Monica, California.

Mr. P. F. Stepler, Johns Hopkins University, has accepted a position as Engineer of Digital Computing with Fairchild Stratos Corporation, Hagerstown, Maryland.

Dr. B. H. Whalen, University of California, Berkeley, has accepted a position as Staff Scientist with RIAS, Baltimore, Maryland.

Professor Margaret V. Rhoads, State Teachers College, Slippery Rock, Pennsylvania, died on July 28, 1962. She was a member of the Association for 6 years.

Dr. W. D. Wray, National Security Agency, Fort George C. Meade, Maryland, died on November 12, 1962. He was a member of the Association for 30 years.

MEETING OF ASSOCIATION FOR COMPUTING MACHINERY

ACM will hold its 18th National Conference on August 27, 28, 29, and 30, 1963 at the Denver Hilton Hotel, Denver, Colorado. The University of Denver is host institution for the conference. In continuation of the policy of the past two years, there will be an International Data Processing Exhibit as a part of the conference. General Chairman is W. H. Eichelberger and Program Chairman is F. P. Venditti. For additional information, contact Professor W. H. Eichelberger of the University of Denver.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-SIXTH ANNUAL MEETING OF THE ASSOCIATION

The Forty-sixth Annual Meeting of the Mathematical Association of America was held at the University of California, Berkeley, California, from Saturday to Monday, January 26 to 28, 1963, in conjunction with the annual meetings of the American Mathematical Society and the Association for Symbolic Logic. There were registered 1,666 persons, including 947 members of the Association.

Sessions of the Association were held on Saturday morning, Sunday morning, and on Monday morning and afternoon in the Auditorium of Wheeler Hall. Presiding officers were Professor Bernard Friedman for the session "On Training Non-Teaching Mathe-

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FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York 53, N. Y. Temporary address from June 15 to August 15, 1963: Dept. of Math., University of California, Berkeley 4, California

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

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THE PUTNAM COMPETITION

L. J. MORDELL, University of Arizona and St. John's College, Cambridge

Editorial Note. We are all aware of Professor Mordell's distinguished career at Cambridge University, but his extensive activities in this country may be less familiar. He went to high school in Philadelphia, belongs to both the MAA and AMS, and has published in most of our journals. He has given talks in forty universities in this country and seven in Canada. He has been Visiting Professor at well-known universities for five years in this country and three years in Canada. He has participated in coaching students for the Putnam Contest. On the basis of this experience, he is evidently well qualified to offer an enlightened opinion on the examinations.

The twenty-second competition was held in December 1961, and it may be assumed that the Putnam competition is well established on a permanent basis. It has been a factor of the utmost importance in arousing and stimulating interest in mathematics in the Colleges and Universities of the United States and Canada. Increasing numbers of students are entering the contest, and 1109 students from 166 institutions took part in the twenty-first competition in December 1960. The competition has undoubtedly played no small part in raising the status, the levels and standards of mathematical education. It is therefore worth while to review recent competitions, and to pay some attention to the kind of pattern that seems to be emerging. Suggestions will be made for increasing the usefulness of the competition and its services to mathematics.

It was customary some years ago to publish in this MONTHLY a syllabus of the subjects of the examination. Thus, for competition 18 held in February 1958, it was stated "The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations and geometry (advanced plane and solid analytical geometry)."

It was stated in this notice that the requirements for the examination would be met by two years (12 semester hours) in the calculus, a half year's work (3 semester hours) each in higher algebra and differential equations and a year's work (6 semester hours) in analytic geometry.

The syllabus would not seem a very formidable one to most students interested in mathematics, and they might well think that the subjects mentioned would give ample opportunity for testing their knowledge, ability and skill. They might hope there was a reasonable chance of not doing too badly in the examination independently of where they may have studied.

They might not have been aware that the syllabus was not always adhered to. Thus, in competition 14 (1954), Part I, Question 7 is: "Prove that there are no integers x and y for which $x^2+3xy-2y^2=122$." The published solution reduces the equation to the form $x^2-17y^2=488$ whence $x^2\equiv 12 \pmod{17}$. It is then stated that by examining a complete residue class or by the theory of quadratic residues, that this congruence is impossible. This question seems to be outside the scope laid down.

It can also be said that the expression "not involving techniques beyond the usual applications" should be taken with a grain of salt. In competition 17 (1957) Part I, Question 2 is: "A uniform wire is bent into a form coinciding with the portion of the curve $y=e^x$, $0\leqq x\leqq a$, $a>1$, and the line segment $a-1\leqq x\leqq a$, $y=e^a$. The wire is then suspended from the point $(a-1, e^a)$ and a horizontal force F is applied at the point $(0, 1)$ to hold the wire in coincidence with the curve and segment. Assuming the x axis is horizontal, show that the force F is directed to the right." It is easy to show that this requires that if $a>1$, then

$$\int_0^a (x - a + 1)\sqrt{(1 + e^{2x})}dx + \tfrac{1}{2} > 0.$$

It requires more than a knowledge of the usual techniques to prove this though it is not too difficult when one knows how.

The procedure for the eighteenth competition held in February 1958 and the later ones was entirely different from the preceding ones. No syllabus was published in the MONTHLY but it was presumably given in the notices circulated to the various institutions. The syllabus and the requirements for the examination seem to have been fundamentally changed without any public announcement. The syllabus for the twenty-second competition, held in December 1961, is as follows:

"The examinations will be constructed to test originality as well as technical competence. It is expected that the contestant will be familiar with the formal theories embodied in undergraduate mathematics through differential equations. It is assumed that such training, designed for mathematics and physical science majors, will include somewhat more sophisticated mathematical concepts than is the case in minimal courses. Thus the differential equations course is proposed to include some references to qualitative existence theorems and subtleties beyond the routine solution devices. Questions will be included which cut across the bounds of various disciplines and self-contained questions which do not fit into any of the usual categories may be included. It will be assumed that the contestant has acquired a familiarity with the body of mathematical lore commonly discussed in mathematics clubs or in courses with such titles as 'survey of the foundations of mathematics.' It is also expected that the self-contained questions involving elementary concepts from group theory, set theory, graph theory, lattice theory, number theory, and cardinal arithmetic, will not be entirely foreign to the contestant's experience."

The new syllabus is open to serious criticism. Emphasis is now laid upon acquiring what can only be described as a smattering of knowledge of various topics, when a student would be better employed in acquiring, say, a greater knowledge of analysis in its various aspects. There is no reason why attendance at mathematical clubs should give one a working knowledge of a particular subject. A survey course in foundations need not add to technical or manipulative skill. How can one expect that a student at this stage of his career should

have a working knowledge of graphs and lattice theory? He can still be a good and promising student without any such knowledge.

The syllabus assumes a very sophisticated type of student, but will handicap very seriously contestants from a large number of institutions who have neither the opportunity nor time for coping with the new syllabus. (Why the special emphasis on differential equations?) It is inevitable that certain institutions will do better than others in any kind of examination. The better students will naturally gravitate to them. These institutions may also devote considerably more time to the study of mathematics than others do, and they may also pursue subjects to a much higher level. It seems undesirable that the examination should accentuate those features in which these institutions are sure to have too great an advantage over others.

It would have been much better to have introduced specifically into the syllabus a subject such as, say, modern algebra. In due course, this might become more and more a subject taught in the earlier years of a mathematics course. In ways of this kind, the competition could exercise a healthy and decisive influence on mathematical curricula, and far more so than by specifying a miscellaneous collection of topics.

There are two very important facts a student would like to know. Does the syllabus include all those topics dealt with in the examination? This is not so since there is no mention of probability though questions are often set on it. It is true that the syllabus says "Questions will be included which cut across the bounds of various disciplines and self-contained questions which do not fit into any of the usual categories may be included." Can one take this to include a reference to probability? Much more important, however, is how much of a subject is the student expected to know? Without some specifications, the word "elementary" is too vague to convey a clear meaning. In fact, questions are set in the examination which by no stretch of the imagination can be described as elementary, and which would be more suitable for graduate students of some maturity. Perhaps one might accept that a rough definition of "elementary" would be the contents of the usual books read by undergraduate students, but not topics found in only one or two books on a subject.

An obvious suggestion is that the content and extent of the syllabus should be referred periodically to a committee representative of every kind of College and University, and also that this committee should draw up a wide list of books useful in preparing for the examination.

Let us now consider in some detail the questions set in the examination. One has every sympathy with the examiners, since setting the questions is not a simple matter. They must not be too easy but also not so difficult as to induce a feeling of dismay and frustration. It must be very discouraging and disheartening to a candidate when he realizes that his unexpected lack of knowledge makes it completely impossible for him to do a number of questions and so to do himself justice in the examination.

It is sometimes rather hard to decide whether a question is easy. It may ap-

pear simple enough when the solution is known. To find this, however, may have taken the proposer considerable time, and even then he may have been in the favorable position of having relevant thoughts and ideas present in his mind. Without these, there is often no reason why one should find the solution in a short time; and it is generally most unlikely that he will do so under the stress of examination conditions. In any case, a question really requiring a new idea will rarely be done in an examination. It can be said, however, that the papers do contain questions which would be considered as easy or difficult by any standard.

It may be remarked that the published solutions are often of a synthetic character, i.e., they show how the question was made up, but of course there is very little reason for a student to know this, and so the given solution may not be representative of what is done in the examination or of the difficulty of the question.

First of all, some general remarks about the examination. It should not be of such a nature as to encourage desultory reading as opposed to reading to obtain a thorough knowledge of the subjects of an undergraduate course. This does not mean, however, that a student should not read widely if he is disposed to do so and can profit by it. The examination should not give him a one-sided view of the scope of mathematics or a false idea of what is worth while. It should not suggest that he spend a great deal of time in acquiring dexterity in topics not really important in mathematics.

Questions are sometimes set which have appeared as problems in journals and were not likely to be done without considerable thought and time. Others are taken from research papers or are really of a research character. The examination room is not the place for research, and such questions are more suitable for contemplation in the quiet of one's study.

One instance may suffice. In competition 19 (1958), Part I, Question 2 is: "Given a set of $n+1$ positive integers none of which exceeds $2n$, show that at least one number of the set must divide another member of the set." This was proposed by Erdös as Problem 3739 in this MONTHLY. A beautifully simple solution appeared in the February issue (1937), and is much nicer than that published for the competition. The simplicity of a solution may have no relation to the difficulty of a question. Thus the very next MONTHLY Problem 3740, also due to Erdös was: "From a point O inside a given triangle ABC, the perpendiculars OP, OQ, OR, are drawn to its sides. Prove that

$$OA + OB + OC \geqq 2(OP + OQ + OR).\text{"}$$

This was considered a very difficult problem, indeed, and was communicated by Erdös to many people who could not find a solution. However, Mordell gave a simple solution which appeared in this MONTHLY for April 1937. This so-called Erdös-Mordell theorem has been ever since the subject of many papers.

Most important is what types of question should be set? Obviously, they should be reasonable in character and this theme will now be developed. Enun-

ciation of questions should be of a helpful kind. There are often questions set in which an answer is asked for. There seems no reason why answers should not be given more frequently, especially in complicated questions. It saves a student time when he sees he has arrived at the correct answer, and he is saved the trouble of verifying his work. It also conduces to a feeling of confidence and improves his morale when he has good reason to think his work is sound. If the answer is too involved to be given, this is perhaps a good reason for either modifying the question or omitting it. The published answers of some questions suggest most emphatically that the questions ought never to have been set in the form in which they appeared.

Questions are frequently set in which one is asked to consider all possible cases or to verify that all solutions have been obtained. It would be desirable sometimes to word a question in such a way as to impress this upon the student and to keep him from going astray. Take competition 21 (1960), Part II Question 1, as a slight instance. "Find all solutions of $n^m=m^n$ in integers n and $m(n\neq m)$. Prove that you have obtained all of them." The form of the equation might easily lead one to believe that only positive integers are required. It would have been better to add after m, "positive or negative" and omit $n\neq m$.

Those who set the papers should have a clear picture in their mind of the qualities they are testing and what they are looking for in the student. Are they, for example, more concerned with the few students who have acquired knowledge far beyond the usual courses, and do they consequently set problems that only these exceptional students can attempt?

Sometimes one is under the impression that they attach undue weight to the ability to carry out troublesome arithmetical calculations in an examination, especially when these add no point to the question, but only lead to spending more time on it with more opportunities for making arithmetical mistakes.

Some of the questions remind one of the proverbial problem of finding a needle in a haystack. If one has some idea of its location or an infallible tool e.g., an electro-magnet, there will be no difficulty. Without some such indications, one will be helpless. Take competition 20 (1959), Part II, Question 3: "Give an example of a continuous real valued function from $[0, 1]$ to $[0, 1]$ which takes every value in $[0, 1]$ an infinite number of times." The given solution notes that this is really a question on finding a Peano space-filling curve and then utilizes the Cantor set C of those numbers in $0\leqq x\leqq 1$ which can be written in the ternary scale using only 0's and 2's. How can any one who has never seen something of functions of this kind before, and probably this applies to most undergraduates, be expected to do such a question? It seems almost impossible that this could be discovered in an examination.

The questions seem to suggest that the examiners are more concerned with a knowledge of the morbid and pathological aspects of mathematics, especially of sets and functions. These of course have led to important mathematical advances but surely the respectable or normal parts of mathematics are worth more attention. The examiners often emphasize the bizarre, the whimsical and

the exotic aspects of mathematics and questions often seem artificial in character. They tend to stress modes of thought which are not possessed by many good mathematicians, (and they are not necessarily the worse for that) and also aspects of mathematics which are not very important nor significant for an undergraduate. This applies in particular to the more difficult questions dealing with the distribution of lines and points, the distances between points in various sets, or covering of sets by others. Of course, such questions may require considerable ingenuity of a sort, and are suitable for a kind of research, but how relevant are they to sound mathematical knowledge and ability? They can easily induce a feeling of repulsion in many people when such emphasis is laid upon them. In some topics, such as combinatorial analysis, there may be considerable difficulty in translating the statement of the question into a mathematical argument. The bizarre aspects of questions occur in those dealing with towers of exponents and an infinity of radicals. There is a real danger that students will be spending too much time on such questions when they would be better employed in studying what is more worth while.

There are many reasons why questions might be called undesirable or unreasonable and a number are now considered in detail.

Questions should not be set depending upon advanced knowledge of which most of the candidates are sure to be ignorant. In competition 18 (1958), Part II, Question 7 is: "Prove that if $f(x)$ is continuous for $a \leqq x \leqq b$ and $\int_a^b x^n f(x)dx = 0$ for $n = 0, 1, 2, \cdots$, then $f(x)$ is identically zero on $a \leqq x \leqq b$."

The published proof of this theorem of Lerch is simple since it uses, but only quotes, the well-known result of Weierstrass on the approximation of continuous functions by polynomials (as is also done in Ford's "Differential Equations"). This result may be found in books dealing with real variables or Fourier series. The question places a great premium on knowledge far beyond what most undergraduates know, for I cannot believe that a student is likely to find a proof during the examination. It might be mentioned that the solution tacitly assumes that a and b are finite. In view of the frequent emphasis in the examination to consider all cases, it might have been mentioned in the question that a and b are finite.

Take competition 17 (1957), Part II, Question 6: "The curve $y = f(x)$ passes through the origin with a slope of 1. It satisfies the differential equation $(x^2+9)y'' + (x^2+4)y = 0$. Show that it crosses the x-axis between $x = 3\pi/2$ and $\pi\sqrt{63}/\sqrt{53}$."

This is a not too difficult application of Sturm's theorem on results of this kind, but involves a little unpleasant arithmetic. The theorem is not found in the usual books on differential equations. It is given, however, in Ford's book on the chapter dealing with Bessel functions—a natural place for such theorems. How much better it would have been for the student to know of this book, but even then the topic might not have been discussed in the course.

Problems should not involve an inordinate amount of arithmetic especially when this adds no value to the question or gives it any special significance. The

mere need for such calculations is the best reason for either rejecting the question or for radical modification. Take competition 20 (1959), Part II, Question 5: "Find the equation of the smallest sphere which is tangent to both of the lines (i) $x=t+1$, $y=2t+4$, $z=-3t+5$, and (ii) $x=4t-12$, $y=-t+8$, $z=t+17$."

The published solution is

$$(502x+915)^2+(502y-791)^2+(502z-8525)^2=5{,}423{,}859.$$

It seems almost cruel to set questions with so many opportunities of making numerical errors. How much better it would have been to present the question rather differently. The point of the question is that the centre of the sphere is at the midpoint of the common perpendicular of the two lines. Call the two lines $x=l_1t+a_1, \cdots$, and $x=l_2t+a_2 \cdots$. The perpendicular has direction cosines L, M, N, proportional to $m_1n_2-m_2n_1, \cdots$. Hence its length i.e., the diameter of the sphere, is the projection of the line joining the points a_1, b_1, c_1 and a_2, b_2, c_2, on the perpendicular and so is easily written down. An elementary knowledge of solid analytical geometry gives at once the equation of the perpendicular, then its intersection with the two given lines. More simply if A, B, C is the centre of the sphere of radius r, the points $A\pm Lr$, $B\pm Mr$, $C\pm Nr$ are on the lines and so A, B, C can be found. Surely the question set is of a type which should be set only in a form requiring no arithmetic.

Sometimes a question seems to involve considerable arithmetic sufficient to discourage a student, and is made worse by requiring technical knowledge. Take competition 21 (1960) Part I, Question 6: "A player throwing a die scores as many points as on the top face of the die and is to play until his score reaches or passes a total n. Denote by $p(n)$ the probability of making exactly the total n, and find the value of $\lim_{n\to\infty} p(n)$."

Here $6p(n)=p(n-1)+p(n-2)+\cdots+p(n-6)$, and it requires some knowledge of probability to see that this is a method of attack. The student might feel unhappy at the prospect of dealing with this recurrence formula of the sixth degree. The solution given is as follows. Write

$$P(s) = 6/6 - s - s^2 \cdots - s^6 = 6/(1-s)Q.$$

Then

$$P(s) = \sum_{n=0}^{\infty} p(n)s^n.$$

Also

$$P(s) = \frac{\frac{2}{7}}{1-s} + \frac{2(15+10s+6s^2+3s^3+s^4)}{7(6+5s+4s^2+3s^3+2s^4+s^5)}\cdot$$

Expanding the two fractions into power series, we obtain from the coefficients of $P(s)$ that $p(n)=\frac{2}{7}+f_n$ where the f_n are coefficients in the expansion of the second fraction. It can be shown that $f_n\to 0$ and therefore that $\lim p(n)=\frac{2}{7}$. What had the examiner in mind for a proof when he states that $f_n\to 0$?

In fact, a very bright student could have done the question with practically no arithmetic. He might have argued that it suffices to find only the term $\frac{2}{7}/(1-s)$ in the decomposition of $P(s)$ and to show that the roots of $Q=0$ have moduli >1. This is obvious since if $|s|\leqq 1$, $|s+\cdots+s^6|\leqq 6$ with equality only when $s=1$.

If such a probability question were thought to be within a candidate's powers, the idea might have retained by using a die with each of the numbers 1, 2, 3 occurring twice. Then if he adopted the published form of solution, $3p(n)=p(n-1)+p(n-2)+p(n-3)$, and now $3-s-s^2-s^3=(1-s)(3+2s+s^2)$, and it is obvious that the roots of $3+2s+s^2=0$ have moduli >1.

Unnecessary complication should be avoided in proposing a question and it is often worth while stripping a question of unessential parts which add no value to the questions. Take competition 20 (1959), Part I, Question 5: "A sparrow, flying horizontally in a straight line, is 50 feet directly below an eagle and 100 feet directly above a hawk. Both hawk and eagle fly directly toward the sparrow, reaching it simultaneously. How far does each bird fly? At what rate does the eagle fly?"

Curves of pursuit lead to differential equations and there is already some work involved in finding the solution. Two pursuers and numerical details are of no special mathematical significance, and merely make the solution more troublesome. How much simpler the question would have been with only one pursuer and asking, say, only for the distance travelled by the prey.

Problems should be stated in such a way that there is no doubt about the meaning of the question or what is required of the candidate. Take competition 22 (1961), Part I, Question 6: "If $J_2=\{0, 1\}$ is the field of integers modulo 2, and if $J_2[x]$ is the integral domain of polynomials in one indeterminate with coefficients in J_2, prove that $p(x)=1+x+\cdots+x^n$ is reducible (factorizable) in case $n+1$ is composite. Is the converse true? That is, if $n+1$ is prime, is $p(x)$ irreducible?"

The first part of the question is simple but what is expected of the candidate in the second part? If he is familiar with the theory of finite fields, he will answer that $f(x)$ is irreducible if and only if 2 is a primitive root of $n+1$, and may wonder if this will be accepted as an answer. As he is unlikely to know this, he might try to see whether $p(x)$ is reducible, for example, to the product of two polynomials $g(x)$, $h(x)$ each of degree $\frac{1}{2}(n+1)$. On changing x into $1/x$, he is led to conditions on the coefficients of $g(x)$, $h(x)$. These suggest that when $n=6$, a possible decomposition may be $(1+x+x^3)(1+x^2+x^3)$, and it is easily verified that these are the factors of $f(x)$. Trial will show that $f(x)$ is not reducible when $n=4$. Would it not have been better to ask to prove the results for $n=4, 6$?

Take competition 21 (1960), Part I, Question 4: "Given two points in the plane P and Q at fixed distances from a line L and on the same side of the line, as indicated, the problem is to find a third point R so that $PR+RQ+RS$ is a minimum where RS is perpendicular to L. Consider all cases."

A diagram is given in which R lies between the perpendiculars from P and Q on L, and also between L and the line through P parallel to L. What is the significance of the diagram and the phrase "consider all cases?" Does it mean that R is to be restricted to the region in which it seems to be? Apparently not. The published solution takes L to be the x-axis and the points P, Q to be $(-a, p)$, (a, q), and we may take $0<p<q$. If R is (x, y), the given solution is an attempt to minimize

$$F(x, y) = |y| + \sqrt{(x+a)^2 + (y-p)^2} + \sqrt{(x-a)^2 + (y-q)^2}.$$

The presence of $|y|$ here means that the phrase "all cases" is meant to include the one when R is below L. Since $|y|$ is not differentiable at $y=0$, it may be remarked that $y=0$ must be considered separately.

The diagram serves no purpose except possibly to mislead candidates. It should have been omitted since the exceptional case may occur when R is at P.

The question would read much better as this: "Given in the plane a line L and two points P and Q on the same side of L. Find a third point R such that the sum of the distances of R from P, Q and the line is a minimum."

Questions dealing with elementary parts of mathematics become difficult if they assume the knowledge of some elementary facts with which the student is not likely to be familiar and which he is not likely to discover in the examination. Take competition 17 (1957), Part I, Question 3: "If A and B are real numbers and k is a positive integer, show that

$$\left(\frac{\operatorname{Cos} kB \operatorname{Cos} A - \operatorname{Cos} kA \operatorname{Cos} B}{\operatorname{Cos} B - \operatorname{Cos} A}\right) < k^2 - 1$$

whenever the left hand side is defined."

It is noted in the published solution that $<$ should be $\leqq$. The result can be deduced easily from the result $|\sin kA/\sin A| \leqq k$, but it seems unlikely that many candidates would know this. It would have been much better if they had been asked first to prove this and then the main part. The published solution starts from $|\sin kA/\sin A| \leqq k$ and shows how the given question arose—a synthesis which makes sense for a student only when the solution has been found otherwise. In fact, it is stated in the results of the competition that no candidate did the question completely.

There is undue emphasis on problems of combinatory analysis. Proficiency in these is not easily developed under examination conditions. Take competition 20 (1959), Part II, Question 4: "Given the following matrix of 25 elements

$$\begin{pmatrix} 11, & 17, & 25, & 19, & 16 \\ 24, & 10, & 13, & 15, & 3 \\ 12, & 5, & 14, & 2, & 18 \\ 23, & 4, & 1, & 8, & 22 \\ 6, & 20, & 7, & 21, & 9 \end{pmatrix},$$

choose five of these elements, no two coming from the same row or column, in such a way that the minimum of these five elements is as large as possible. Prove that your answer is correct."

Probably thoughts about such matrices have not been present in the minds of many contestants. Confronted with an apparently meaningless collection of numbers, whose significance or relation to each other is not easily grasped, most students would be helpless. It seems most undesirable that a mathematician's competence should be tested or judged by this kind of question.

Too much emphasis is sometimes placed upon topics not justified by the role they play in undergraduate work, or their importance in mathematics as a whole. It has already been stated that nowhere is probability mentioned as a subject of the examination. When or where is the student to acquire a knowledge of this? He may acquire a little from some books on algebra or calculus. There are institutions in which probability appears as an undergraduate course, but there may have been no reason for a student to attend this if other courses seemed more attractive to him. It may be remarked that almost every subject will be taught in undergraduate courses in some place or other. Time spent in designing a well balanced curriculum would be worth while.

If questions on probability are to be set, it seems undesirable to assume more than a simple common-sense knowledge. This is done for Question 3 in Part I of Competition 18 (1958): "Real numbers are chosen at random from the interval $0 \leqq x \leqq 1$. If after choosing the nth number, the sum of the numbers so chosen first exceeds 1, show that the expected or average value of n is e."

Question 4 in Part II of the same competition is: "What is the average straight line distance between two points on a sphere of radius 1?" Two questions on average values are too many.

What is to be the future of this competition which has to cater for increasing numbers of students of all kinds? Most of them cannot expect to profit financially or to do brilliantly in the examination. They probably regard it as a kind of sporting, free-for-all contest, and think the effort worth while. They might be happier if the competition was split into two distinct ones, say an A and a B competition, with adjustments in the value of the prizes.

The B competition should have a syllabus comparable with the previous one prior to the change. Most of the entries for this would come from the smaller institutions and others in which mathematics is not taught on a high level. These institutions might feel that honor is satisfied if their students feel they have made a creditable showing. The A competition would be primarily for those from the larger institutions and others in which mathematics is intensively studied. It should, however, be open to all who wish to enter independently of any place of study. The A syllabus should cover a greater range than the B one and make wider demands on the contestants. Both examinations should be such as not to give the student a jaundiced mathematical outlook. They should give him the feeling that mathematics is real, mathematics is earnest.

THE PUTNAM COMPETITION—REJOINDER

L. M. KELLY, Michigan State University

The very searching analysis of the Putnam examinations of the past few years by such an eminent mathematician as L. J. Mordell warrants the interest and concern of all who feel, as I do, that this distinguished institution has played, and should continue to play, an important role in undergraduate mathematical training in this country and Canada. My interest in the competition dates from my undergraduate days and became quite intense, of course, when I served on the examination committee some years ago. The examination committee consists of three members appointed by the president of the Association for three year terms. The appointments are on a rotating schedule, one member retiring and one new member joining each year. During my tenure the revision of what Professor Mordell refers to as the syllabus was suggested and I actively participated in drawing up the new version. Before I go into a detailed discussion of this and other matters it may be in order to point out that these opinions are entirely my own and are not to be given any official status. I know that numerous colleagues who have served on previous examination committees share many of the sentiments which are expressed herein but I claim only to speak for myself unless specific quotations are employed. In particular I will make liberal use of a letter which Professor Ivan Niven has made available.

Professor Mordell's criticisms seem to center on four main points:

1. The present syllabus is not well conceived.
2. The examinations do not always adhere to the syllabus.
3. There is excessive emphasis on the bizarre, the exotic and the pathological.
4. Some problems are poorly constructed, are occasionally ambiguous, and involve tedious calculations to no good end.

In order to maintain some perspective on the effect of the appearance of an occasional unfortunate problem perhaps the following observations may be useful. Experience has shown that in order to discriminate among the top ten or so contestants the examination must be of the order of difficulty of those which have been given. As those who have worked with competitions know, however, it is very difficult to devise an examination which will provide a significant linear ordering very far down the list. In these examinations it has been the custom to present the contestant with 14 problems from which they were to select 12. As a matter of fact the successful solving of 6 or 7 of the 14 quite usually placed the contestant among the top 30 or 40. Thus the inclusion of as many as 2 or 3 "bad" problems on a single examination will not seriously affect the chances of success of the vast majority of those participating. I am not contending that we should not be concerned about such a state of affairs, that it may not have some psychological effect, and that it would not have been better to have more suitable problems in their stead, but I think it is unrealistic to make the claim that the presence of an occasional unreasonable problem is a decisive factor in

the performance of students from the small or less distinguished schools whose fate seems to be one of Professor Mordell's principal concerns.

My principal disagreement with the contention of point 4 is the extent to which such problems occur and the degree of their effect on the performance of most of the contestants. A similar observation holds for point 2 as well. Many people will take issue with some of the examples which Professor Mordell has chosen to illustrate his points. For example, Professor Niven notes that the problem concerning the sphere tangent to the skew lines was not a calculated form of cruelty but became such because of a misprint in one of the signs. These things happen, of course, and should not form the basis for serious criticism. My own perusal of old examinations reveals relatively few problems involving onerous calculations. Professor Niven further observes that he had in mind a much simpler solution of the probability problem concerning the die than that envisaged by either Professor Mordell or the graders. He suggests that it might be a good idea to have the examination committee submit its solutions for publication rather than leaving this entirely in the hands of the graders as is now done. It is my understanding that Professor Bush intends to follow this procedure in the future.

I also would disagree with some of the illustrations. The problem concerning the 5×5 matrix is quite a trivial problem and has been so evaluated by several contestants. I have frequently presented this to high school students who were neither "helpless" nor dismayed. In fact they found the problem rather entertaining and its solution not too difficult. Or again, the problem involving the force on the bent wire amounts to establishing the inequality

$$\int_0^a x\sqrt{1+e^{2x}}\,dx > (a-1)\int_0^a \sqrt{1+e^{2x}}\,dx, \qquad a > 1.$$

While this may involve a degree of ingenuity it hardly qualifies as "outside the range of the usual applications."

But it is not my intention to take issue with each of Professor Mordell's examples of problems he considers inappropriate. I readily agree that there have been many such examples in the 23 year history of the competition and not all in the past few years. It would, in fact, be a striking achievement if a committee effort over such an extended period had not produced such instances. I do claim, however, that on the whole the examinations have been remarkably good, have stimulated interest, and have evoked the respect of the mathematics community in the competition.

Professor Mordell seems to suggest that something slightly irregular happened in 1958. "The syllabus and the requirements for the examination seem to have been fundamentally changed without any public announcement," he writes. The announcement was given, of course, in the brochure announcing the forthcoming examination but apparently Professor Mordell feels that there should have been more publicity signalling the event. The fact of the matter is that those of us involved in the revision did not regard this as an effort to change

the pattern and nature of subsequent examinations but rather to recognize and describe properly what had been going on for some years. It was apparent to us that almost from the outset, problems had been included which would fit within the announced scope only by the most flexible interpretation.

On the ninth competition we find: "Assume the complex numbers a_1, a_2, $a_3, \cdots, a_n, \cdots$ are all different from zero and that $|a_r - a_s| > 1$ for $r \neq s$. Show that $\sum 1/a_n^3$ converges." Does this fall within the range of advanced calculus, vintage 1949?

Or again in the 11th competition this problem appears: "Find the volume of a four dimensional hypersphere $x^2+y^2+z^2+t^2=r^2$ and also the hypervolume of its interior $x^2+y^2+z^2+t^2<r^2$." What in the syllabus covers this? A 1950 course in advanced calculus or a course in solid analytics?

As a final example consider PM 3 again from the 9th competition: "Let K be a closed plane curve such that the distance between any two points of K is less than 1. Show that K lies inside a circle of radius $1/\sqrt{3}$." I will comment more on this problem shortly. For the moment I merely observe that it is hard to justify this as being in the realm of the original examination syllabus.

In the light of such examples the committee sought advice from several competent sources including Professor Bush and drafted the current version to which Professor Mordell objects. In a general way, the committee was attempting to introduce more flexibility into the description, realizing that such descriptions are necessarily vague and should serve only as a broad suggestion of what to expect on any given examination and not as some sort of legal straight jacket within which the committee must operate. Some effort was made to anticipate slightly possible desirable trends but largely the motive was to describe the state of affairs at that time.

No doubt an improvement in the present language can be effected but whatever is done should not rule out such elegant and suitable problems as AM 4 of the 15th competition, PM 5 of the 16th competition, and PM 5 of the 17th (see note at end of this paper) which can only be classified as combinatorial analysis, the theory of linear graphs, and set (or lattice) theory respectively. Most committees, I am sure, make an effort to conform to the tradition of the examination while introducing some element of novelty. Thus the best syllabus is a reference to recent past examinations.

I am unable to subscribe to the picture of frustration painted by Professor Mordell, of the contestant who innocently reads the syllabus and decides on this basis that he is adequately prepared, only to be rudely shocked when confronted with the actual examination. Surely any serious contestant would glance over some previous examinations and would in addition have the advice and assistance of a faculty member of some experience.

I close consideration of the syllabus by remarking that Professor Mordell weakens his case when he asks where a student is supposed to acquire a "working knowledge" of graph theory. This is a grossly unfair reading of the syllabus. The syllabus reads, "It is also expected that *self-contained* questions involving

elementary concepts from group theory, set theory, graph theory, lattice theory, number theory, and cardinal arithmetic will not be *entirely foreign* to the contestant's experience."

The emphasis here is on the *entirely foreign* aspect. Surely no essential fault can be found with a problem such as PM 5 from the 17th competition. It involves no technical knowledge not possessed by a good freshman. Yet if the contestant has no *prior experience* with this type of problem, he is at a definite disadvantage. The committee was merely trying to suggest to the faculty advisers that the prospective contestants should be prepared, psychologically at least, for questions of this character. Again we remark that such broad descriptions take on more meaning by reference to a few past examinations.

The objection to the inclusion of *research* problems as illustrated by the Erdös-Anning theorem (see note) is well taken but such problems are hardly characteristic of recent examinations. PM 3 from the ninth competition, referred to above, is essentially the celebrated theorem of Jung and in some respects more difficult to prove than the Erdös-Anning result. Certainly knowing the very simple and elegant solution of the Erdös-Anning theorem a committee would be hard put to decide whether this should be regarded as suitable. The history of the result suggests not, since Erdös and Anning first published a rather involved solution before Erdös observed the almost trivial argument that he eventually published. In addition, the problem has had quite a wide circulation and it is quite possible that a number of the contestants had already seen the solution.

In the final analysis the way to produce a good examination is to have a good committee. The composition of the committees these past few years has been in my opinion quite good and quite representative. The present committee, for example, has members representing Williams College, M.I.T., and Southern Illinois University. Among the types of schools usually participating in the examination can one imagine a better cross sectional representation?

In criticising problems as being "bizarre, whimsical, and exotic," we enter a very subjective area and one which is certainly hard to debate. As Professor Niven puts it, "It seems to me that Mordell, in criticising the selection of problems used in Putnam examinations, is really criticising the current interests, fashions, outlooks and standpoints of contemporary American mathematics. The difficulty is that what is respectable and normal to one is exotic to another." Keeping in mind that the examination committee changes each year and the president of the association who appoints them, every two years, certainly any "trend" which Professor Mordell has noted over the past, say ten years' represents the judgment of a large number of competent mathematicians.

As a partial remedy for some of the faults which he sees in the present arrangements, Prof. Mordell suggests a second and simpler examination for the less talented or less sophisticated. This may be an attractive "solution" to the progressive educator who visualizes all examinations as a form of group therapy but I am surprised to find it advocated by a member of the old school, particu-

larly one trained in the British tradition. Students, I believe, are proud to participate in this competition even if they do not do too well because they respect the examination and they recognize that doing well in it represents an achievement of high order. Resorting to some classification or handicapping scheme will, in my view, cheapen the entire program and undermine the respect which this splendid competition has generated over the past 25 years. I have no objection to some sort of minor league competition if some other group wishes to sponsor it but let's leave the Putnam competition simple and unencumbered, the winning of which represents one of the most significant achievements of American undergraduate mathematics.

Note. For the convenience of the reader we include here the statements of the problems referred to in the rejoinder.

AM 4, 15th competition. On a circle n points are selected and the chords joining them in pairs are drawn. Assuming that no three of these chords are concurrent (except at the end points) how many points of intersection are there?

PM 5, 16th competition. Consider $2n$ points in space, $n>1$. Suppose they are joined by at least n^2+1 segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 segments without any triangle being formed.

PM 5, 17th competition. With each subset X of a set is associated a second subset $f(X)$. The association is such that whenever X contains Y then $f(X)$ contains $f(Y)$. Show that for some set $A, f(A)=A$.

The Anning-Erdös Theorem. If the distances between pairs of points of an infinite plane set are all integers, the set is linear.

AN ANALOGUE OF THE BROCARD POINTS

PETER YFF, American University of Beirut

1. Introduction. The triangle $A_1A_2A_3$, or (T), has Brocard points Ω and Ω'. A well-known property of Ω is that $\angle\Omega A_1A_2=\angle\Omega A_2A_3=\angle\Omega A_3A_1=\omega$; similarly, $\angle\Omega' A_2A_1=\angle\Omega' A_3A_2=\angle\Omega' A_1A_3=\omega$.

A pair of analogous points will now be noted. Let the point U be situated so that A_iU meets the opposite side of (T) at B_i $(i=1, 2, 3)$ and $A_1B_3=A_2B_1=A_3B_2=x$. Then if U' is the isotomic conjugate of U, the line A_iU' meets the opposite side at B_i' $(i=1, 2, 3)$ in such a way that $A_1B_2'=A_2B_3'=A_3B_1'=x$. This follows readily from Ceva's theorem.

It will be shown that U and U' are interior to (T) and that each is unique. If the side opposite A_i is of length a_i $(i=1, 2, 3)$, application of Ceva's theorem to the lines A_iB_i yields

$$x^3=(a_1-x)(a_2-x)(a_3-x)$$

or

$$(x)\equiv 2x^3-px^2+qx-r=0,$$

in which $p=a_1+a_2+a_3$, $q=a_2a_3+a_3a_1+a_1a_2$, $r=a_1a_2a_3$. Evidently $f(x)=0$ has no negative roots, which means that U is an interior point. To prove that there exists exactly one real root, we note that

$$a_1^2 < a_1(a_2+a_3);$$
$$a_2^2 < a_2(a_3+a_1);$$
$$a_3^2 < a_3(a_1+a_2).$$

These are added to obtain

$$a_1^2+a_2^2+a_3^2 < 2q,$$

and addition of $2q$ to each side gives

$$p^2 < 4q.$$

Hence the discriminant of $f'(x)$, which is $4(p^2-6q)$, is negative, so that $f'(x)>0$ and $f(x)=0$ has a unique real root u.

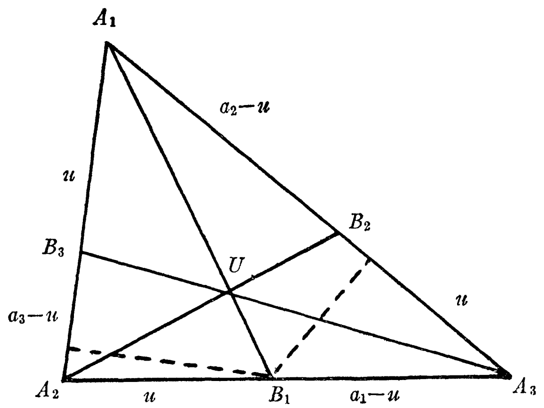

FIG. 1

Ordinarily $f(x)$ is irreducible, so its roots, as well as U and U', may not be constructed by ruler and compass. By way of contrast, the corresponding cubic for Ω and Ω' involves functions of the angles of (T) and may be reduced by means of certain trigonometric identities.

Trilinear coordinates with (T) as the triangle of reference will now be used to locate U. Letting α_i denote the angle at A_i $(i=1, 2, 3)$, the distances of B_1 from A_3A_1 and A_1A_2 are $(a_1-u)\sin\alpha_3$ and $u\sin\alpha_2$ respectively (Fig. 1). The second and third trilinear coordinates of B_1 (and U) are therefore in the ratio $a_3(a_1-u):a_2u$. By cyclic permutation the first and second coordinates of U are

found to be in the ratio $a_2(a_3-u):a_1u$. These results are combined to give the coordinates of U as

$$\left(\frac{a_3-u}{a_1u}, \frac{1}{a_2}, \frac{u}{a_3(a_1-u)}\right).$$

Permutation of subscripts and coordinates yields alternative sets of coordinates:

$$\left(\frac{u}{a_1(a_2-u)}, \frac{a_1-u}{a_2u}, \frac{1}{a_3}\right)$$

and

$$\left(\frac{1}{a_1}, \frac{u}{a_2(a_3-u)}, \frac{a_2-u}{a_3u}\right).$$

Symmetry may be obtained by termwise multiplication of these results, then taking the cube root:

$$\left(\frac{1}{a_1}\left(\frac{a_3-u}{a_2-u}\right)^{1/3}, \frac{1}{a_2}\left(\frac{a_1-u}{a_3-u}\right)^{1/3}, \frac{1}{a_3}\left(\frac{a_2-u}{a_1-u}\right)^{1/3}\right).$$

The coordinates of U' are found by the same method to be

$$\left(\frac{1}{a_1}\left(\frac{a_2-u}{a_3-u}\right)^{1/3}, \frac{1}{a_2}\left(\frac{a_3-u}{a_1-u}\right)^{1/3}, \frac{1}{a_3}\left(\frac{a_1-u}{a_2-u}\right)^{1/3}\right).$$

2. Miscellaneous properties. (a) The line UU' is perpendicular to the line of the incenter I and the circumcenter O.

Deriving the equation of UU', we find after simplification that it may be given by

$$\sum \frac{a_i^2(a_i-2u)}{a_i-u} x_i = 0. \tag{1}$$

(Here and later, summation is from 1 to 3). This line is irrational in the sense that the coefficients of x_i $(i=1, 2, 3)$ can not be expressed as rational functions of the a_i's or of the trigonometric functions of the α_i's. It contains, however, one rational point $(a_2-a_3, a_3-a_1, a_1-a_2)$, which is on the line at infinity $\Sigma a_ix_i=0$. This point is also on the line $\Sigma x_i=0$, the trilinear polar of I. Therefore UU' is parallel to $\Sigma x_i=0$ and perpendicular to IO. [1, p. 246].

(b) The length of the segment UU' is

$$\frac{4ud\Delta}{u^3+a_1a_2a_3},$$

in which Δ is the area of (T) and d is the length of the segment IO.

If the coordinates of $P(x_1, x_2, x_3)$, or (x_i), and $Q(y_i)$ have been adjusted so that they represent the actual distances of the respective points from the sides

of (T), the length of PQ may be shown to be given by

$$2R\sqrt{-\frac{1}{a_1a_2a_3}\left[a_1(x_2-y_2)(x_3-y_3)+a_2(x_3-y_3)(x_1-y_1)+a_3(x_1-y_1)(x_2-y_2)\right]},$$

in which R is the circumradius of (T). Considerable simplification is required in the case of UU'.

(c) Any line perpendicular to IO is the locus of points for which the sum of the distances from the sides of (T) is constant [**4**]. For UU' this constant is

$$2\Delta\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}-\frac{1}{u}\right).$$

(d) The area of the inscribed triangle $B_1B_2B_3$ is $u^3/2R$.

(e) It is known that the isotomic conjugate of (x_i) is $(1/a_i^2x_i)$. If x_i is replaced by $1/a_i^2x_i$ in (1), the result is a conic

$$\sum\frac{a_i-2u}{(a_i-u)x_i}=0. \tag{2}$$

Since this conic is the locus of isotomic conjugates of points on UU', it contains both U and U'. Also, multiplication by $x_1x_2x_3$ adds A_1, A_2, A_3 to the curve. Since U and U' are interior to (T), the conic is a hyperbola. Its center (found by considering it as the pole of the line at infinity) is given by

$$\left(\frac{(a_i-2u)^2}{a_i(a_i-u)}\right).$$

The hyperbola may also be regarded as the polar conic [**2**, p. 114] of

$$\left(\frac{a_i-2u}{a_i-u}\right),$$

the intersection point of the trilinear polars of U and U'. This point, though irrational, lies on a rational line through the centroid of (T).

3. Analogous relations. (a) A well-known property of the Brocard angle is given by $\omega\leqq\pi/6$, and the analogous inequality $u\leqq p/6$ is also true. To prove this, consider

$$f\left(\frac{p}{6}\right)=\frac{p^3}{108}-\frac{p^3}{36}+\frac{qp}{6}-r,$$

in which the last two terms constitute a function of two independent variables (e.g., a_1 and a_2) if p is fixed. This function is found by ordinary methods to have $p^3/54$ as its minimum value. Hence $f(p/6)\geqq 0$, and $u\leqq p/6$. Equality holds only in an equilateral triangle.

(b) There also exist the relations

$$\omega < \alpha_i < \pi - 3\omega \qquad (i = 1, 2, 3)$$

and

$$u < a_i < p - 3u.$$

The inequalities $\omega < \alpha_i$ and $u < a_i$ are obvious because Ω and U are respectively interior to (T). It is also clear that $p - 3u \geqq p - (p/2) > a_i$. To prove $\alpha_i < \pi - 3\omega$, we use the angles δ and δ' defined by $\cot(\delta/2) = \cot\omega - \sqrt{(\cot^2\omega - 3)}$ and $\cot(\delta'/2) = \cot\omega + \sqrt{(\cot^2\omega - 3)}$ [**3**, p. 288]. For a given ω, δ and δ' are respectively the maximum and minimum possible values for α_i. It is easy to show that $\delta + \delta' = \pi - 2\omega$ and $\delta' > \omega$, whence $\pi - 3\omega > \delta \geqq \alpha_i$.

(c) Now if the angles are ordered by $\alpha_1 \leqq \alpha_2 \leqq \alpha_3$, it may be proved that $\alpha_1 \leqq 2\omega$. Using the fact that the cotangent is a continuous decreasing function in the open interval from 0 to π, we have

$$\tan\frac{\alpha_1}{2} = \cot\frac{\pi - \alpha_1}{2} = \cot\frac{\alpha_2 + \alpha_3}{2} \geqq \cot\alpha_3$$

and

$$\cot\alpha_1 \geqq \cot\alpha_2,$$

whence

$$\cot\alpha_2 + \cot\alpha_3 \leqq \cot\alpha_1 + \tan\tfrac{1}{2}\alpha = \csc\alpha_1.$$

Therefore

$$\cot\omega = \sum \cot\alpha_i \leqq \csc\alpha_1 + \cot\alpha_1 = \cot\tfrac{1}{2}\alpha,$$

or

$$\omega \geqq \tfrac{1}{2}\alpha.$$

The analogous relation is $a_1 \leqq 2u$, which is verified by

$$\begin{aligned} f\left(\frac{a_1}{2}\right) &= \frac{a_1^3}{4} - \frac{pa_1^2}{4} + \frac{qa_1}{2} - r \\ &= \frac{a_1}{4}(a_1a_2 + a_1a_3 - 2a_2a_3) \\ &= \frac{a_1}{4}[a_2(a_1 - a_3) + a_3(a_1 - a_2)] \leqq 0. \end{aligned}$$

Under the same ordering, one may prove $\frac{1}{2}\pi - \omega \leqq \alpha_3 < \pi - 3\omega$ and $\frac{1}{2}p - u \leqq a_3 < p - 3u$, as well as many other pairs of analogous relations.

(d) Simultaneous solution of the equations of the hyperbola (2) and the

line at infinity involves the expression $\sqrt{(a_1a_2a_3-8u^3)}$. Since the intersections are real, it follows that $8u^3 \leqq a_1a_2a_3$. Equality occurs only in the equilateral case, when the hyperbola is degenerate.

The corresponding relation, $8\omega^3 \leqq \alpha_1\alpha_2\alpha_3$, is apparently true but remains to be proved generally. If valid, it would be a refinement of $\omega \leqq \pi/6$.

4. Exceptions to the analogy. There is a basic difference between the Brocard theory and the present one, in that each side of (T) is less than the sum of the others, whereas the angles are not similarly restricted. One may therefore expect further discrepancies.

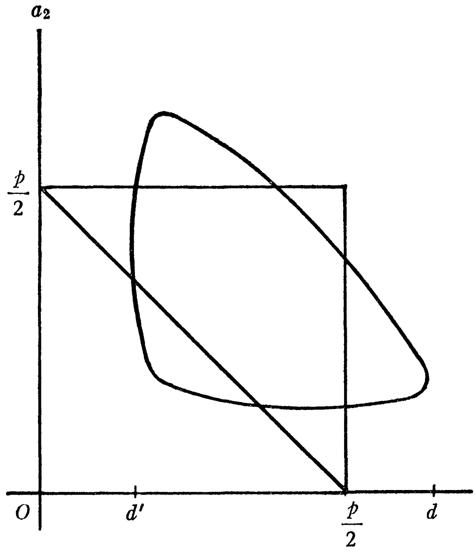

FIG. 2

By means of the substitution $a_3=p-a_1-a_2$, the equation $f(u)=0$ becomes $a_1a_2(a_1+a_2)-pa_1a_2-u(a_1^2+a_1a_2+a_2^2)+pu(a_1+a_2)-pu^2+2u^3=0$. The relevant part of its graph is shown in Fig. 2; indeed the only portion to be considered is within the triangular region defined by $a_1<\frac{1}{2}p$, $a_2<\frac{1}{2}p$, $a_1+a_2>\frac{1}{2}p$. Ignoring this restriction for the moment, one finds by calculus that the extremes of a_1 occur when $a_1+2a_2=p$ or $a_2=a_3$. Letting $a_2=\frac{1}{2}(p-a_1)$ in (2), one obtains $\phi(a_1)\equiv a_1^3-(2p-3u)a_1^2+p(p-2u)a_1-u(p^2-4pu+8u^2)=0$, two of whose roots are the minimum d' and the maximum d. The third root belongs to an extraneous part of the graph not shown in Fig. 2.

The point at which $a_1=d'$ is easily found to be within the triangular region in Fig. 2, so d' is the actual minimum length of any side of (T) for a given u. In the Brocard geometry there is a relation $\omega<\delta'\leqq 2\omega$, which has as its counterpart $u<d'\leqq 2u$. The latter is also true since $\phi(u)=-4u^3<0$, while $\phi(2u)=u(p-2u)(p-6u)\geqq 0$.

A similar relation is $\pi-4\omega \leqq \delta < \pi-3\omega$, which suggests $p-4u \leqq d < p-3u$. Now $p-4u \leqq d$ is valid because $\phi(p-4u)=4u^2(p-6u) \geqq 0$. However, $\phi(p-3u) = u^2(p-8u)$; thus $d<p-3u$ only when $p<8u$. Since $d \geqq p-3u$ is possible, and $u \leqq p/6$, it follows that $d \geqq p/2$ is possible. In this case the least upper bound of a_1 is not d but $p/2$.

Finally it is easy to show that the curve in Fig. 2 lies entirely within the triangle when $d<p/2$. To find the critical value of u when $d=p/2$, we calculate

$$\phi\left(\frac{p}{2}\right) = \frac{1}{8}(p^3 - 10p^2u + 32pu^2 - 64u^3),$$

which vanishes when $u=0.1508p$ approximately.

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A COMPLETE EXTENSION OF ORDINAL NUMBERS

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This paper proposes a complete extension of ordinal numbers with their non-commutative operations. We are led to a dense extension of ordinal numbers analogous to the usual extension of natural numbers to rationals. The difficulty in introducing density in the extension, due to noncommutativity, will be overcome through the introduction of ordinal continued fractions. Such an extension and its completion will not be the rational and real numbers in the usual sense but will be something analogous to these systems and may here be called "ordinal rational numbers" and "ordinal real numbers."

The theory of ordinal numbers is usually based on the theory of well ordering, but R. M. Robinson [1], P. Bernays [2], and K. Gödel [3] among others, have introduced an independent theory of ordinal numbers without referring in the definition to the concept of order. Addition $(+)$ and multiplication $(\cdot)$ of ordinal numbers can be defined inductively with the following properties:

(1) Each operation is associative but not commutative;

(2) Left cancellation is always possible (except for a zero factor), i.e., $a<b$ implies the existence of a unique ordinal number $c>0$ such that $a+c=b$; c is designated as $-a+b$, and the difference $\triangle(a, b)$ is $-a+b$ or $-b+a$ according as $a \leqq b$ or $a \geqq b$.

1. A dense extension. Natural numbers can be extended directly to rationals by means of triplets. This suggests an analogous extension of ordinal numbers.

1.1. *Ordinal triplets.* (a, b, c) is called an ordinal triplet if and only if a and b are any two ordinal numbers while c is a nonzero ordinal number.

A triplet (a, b, c) can be brought to a unique canonical form, designated $(a, b, c)^*$ by cancelling the greatest common left summand from a and b and cancelling the greatest common left factor from $\triangle(a, b)$ and c. If $\triangle(a, b)=p\cdot u$ and $c=p\cdot v$, then $(a, b, c)^*=(u, 0, v)$ or $(0, u, v)$ or $(0, 0, 1)$ according as $a>b$ or $a<b$ or $a=b$.

An equivalence relation "$\simeq$" is introduced in terms of canonical forms.

DEFINITION 1.1. *$(a_1, b_1, c_1)\simeq(a_2, b_2, c_2)$ if and only if $(a_1, b_1, c_1)^*=(a_2, b_2, c_2)^*$.*

This relation is clearly reflexive, symmetric and transitive.

1.2. *Equivalence class of triplets.* An equivalence class $\{(a, b, c)\}$ defined by $\simeq$ is said to be an ordinal rational number and the class of all such rationals is denoted by R.

DEFINITION 1.2. *An equivalence class $\{(a, b, c)\}$ is said to be:*

(i) *finite if at least one of the equivalence class representatives consists of finite ordinal numbers;*

(ii) *nonfinite otherwise; and*

(iii) *positive or negative or null (zero) according as $a>b$ or $a<b$ or $a=b$ for any equivalence class representative.*

Remark 1.1. $\{(a, b, c)\}$ is finite or nonfinite according as the elements of $(a, b, c)^*$ are all finite or not.

Remark 1.2. $\{(a, b, c)\}$ is positive or negative or null according as the second or first or both those elements of $(a, b, c)^*$ are zero.

1.3. *Mapping of R onto continued fractions.* The class R can be mapped onto the set of continued fractions (CF's) by a one to one correspondence by developing each element $(a, b, c)^*$ in a simple CF as follows. Let

$$\text{(A)}\qquad \begin{aligned} \triangle(a, b) &= cx_0 + r_0, & r_0 &< c \\ c &= r_0x_1 + r_1, & r_1 &< r_0 \\ r_0 &= r_1x_2 + r_2, & r_2 &< r_1 \\ r_1 &= r_2x_3 + r_3, & r_3 &< r_2 \\ &\cdots\cdots\cdots\cdots & & \\ r_{m-3} &= r_{m-2}x_{m-1} + r_{m-1}, & r_{m-1} &< r_{m-2} \\ r_{m-2} &= r_{m-1}x_m + 0. & & \end{aligned}$$

The Euclidean algorithm terminates after a finite number of steps since the r_i form a strictly decreasing sequence and will therefore contain only a finite number of ordinals. $(\triangle(a, b), 0, c)$ is therefore developed in a simple positive CF $x_0+1/x_1+1/x_2+\cdots+1/x_m$, while $(0, \triangle(a, b), c)$ is developed in the corresponding negative CF. To a given $(\triangle(a, b), 0, c)$ or $(0, \triangle(a, b), c)$ there corresponds a unique CF but not conversely. For, in (A), $\triangle(a, b)$ and c alone determine

x_i $(i=0, 1, \cdots, m)$. But, on the other hand, x_i as well as r_{m-1} determine $\triangle(a, b)$ and c. The r_{m-1} is also the greatest common left factor of $\triangle(a, b)$, and c and the x_i are not affected at all by cancelling r_{m-1} from $\triangle(a, b)$ and c. $(a, b, c)^*$ is therefore also developed in the same CF with or without the negation sign placed over it according as $a<b$ or $a>b$. For the sake of brevity a CF will also be denoted by $\{(x_0, x_1, x_2, \cdots, x_m)\}$ and the zero CF by $\{(0)\}$. If $x_m=x_m'+1$, then the $CF=\{(x_0, x_1, \cdots, x_m', 1)\}$.

Associating $\{(a, b, c)\}$ with the CF corresponding to $(a, b, c)^*$, we map R onto the set of CF's by a one to one correspondence.

1.4. *Density and incompleteness of* $(R, <)$. With the aid of the following definitions an order is introduced into the system of ordinal rational numbers which makes it dense and incomplete.

DEFINITION 1.3. $\alpha=\{(x_0, x_1, \cdots, x_m)\}<\{(y_0, y_1, \cdots, y_n)\}=\beta$ *if and only if*

A. α *is negative and* β *is nonnegative* (*positive or zero*); *or*

B. α *is zero and* β *is positive*; *or*

C. α *and* β *are both positive and any one of the following holds*:

(i) $x_0<y_0$, *or*

(ii) *for some* $r<m$ *and* $r<n$, $x_i=y_i$ *for* $i=0, 1, \cdots, r$ *and further*

(a) r *is zero or even and* $x_{r+1}>y_{r+1}$, *or*

(b) r *is odd and* $x_{r+1}<y_{r+1}$, *or*

(iii) $m<n$, $x_i=y_i$ *for* $i=0, 1, \cdots, m$ *and* m *is even, or*

(iv) $m>n$, $x_i=y_i$ *for* $i=0, 1, \cdots, n$ *and* n *is odd*; *or*

D. α *and* β *are both negative and* $\{+(y_0, y_1, \cdots, y_n)\}<\{+(x_0, x_1, \cdots, x_m)\}$.

DEFINITION 1.4. $\{(a, b, c)\}<\{(d, e, f)\}$ *if and only if the CF associated with* $\{(a, b, c)\}$ *is less than the CF associated with* $\{(d, e, f)\}$.

THEOREM 1.1. *The set of CF's is dense with respect to "<."*

Proof. Let $\{(x_0, x_1, \cdots, x_m)\}<\{(y_0, y_1, \cdots, y_n)\}$. The following cases may arise in determining the existence of a CF lying between them.

Case 1. At least one is $\{(0)\}$. Clearly there will then exist a $\{-(x_0)\}$ or $\{+(y_0)\}$ lying between them according as the first is negative or the second is positive.

Case 2. Both are of the opposite sign. Clearly then $\{(0)\}$ will lie between them.

Case 3. Both are positive and:

(i) if $x_0 \neq y_0$, then there exists a $\{+(y_0)\}$ lying between them;

(ii) if for some $r \leqq m-1$ and $r<n$, $x_i=y_i$ for $i=0, 1, \cdots, r$ and further r is zero or even and $x_{r+1}>y_{r+1}$, then there exists a positive

$\{(x_0, x_1, \cdots, x_r, x'_{r+1})\}$, where $y_{r+1}<x_{r+1}<x_{r+1}$ and $x_{r+1}>y_{r+1}+1$, or

$\{(x_0, x_1, \cdots, x_r, x_{r+1}, x'_{r+2})\}$, where $x'_{r+2}>x_{r+2}$, $x_{r+1}=y_{r+1}+1$ and $r<m-1$, or

$\{(x_0, x_1, \cdots, x_{m-1}, y_m, 1, x'_{m+2})\}$, where $x'_{m+2} > 1$, $x_m = y_m + 1$ and $r = m - 1$;

(iii) if in (ii) r is odd and $x_{r+1} < y_{r+1}$, then there exists a positive

$\{(x_0, x_1, \cdots, x_r, x''_{r+1})\}$, where $x_{r+1} < x''_{r+1} < y_{r+1}$ and $y_{r+1} > x_{r+1} + 1$, or

$\{(x_0, x, \cdots, x_{r+1}, x''_{r+2})\}$, where $x''_{r+2} < x_{r+2}$, $y_{r+1} = x_{r+1} + 1$ and $r < m - 1$, or

$\{(x_0, \cdots, x_{m-1}, x_m, x''_{m+1})\}$, where $x''_{m+1} > 1$, $y_m = x_m + 1$ and $r = m - 1$;

(iv) if $x_i = y_i$ for $i = 0, 1, \cdots, m$ and further m is even $< n$, then there exists a

$$\{(x_0, x_1, \cdots, x_m, x'_{m+1})\}, \quad \text{where} \quad x'_{m+1} > y_{m+1};$$

(v) if $x_i = y_i$ for $i = 0, 1, \cdots, n$ and n is odd $< m$, then there exists a

$$\{(x_0, x_1, \cdots, x_n, x'_{n+1})\} \quad \text{where} \quad x'_{n+1} > x_{n+1}.$$

Case 4. Both are negative. The existence of a CF lying between two negative CF's can easily be deduced from the Definition 1.3.D and the Case 3.

COROLLARY 1.1. *R is dense with respect to "$<$".*

For the density of R follows from the density of CF's.

THEOREM 1.2. *$(R, <)$ is incomplete.*

Proof. To show incompleteness of R we shall examine the nature of a Dedekind section $[A, B]$ associated with a sequence (α_n) where

$$\alpha_n = \{+(x_0, x_1, \cdots, x_n)\}.$$

Since α_n increases or decreases according as n runs over the increasing sequence of even or odd natural numbers, (α_{2n}) is monotonic increasing and (α_{2n+1}) is monotonic decreasing. Now consider two classes A and B such that every negative, the null and all positive CF's which are either $<\alpha_0$ or $<\alpha_2$ or $<\alpha_4 \cdots$ or $<\alpha_{2n} \cdots$ belong to A, while all positive CF's which are either $>\alpha_1$ or $>\alpha_3$ or $>\alpha_5 \cdots$ or $>\alpha_{2n+1} \cdots$ belong to B. Then every α_{2n} belongs to A, while every α_{2n+1} belongs to B. There cannot exist a CF lying between α_n and α_{n+1} for every n. If it were possible, let $\beta = \{+(y_0, y_1, \cdots, y_r)\}$ be such a CF. Then β belongs to neither A nor B.

Since $\beta \overline{\in} A$, we have $y_0 \geqq x_0$. For otherwise $y_0 < x_0$, $\beta < \alpha_0$ and $\beta \in A$, a contradiction. Again since $\beta \overline{\in} B$, we have $y_0 \leqq x_0$. For otherwise $y_0 > x_0$, $\beta > \{+(x_0, x_1)\}$ and $\beta \in B$, again a contradiction. Obviously then $y_0 = x_0$.

Following the same arguments with the added assumption that $y_0 = x_0$, it can easily be verified that $y_1 \leqq x_1$ and $y_1 \geqq x_1$ since β belongs to neither A nor B. Clearly then $y_1 = x_1$. By a chain of similar arguments, it then follows that $y_2 = x_2$, $y_3 = x_3 \cdots$ and $y_r = x_r$. Obviously β is simply α_r which belongs to either A or B.

Since A and B are induced by two infinite monotone sequences (α_{2n}) and

(α_{2n+1}) it is clear that there is no maximal element in A and no minimal element in B and (α_n) thus induces a gap section. Further the first of (α_{2n}) and (α_{2n+1}) is monotonic increasing and bounded above but has no supremum, while the second is monotonic decreasing and bounded below but has no infimum. It therefore follows that R is incomplete.

2. Completeness of $(R, <)$.

2.1. *Russell's method of completion.* It is convenient to follow Russell's method of finding the completion of a dense system. In this section $\alpha, \beta, \gamma, \delta \cdots$ will be used for the elements of R and $A, B, C, D, \cdots$ for the elements of the completion $\hat{R}$ of R.

DEFINITION 2.1. *A nonempty class X is said to be an ordinal real number if whenever $\alpha_1 \in X$, and $\alpha_2 \leqq \alpha_1$, then $\alpha_2 \in X$ and also there exists an ordinal rational $\alpha_3 \in X$ such that $\alpha_3 > \alpha_1$.*

The class of all such real numbers is denoted by $\hat{R}$. The usual concept of $\leqq$ and $<$ between real numbers is translated into the concept of subset $\subseteq$ and proper subset $\subset$, and familiarity with these concepts and their properties will be assumed.

2.2. *Theorem of completeness and fundamental theorem of analysis.* We will need the following definitions.

DEFINITION 2.2. *If $A < B$, then the set of all elements X such that $A < X < B$ constitutes an (open) interval (A, B). An (A, B) containing an element C is said to be a neighbourhood $Nb(C)$ of C. L is said to be a limiting element of a set $\hat{M}$ of real numbers if and only if every $Nb(L)$ contains at least one element of $\hat{M}$ other than L.*

Since the system is dense it is trivial that $Nb(L)$ contains infinitely many elements of $\hat{M}$.

THEOREM 2.1. *Every class $\hat{M}$ of ordinal real numbers which is bounded above (below) has a unique supremum (infimum).*

THEOREM 2.2. *Every bounded infinite set $\hat{M}$ of ordinal real numbers has a limiting element.*

The proofs of the theorems are the same as for the ordinary real numbers.

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Editorial Note. The referee kindly supplied the following further references, obtained from Fraenkel's "Abstract Set Theory," North-Holland Publishing Company, Amsterdam, 1953 (p. 294) and 2nd Ed., 1961 (pp. 211, 267).

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THE ORDER OF ARITHMETICAL FUNCTIONS OF GENERALIZED INTEGERS

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1. Introduction. In a previous paper, [1] I have defined generalized integers as follows. Suppose there is given a finite or infinite sequence $\{p\}$ of real numbers (generalized primes) such that

$$1 < p_1 < p_2 < \cdots.$$

Form the set $\{b\}$ of all possible p-products, i.e., products $p_1^{v_1}p_2^{v_2}\cdots$, where $v_1, v_2, \cdots$, are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalized integers and suppose that no two generalized integers are equal if their v's are different. Then arrange $\{b\}$ as an increasing sequence:

$$1 = b_1 < b_2 < b_3 < \cdots.$$

Note. In [1] the generalized integers were denoted by l_n instead of b_n as here. Let $[x]$ = the number of generalized integers $\leqq x$, where x is any real number.

Assume

$$[x] = x + R(x), \tag{1.1}$$

where $R(x) = O(x^\alpha)$ and $\alpha < 1$. This assumption is fundamental to the paper. Without it only results depending on the method of construction of the generalized integers and their ordering may be found as, for example, those given in [1]. The assumption means that we choose our generalized integers in such a way that the size of the nth generalized integer, b_n, is about that of the integer n. For, from (1.1) $[x] \sim x$, and $[b_n] = n$, and so $b_n \sim n$.

The aim of this paper is to extend the classical work done on the order of magnitude of arithmetical functions (see e.g. [2]) to generalized integers with the help of assumption (1.1).

2. Supplementary results.

THEOREM 1. *The number of generalized primes is infinite.*

Proof. Suppose there are only a finite number of generalized primes, say $p_1, p_2, \cdots, p_k$. Then any $b_n = p_1^{v_1}p_2^{v_2}\cdots p_k^{v_k}$. Also, if $b_n \leqq x$ then $p_r^{v_r} \leqq x$, for $r = 1, 2, \cdots, k$. Hence

$$v_r \leqq \frac{\log x}{\log p_r} \leqq \frac{\log x}{\log p_1}.$$

Denote by $\{x\}$ the number of *integers* $\leqq x$ in this section only. Since v_r must be integral, the number of possible products formed from $p_1, p_2, \cdots, p_k$

and $\leqq x$ is less than or equal to

$$\left(\left\{\frac{\log x}{\log p_1}\right\} + 1\right)^k.$$

Hence

$$[x] \leqq \left(\left\{\frac{\log x}{\log p_1}\right\} + 1\right)^k = O(\log^k x).$$

This is contrary to assumption (1.1) and Theorem 1 is proved. Again, since $p_n > b_n$ and $b_n \to \infty$ with n, $p_n \to \infty$ with n, and so there exists a

(2.1) $$p_a \text{ (say) such that } p_a \geqq 2.$$

The Möbius function. Define $\mu(b_n) = 0$ if b_n has a square factor; $\mu(b_n) = (-1)^k$, where k denotes the number of prime divisors of b_n and b_n has no square factor; $\mu(1) = 1$. Then

(2.2) $$\sum_{d \mid b_n} \mu(d) = \begin{cases} 0 & \text{when } b_n \neq 1 \\ 1 & \text{when } b_n = 1. \end{cases}$$

This is proved in [1]. The following inversion formula is also proved in [1]. If $G(b_n) = \sum_{d \mid b_n} F(d)$, then $F(b_n) = \sum_{d \mid b_n} \mu(b_n/d) G(d)$.

In this paper, however, another inversion formula is needed. It is

THEOREM 2. *If* $G(x) = \sum_{b_n \leqq x} F(x/b_n)$, *then* $F(x) = \sum_{b_n \leqq x} \mu(b_n) G(x/b_n)$.

Proof.

$$\begin{aligned} \sum_{b_n \leqq x} \mu(b_n) G(x/b_n) &= \sum_{b_n \leqq x} \mu(b_n) \sum_{b_m \leqq x/b_n} F(x/b_m b_n) = \sum_{b_k \leqq x} F(x/b_k) \sum_{b_n \mid b_k} \mu(b_n) \\ &= F(x) \quad \text{from (2.2).} \end{aligned}$$

Various estimates. Some estimates will be needed and Abel's transformation, in the following form, will be used.

THEOREM 3. *Suppose two sequences* $\{\lambda_n\}$ *and* $\{a_n\}$ *given with* $\lambda_1 \leqq \lambda_2 \leqq \cdots$; $\lambda_n \to \infty$. *Let* $A(x) = \sum_{\lambda_n \leqq x} a_n$. *Suppose* $\phi(x)$ *has a continuous derivative* $\phi'(x)$ *for all* x *involved except possibly for* $x = \lambda_r$, $(r = 1, 2, \cdots)$. *Then*

$$\sum_{\lambda_n \leqq x} a_n \phi(\lambda_n) = A(x)\phi(x) - \int_{\lambda_1}^{x} A(y)\phi'(y)dy.$$

Proof.

(2.3) $$\sum_{\lambda_n \leqq x} a_n \{\phi(x) - \phi(\lambda_n)\} = \sum_{\lambda_n \leqq x} a_n \int_{\lambda_n}^{x} \phi'(y)dy = \int_{\lambda_1}^{x} \sum_{\lambda_n \leqq y} a_n \phi'(y)dy.$$

The interchange of the order of summation and integration is effected by using

the formula

$$\sum_{\lambda_n \leqq x} \int_{\lambda_1}^{x} f_n(y)dy = \int_{\lambda_1}^{x} \sum_{\lambda_n \leqq x} f_n(y)dy,$$

where

$$f_n(y) = \begin{cases} 0 & \text{for } \; y < \lambda_n \\ a_n\phi'(y) & \text{for } \; y \geqq \lambda_n. \end{cases}$$

Rewriting (2.3) gives $A(x)\phi(x) - \sum_{\lambda_n \leqq x} a_n\phi(\lambda_n) = \int_{\lambda_1}^{x} A(y)\phi'(y)dy$. This proves Theorem 3.

Applications of Theorem 3. Put $\lambda_n = b_n$, $a_n = 1$ and $\phi(x) = x^{-\beta}$. Then, from (1.1)

$$A(x) = \sum_{b_n \leqq x} 1 = [x] = x + R(x).$$

Hence

$$(2.4) \qquad \begin{aligned} \sum_{b_n \leqq x} \frac{1}{b_n^{\beta}} &= (x + R(x))x^{-\beta} + \int_1^x \beta(y + R(y))y^{-\beta-1}dy, \\ &= \frac{x^{1-\beta}}{1-\beta} + \gamma_\beta + O(x^{\alpha-\beta}), \end{aligned}$$

where $\beta \neq 1$, $\beta \neq \alpha$, and γ_β is a constant.

Later we shall use (2.4) in the form

$$(2.41) \qquad \sum_{b_n \leqq x} \frac{1}{b_n^{\beta}} = \frac{x^{1-\beta}}{1-\beta} + O(1) + O(x^{\alpha-\beta}).$$

In (2.4) replace x by y, where $y > x$, giving

$$(2.5) \qquad \sum_{b_n \leqq y} \frac{1}{b_n^{\beta}} = \frac{y^{1-\beta}}{1-\beta} + \gamma_\beta + O(y^{\alpha-\beta}).$$

Now subtract (2.4) from (2.5), take $\beta > 1$, and let $y \to \infty$, obtaining

$$(2.6) \qquad \sum_{b_n > x} \frac{1}{b_n^{\beta}} = O(x^{1-\beta}) \quad \text{for} \quad \beta > 1.$$

Again, in Theorem 3, put $\lambda_n = b_n$, $a_n = 1$ and $\phi(x) = [b_m/x]$, where b_m is a fixed generalized integer. Then

$$(2.7) \qquad \begin{aligned} \sum_{b_n \leqq b_m} \left[\frac{b_m}{b_n}\right] &= (b_m + R(b_m))[1] - \int_1^{b_m} (y + R(y)) \frac{d}{dy}\left[\frac{b_m}{y}\right] dy \\ &= b_m \log b_m + O(b_m). \end{aligned}$$

The generalized zeta function. Define $\zeta(s)=\sum_{n=1}^{\infty} b_n^{-s}$, $(s>1)$. (2.6) may now be written

$$\sum_{b_n>x}\frac{1}{b_n^\beta}=\zeta(\beta)-\sum_{b_n\leqq x}\frac{1}{b_n^\beta}=O(x^{1-\beta}) \qquad \text{if } \beta>1. \tag{2.8}$$

It is proved in [3], using an assumption equivalent to (1.1) that

$$\zeta(s)=\prod_{r=1}^{\infty}\frac{1}{1-p_r^{-s}}\cdot$$

Hence

$$\frac{1}{\zeta(s)}=\prod_{r=1}^{\infty}(1-p_r^{-s})=\sum_{n=1}^{\infty}\mu(b_n)b_n^{-s}, \qquad s>1. \tag{2.9}$$

3. The order of magnitude of $\tau(b_n)$, the number of divisors of b_n. Obviously, since $\tau(p_n)=2$, we have

$$\liminf_{n\to\infty}\tau(b_n)=2. \tag{3.1}$$

THEOREM 4. *The order of magnitude of $\tau(b_n)$ is sometimes larger than any power of* $\log b_n$; *i.e., the equation*

$$\tau(b_n)=O\{\log b_n)^\Delta\} \tag{3.2}$$

is false for every Δ.

Proof. If $b_n=p_1^m$, then $\tau(b_n)=m+1\sim(\log b_n/\log p_1)$. If $b_n=(p_1p_2)^m$, then $\tau(b_n)=(m+1)^2\sim(\log b_n/\log p_1p_2)^2$; and so on. If $k\leqq\Delta<k+1$ and $b_n=(p_1p_2\cdots p_{k+1})^m$, then

$$\tau(b_n)=(m+1)^{k+1}\sim\left\{\frac{\log b_n}{\log p_1p_2\cdots p_{k+1}}\right\}^{k+1}>C(\log b_n)^{k+1},$$

where C is independent of b_n. Hence (3.2) is false for an infinite sequence of values of b_n.

THEOREM 5. $\tau(b_n)=O(b_n^\delta)$ *for all positive real numbers* δ.

Proof. Let $b_n=\prod_{j=1}^{k} p_j^{v_j}$. Then $\tau(b_n)=\sum_{d|b_n}1=\prod_{j=1}^{k}(1+v_j)$, and so

$$\frac{\tau(b_n)}{b_n^\delta}=\prod_{j=1}^{k}\left(\frac{1+v_j}{p_j^{v_j\delta}}\right). \tag{3.3}$$

Let p_a be the least prime $\geqq 2$, (from (2.1)). Now for any prime $p\geqq p_1$, $v\delta\log p_1\leqq e^{v\delta\log p_1}=p_1^{v\delta}\leqq p^{v\delta}$. Hence

$$\frac{v+1}{p^{v\delta}}\leqq 1+\frac{v}{p^{v\delta}}\leqq 1+\frac{v}{v\delta\log p_1}\leqq\exp\left(\frac{1}{\delta\log p_1}\right).$$

We use this in (3.3) for those primes p for which $p_1 \leqq p < p_a^{1/\delta}$. There are less than $[p_a^{1/\delta}]$ such primes. If $p \geqq p_a^{1/\delta}$ then $p^\delta \geqq p_a$ and, since $p_a \geqq 2$,

$$\frac{v+1}{p^{v\delta}} \leqq \frac{v+1}{p_a^v} \leqq 1.$$

Hence

$$\frac{\tau(b_n)}{b_n^\delta} \leqq \prod_{p<p_a^{1/\delta}} \exp\left(\frac{1}{\delta \log p_1}\right) \leqq \exp \frac{[p_a^{1/\delta}]}{\delta \log p_1} = O(1). \tag{3.4}$$

This is Theorem 5.

4. The average order of $\tau(b_n)$. Let $T(x) = \sum_{b_n \leqq x} \tau(b_n)$. Then

$$T(b_n) = \sum_{h=1}^{n} \left[\frac{b_n}{b_h}\right]. \tag{4.1}$$

This is Theorem 5 of [1] but an alternative proof is as follows:

Construct a rectangular lattice with axes b_x, b_y, i.e., generalized integers $b_1, b_2, \cdots$ are equally spaced along b_x starting from an origin O, and again along b_y which is perpendicular to b_x also starting from the origin O. (A helpful picture is Fig. 9, page 265 of [2].) Then all the integral points (not counting those on the axes) have co-ordinates (b_i, b_j). For fixed b_n, draw the hyperbola

$$b_x b_y = b_n.$$

Denote by D the region contained between the axes and this hyperbola. Now every lattice point in D appears on a hyperbola

$$b_x b_y = b_s \qquad (1 \leqq b_s \leqq b_n)$$

and the number on such a hyperbola is $\tau(b_s)$. Hence the number of lattice points in D is

$$\tau(b_1) + \tau(b_2) + \cdots + \tau(b_n) = T(b_n).$$

Of these points $[b_n]$ have b_x coordinate b_1, $[b_n/b_2]$ have b_x coordinate b_2, and so on. So again the total number of lattice points in D is

$$[b_n] + \left[\frac{b_n}{b_2}\right] + \left[\frac{b_n}{b_3}\right] + \cdots = \sum_{h=1}^{n} \left[\frac{b_n}{b_h}\right].$$

Hence $T(b_n) = \sum_{h=1}^{n} [b_n/b_h]$ and (4.1) is proved. Now using (4.1) and (2.7) we have

THEOREM 6. $T(b_n) = b_n \log b_n + O(b_n)$.

In other words, the average order of $\tau(b_n)$ is $\log b_n$.

5. The average order of $\sigma(b_n)$, the sum of the divisors of b_n. Since $1 | b_n$ and $b_n | b_n$ we have

$$\sigma(b_n) > b_n. \tag{5.1}$$

THEOREM 7. $\sum_{r=1}^{n} \sigma(b_r) = \frac{1}{2}\zeta(2)b_n^2 + O(b_n^{1+\alpha})$.

Proof. We again use the lattice described in Section 4.

$$\sigma(b_1) + \sigma(b_2) + \cdots + \sigma(b_n) = \sum b_y,$$

where the r.h.s. is the sum of the ordinates of every lattice point of D. Hence, from (2.41) with $\beta = -1$ and $x = b_n/b_x$,

$$\sum_{b_r \leq b_n} \sigma(b_r) = \sum_{b_x \leq b_n} \sum_{b_y \leq b_n/b_x} b_y = \sum_{b_x \leq b_n} \left\{ \frac{1}{2}\left(\frac{b_n}{b_x}\right)^2 + O(1) + O\left(\frac{b_n}{b_x}\right)^{\alpha+1} \right\}. \tag{5.2}$$

But from (2.8) with $\beta = 2$ and $x = b_n$;

$$\sum_{b_x \leq b_n} \frac{1}{2}\left(\frac{b_n}{b_x}\right)^2 = \frac{b_n^2}{2} \sum_{b_x \leq b_n} \frac{1}{b_x^2} = \frac{b_n^2}{2}\left\{\zeta(2) + O\left(\frac{1}{b_n}\right)\right\};$$

and

$$\sum_{b_x \leq b_n} O(1) = O\left(\sum_{b_x \leq b_n} 1\right) = O(b_n)$$

from (1.1); and from (2.41) with $\beta = 1 + \alpha$ and $x = b_n$

$$\sum_{b_x \leq b_n} O\left(\frac{b_n}{b_x}\right)^{\alpha+1} = O\left(b_n^{\alpha+1} \sum_{b_x \leq b_n} \left(\frac{1}{b_x}\right)^{\alpha+1}\right) = O(b_n^{\alpha+1} \cdot (b_n^{-\alpha} + O(1))).$$

Substituting these results in (5.2) we obtain

$$\sum_{b_r \leq b_n} \sigma(b_r) = \tfrac{1}{2}\zeta(2)b_n^2 + O(b_n) + O(b_n^{1+\alpha}).$$

This proves Theorem 7.

6. The number of square-free generalized integers. Let $Q(x)$ be the number of square-free generalized integers $\leqq x$. We can arrange all the numbers $b_n \leqq y^2$ in sets $S_{b_1}, S_{b_2}, \cdots$, such that S_{b_d} contains just those b_n whose largest square factor is b_d^2. Thus S_1 is the set of all square-free $b_n \leqq y^2$.

The number of b_n belonging to S_{b_d} is $Q(y^2/b_d^2)$, and, when $b_d > y$, S_{b_d} is empty. Hence

$$[y^2] = \sum_{b_d \leq y} Q(y^2/b_d^2).$$

Using the inversion formula of Theorem 2 we have

$$Q(y^2) = \sum_{b_n \leqq y} \mu(b_n)[y^2/b_n^2] = \sum_{b_n \leqq y} \mu(b_n)\left(\frac{y^2}{b_n^2} + O\left(\frac{y}{b_n}\right)^{2\alpha}\right) \qquad \text{(from (1.1))}$$

$$= y^2 \sum_{b_n \leqq y} \frac{\mu(b_n)}{b_n^2} + O\left(y^{2\alpha} \sum_{b_n \leqq y} \frac{\mu(b_n)}{b_n^{2\alpha}}\right)$$

$$= y^2 \cdot \frac{1}{\zeta(2)} + O\left(y^2 \sum_{b_n > y} \frac{1}{b_n^2}\right) + O\left(y^{2\alpha} \sum_{b_n \leqq y} \frac{\mu(b_n)}{b_n^{2\alpha}}\right)$$

from (2.9). But

$$O\left(y^2 \sum_{b_n > y} \frac{1}{b_n^2}\right) = O\left(y^2 \cdot \frac{1}{y}\right) \qquad \text{(from (2.6))}$$

and

$$O\left(y^{2\alpha} \sum_{b_n \leqq y} \frac{\mu(b_n)}{b_n^{2\alpha}}\right) = O\left(y^{2\alpha} \sum_{b_n \leqq y} \frac{1}{b_n^{2\alpha}}\right)$$

$$= O(y^{2\alpha} \cdot (y^{1-2\alpha} + O(1))) \qquad \text{(from (2.41))}$$

$$= O(y) + O(y^{2\alpha}).$$

Hence

$$Q(y^2) = y^2 \cdot \frac{1}{\zeta(2)} + O(y) + O(y^{2\alpha}).$$

We therefore obtain

THEOREM 8.

$$Q(x) = \frac{x}{\zeta(2)} + O(\sqrt{x}) + O(x^{\alpha}).$$

A number b_n is square-free if $\mu(b_n) = \pm 1$, or $|\mu(b_n)| = 1$. Hence an alternative statement of Theorem 8 is

$$\sum_{b_n \leqq b_m} |\mu(b_n)| = \frac{b_m}{\zeta(2)} + O(\sqrt{b_m}) + O(b_m^{\alpha}). \tag{6.1}$$

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SOME CHARACTERIZATIONS OF ABSOLUTE CONTINUITY

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Let X be a set, and let $\mathbf{X}$ be a completely additive class of subsets of X. Let μ and ν be measures on $\mathbf{X}$. (See [2] for all terminology. The notation of [2] has been modified slightly. For example, boldface capitals are used instead of German capitals.) The following definition of absolute continuity is often given:

DEFINITION 1. *ν is absolutely continuous with respect to μ if and only if whenever $E \in \mathbf{X}$ and $\mu(E)=0$, then $\nu(E)=0$.*

Suppose now that F is a nondecreasing function of a real variable. Then we may use F to construct a Caratheodory outer measure F^* and a family $\mathbf{L}(F)$ of measurable sets. F^* will be a measure on the completely additive class $\mathbf{L}(F)$. (Here and in all that follows we shall assume that all nondecreasing functions F referred to are normalized so that they are continuous from the right and so that $F(0)=0$. This does not affect the family $\mathbf{L}(F)$ nor the measure F^*.) If we have two such functions F_1 and F_2, we may talk about the absolute continuity of F_1^* with respect to F_2^*, in the sense of Definition 1. It is frequently not pointed out that in this context Definition 1 involves certain relations between the classes $\mathbf{L}(F_1)$ and $\mathbf{L}(F_2)$. It is the purpose of this note to point out some of these relations. For example, if F_1 is continuous, then the relation $\mathbf{L}(F_2) \subset \mathbf{L}(F_1)$ is equivalent to the absolute continuity of F_1^* with respect to F_2^* in the sense of Definition 1.

We first observe that in Definition 1, we consider the measures μ and ν on a common completely additive class $\mathbf{X}$. If we are to say that F_1^* is absolutely continuous with respect to F_2^* in the sense of Definition 1, what should we take as $\mathbf{X}$? Since $\mathbf{B} \subset \mathbf{L}(F)$ for any nondecreasing function F (here $\mathbf{B}$ stands for the Borel sets), it seems natural to take $\mathbf{X}$ to be $\mathbf{B}$. Definition 1 would then be expressed as

DEFINITION 2. *F_1^* is absolutely continuous with respect to F_2^* if and only if whenever $E \in \mathbf{B}$ and $F_2^*(E)=0$, then $F_1^*(E)=0$. For brevity, we shall write "F_1^* is A.C. ($\mathbf{B}$, F_2^*)."*

That Definition 2 is the appropriate form of Definition 1 in this situation is further borne out by the following two theorems. Theorem 2 expresses absolute continuity in a form analogous to the way in which it is expressed for a function of a real variable.

THEOREM 1. *Let $\mathbf{X}$ be any completely additive class such that $\mathbf{B} \subset \mathbf{X} \subset \mathbf{L}(F_1) \cap \mathbf{L}(F_2)$. Then, F_1^* is A.C. ($\mathbf{B}$, F_2^*) if and only if F_1^* is A.C. ($\mathbf{X}$, F_2^*).*

THEOREM 2. *F_1^* is A.C. ($\mathbf{B}$, F_2^*) if and only if for each interval I and for each $\epsilon>0$, there exists an $\eta>0$ such that if $\{[a_i, b_i]\}_{i=1}^n$ is a set of non-overlapping intervals in I for which $\sum_{i=1}^n [F_2(b_i)-F_2(a_i)]<\eta$, then $\sum_{i=1}^n [F_1(b_i)-F_1(a_i)]<\epsilon$.*

Having determined the appropriate form of Definition 1 for the situation of two nondecreasing functions F_1 and F_2, we now indicate some implications of Definition 2 with respect to $\mathbf{L}(F_1)$ and $\mathbf{L}(F_2)$. Note that Theorems 3, 4, 5, and 6 appear in [**3**], in which, however, it is assumed that the nondecreasing functions F_1 and F_2 are of bounded variation on the whole real axis, and in which Theorem 5 is taken as the definition of absolute continuity. The proofs of Theorems 1–6 are straightforward applications of the methods and results of [**2**] and [**3**] and are therefore omitted. The basis of many of the omitted proofs is that given any set A there is a Borel set $B\supset A$ such that $F^*(B)=F^*(A)$.

THEOREM 3. *Let $\mathbf{N}(F_i)$ stand for the class of all sets E for which $F_i^*(E)=0$. Then, F_1^* is A.C. $(\mathbf{B}, F_2^*)$ if and only if $\mathbf{N}(F_2)\subset\mathbf{N}(F_1)$, i.e., $F_2^*(E)=0$ implies $F_1^*(E)=0$.*

THEOREM 4. *If F_1^* is A.C. $(\mathbf{B}, F_2^*)$, then $\mathbf{L}(F_2)\subset\mathbf{L}(F_1)$.*

THEOREM 5. *F_1^* is A.C. $(\mathbf{B}, F_2^*)$ if and only if there is a nonnegative function $\alpha(x)$ which is integrable (F_2) and for which $F_1^*(E)=\int_E\alpha(x)dF_2(x)$ for every bounded set E in $\mathbf{L}(F_2)$.*

THEOREM 6. *Suppose that F_1^* is A.C. $(\mathbf{B}, F_2^*)$. Let α be the function mentioned in Theorem 5, and let $A=\{x:\alpha(x)=0\}$. Then $E\in\mathbf{L}(F_1)$ if and only if $E=E_1+E_2$, where $E_1\cap E_2=\phi$, $E_1\in\mathbf{L}(F_2)$, and $E_2\subset A$. In this case, $F_1^*(E_2)=0$, and $F_1^*(E)=F_1^*(E_1)$. If $E\in\mathbf{L}(F_1)$, we may take $E_1=E\cap A^c$ and $E_2=E\cap A$.*

Theorem 6 is useful in proving the following:

THEOREM 7. *Under the conditions of Theorem 6 there exists a unitary transformation U of $L^2(F_1;(-\infty,\infty))$ onto $L^2(F_2;A^c)$ given for each $f\in L^2(F_1;(-\infty,\infty))$ by $(Uf)(x)=\{\alpha(x)\}^{1/2}f(x)$.* (Here, for any set $S\in\mathbf{L}(F_i)$, $L^2(F_i;S)$ stands for all functions f which are measurable $(\mathbf{L}(F_i))$ on S and for which $\int_S|f(x)|^2dF_i(x)<\infty$.)

As we have mentioned, a converse of Theorem 4 holds, namely,

THEOREM 8. *Suppose that the discontinuity points of F_1 are also discontinuity points of F_2 and that $\mathbf{L}(F_2)\subset\mathbf{L}(F_1)$. Then, F_1 is A.C. $(\mathbf{B}, F_2^*)$. Thus, if F_1 is continuous, then F_1 is A.C. $(\mathbf{B}, F_2^*)$ if and only if $\mathbf{L}(F_2)\subset\mathbf{L}(F_1)$.*

Before giving the proof of Theorem 8 we state several lemmas.

LEMMA 1. *Let F_1 and F_2 be nondecreasing functions. Then, $F_1(x)=F_{11}(x)+F_{12}(x)+F_{13}(x)$, where F_{11}, F_{12}, F_{13} are nondecreasing functions, F_{13} is a jump function having jumps equal to the jumps of F_1 at the points of discontinuity of F_1 which are not points of discontinuity of F_2, F_{11}^* is A.C. $(\mathbf{B}, F_2^*)$, F_{12} is continuous, and F_{12}^* is singular $(\mathbf{B}, F_2^*)$. Such a decomposition is unique.* (Recall our convention about normalization of nondecreasing functions.)

The proof of Lemma 1 follows from [**2**], Ch. I, (14.6).

LEMMA 2. *For any set E, $|F[E]| = F^*(E)$, where $F[E]$ is the image of E under F, and $|F[E]|$ is the outer Lebesgue measure of $F[E]$; $E \in \mathbf{L}(F)$ if and only if $F[E] \in \mathbf{L}$, where $\mathbf{L}$ stands for the Lebesgue measurable sets.*

The proof of Lemma 2 follows from [**2**], Ch. III, (13.3).

LEMMA 3. *Suppose that F_1 is a continuous, not identically constant function such that F_1^* is singular ($\mathbf{B}$, F_2^*). Then, there is a set E such that $E \in \mathbf{L}(F_2)$ but $E \notin \mathbf{L}(F_1)$.*

Proof. Since F_1 is not identically constant, there are numbers a, b such that $a < b$ and $F(a) < F(b)$. Since F_1 is singular ($\mathbf{B}$, F_2^*), there is a set $E_0 \subset [a, b]$ such that $E_0 \in \mathbf{B}$, $F_2^*(E_0) = 0$, $F_1^*([a, b] - E_0) = 0$. Thus, $F_1(b) - F_1(a) = F_1^*([a, b]) = F_1^*([a, b] - E_0) + F_1^*(E_0) = F_1^*(E_0)$, so that $F_1^*(E_0) = F_1(b) - F_1(a) > 0$. By Lemma 2, $F_1[E_0] \in \mathbf{L}$, and $|F_1[E_0]| = F_1^*(E_0) > 0$. By [**1**], Ch. VI, Sec. 44, there is a set $Z \subset F_1[E_0]$ such that $Z \notin \mathbf{L}$. But then there exists a set $E \subset E_0$ such that $F_1[E] = Z$, and by Lemma 2, $E \notin \mathbf{L}(F_1)$. Since $E \subset E_0$, $F_2^*(E) = 0$. Thus, $E \in \mathbf{L}(F_2)$, but $E \notin \mathbf{L}(F_1)$. This proves the lemma.

We now prove Theorem 8. Since the discontinuity points of F_1 are also discontinuity points of F_2, by Lemma 1 we may write $F_1(x) = F_{11}(x) + F_{12}(x)$, where F_{11}^* is $A.C.$ ($\mathbf{B}$, F_2^*), F_{12} is continuous, and F_{12}^* is singular ($\mathbf{B}$, F_2^*).

Suppose that F_{12} is not identically zero. Then by Lemma 3 there is a set E such that $E \in \mathbf{L}(F_2)$ but $E \notin \mathbf{L}(F_{12})$. By [**1**], Theorem 50.1, $\mathbf{L}(F_1) = \mathbf{L}(F_{11}) \cap \mathbf{L}(F_{12})$. Hence, $E \in \mathbf{L}(F_2)$, but $E \notin \mathbf{L}(F_1)$. This contradicts our hypothesis that $\mathbf{L}(F_2) \subset \mathbf{L}(F_1)$. It follows, then, that $F_{12}(x) \equiv 0$, and that $F_1(x) \equiv F_{11}(x)$, so that F_1^* is $A.C.$ ($\mathbf{B}$, F_2^*). This proves the theorem.

Remark 1. We cannot dispense with the assumption that the discontinuity points of F_1 are also discontinuity points of F_2 in Theorem 8. Suppose, for example, that F_2^* is Lebesgue measure, and F_1 is any jump function. Then, $\mathbf{L}(F_1)$ consists of all subsets of the real line, so that $\mathbf{L}(F_2) \subset \mathbf{L}(F_1)$, but by Theorem 2 it is not true that F_1^* is $A.C.$ ($\mathbf{B}$, F_2^*).

Remark 2. We can say, however, that if $\mathbf{L}(F_2) \subset \mathbf{L}(F_1)$, then $F_1 = F_{11} + F_{13}$, where F_{11}^* is $A.C.$ ($\mathbf{B}$, F_2^*) and F_{13} is a jump function having jumps equal to those of F_1 at the points of discontinuity of F_1 which are not points of discontinuity of F_2. F_{13}^* is therefore singular ($\mathbf{B}$, F_2^*).

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ON THE DISTRIBUTION OF CERTAIN SEQUENCES OF INTEGERS

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1. Introduction. Let S represent the set of positive integers n such that each prime factor of n occurs with multiplicity at least 2. For real $x \geqq 1$, let $S(x)$ denote the number of n contained in S and not exceeding x.

In 1933 Feller and Tornier [6, Section 9] proved that S has asymptotic density zero, that is,

$$\lim_{x\to\infty} \frac{S(x)}{x} = 0. \tag{1.1}$$

This result was proved independently by Schoenberg [10, Section 12] by an entirely different approach. Wintner [12, Section 62] gave the improved order result,

$$S(x) = O(x^{1/2+\epsilon}) \qquad \text{for all } \epsilon > 0; \tag{1.2}$$

this estimate can be improved to $S(x) = O(\sqrt{x})$. In fact, Wintner [12, (140) Sections 63, 64] showed, on the basis of generating functions, that

$$\lim_{x\to\infty} \frac{S(x)}{\sqrt{x}} = \frac{\zeta(3/2)}{\zeta(3)}, \tag{1.3}$$

where $\zeta(s)$ denotes the Riemann Zeta function. For a recent proof of (1.3), see Hornfeck [8, Satz 3, $k=2$]. Using elementary methods, Erdös and Szekeres [5, Section 2] obtained the following refinement of (1.3):

$$S(x) = \frac{\zeta(3/2)\sqrt{x}}{\zeta(3)} + O(\sqrt[3]{x}). \tag{1.4}$$

A very simple proof of (1.4) was given later by Sklar [11]. By means of the Euler-Maclaurin sum-formula, Bateman [1] obtained the further refinement,

$$S(x) = \frac{\zeta(3/2)\sqrt{x}}{\zeta(3)} + \frac{\zeta(2/3)\sqrt[3]{x}}{\zeta(2)} + O(\sqrt[5]{x}). \tag{1.5}$$

Even more refined estimates of $S(x)$ have been established by means of deep analytical methods (cf. Bateman and Grosswald [2, Section 5]).

It should be observed that Erdös and Szekeres [5], Hornfeck [8], and also Bateman and Grosswald [2], actually considered a generalization of the problem discussed above. A second generalization was considered by Wintner [12, Section 62], and two other generalizations have been treated by this writer [3], [4]. It is the object of the present paper to consider still another generalization of the sequence S.

Let n have distinct prime factors $p_1, \cdots, p_r$ ($r=0$ if $n=1$),

$$n = p_1^{e_1} \cdots p_r^{e_r}. \tag{1.6}$$

For positive integers k, let S_k denote the sequence consisting of those n for which *none* of the canonical exponents e_i in (1.6) is an *odd* number $\leqq 2k-1$. Further, let $S_k(x)$ denote the number of $n \in S_k$ such that $n \leqq x$. Clearly, $S_1 = S$ and $S_1(x) = S(x)$.

The set S_k arises as the special case $a=2$, $b=2k+1$ of the sequence $R_{a,b}$ considered in [3]. In this case the main result [3, Theorem 2.1] of the latter paper becomes

$$(1.7)\quad S_k(x) = \left(\frac{\zeta(k+1/2)}{\zeta(2k+1)}\right)\sqrt{x} + \left(\frac{\zeta\{2/(2k+1)\}}{\zeta(2)}\right)x^{1/(2k+1)} + O(x^{1/(2k+3)}).$$

The principal result of the present paper (Theorem 3.1) has to do with the distribution of the general set of integers $S_{a,b}$, defined in Section 2, which like $R_{a,b}$ reduces to S_k in the case $a=2$, $b=2k+1$ (Remark 2.2). As an application of Theorem 3.1, we mention the result contained in Theorem 3.2, which is concerned with another set of integers $S^*_{a,b}$, also defined in Section 2. The proofs are based on identities proved in Section 2, in connection with an estimate (3.1), due to Franel and Landau, concerning generalized divisor functions.

Remark 1.1. The O-estimates of this paper in general depend upon the parameters a, b (or k).

Remark 1.2. The historical discussion at the beginning is not in a strictly chronological order (cf. the Bibliography).

2. Identities. First we prove

LEMMA 2.1. *If* $(a, b)=1$, *then the set of integers*,

$$(2.1)\quad \begin{array}{ll} ai & (0 \leqq i \leqq b-1) \\ bj & (1 \leqq j \leqq a), \end{array}$$

forms a complete set of residues (mod $(a+b)$).

Remark 2.1. Since $(a, b)=1$, it follows that (2.1) comprises $a+b$ distinct integers.

Proof. Let i_1, i_2 denote integers i in (2.1), $i_1 > i_2$. Suppose that $ai_1 \equiv ai_2$ (mod $(a+b)$). Then there exists an integer $d>0$ such that $0 < a(i_1-i_2-d) = bd$. But this is impossible, because $(a, b)=1$, $b > i_1 - i_2 - d > 0$. Hence $ai_1 \not\equiv ai_2$ (mod $(a+b)$) and the numbers ai in (2.1) are distinct (mod $(a+b)$). A similar statement applies to those integers bj contained in (2.1).

Let i and j denote integers defined as in (2.1), and suppose that $ai \equiv bj$ (mod $(a+b)$), $ai > bj$. Then there must exist a $d>0$ such that $i>d$, $a(i-d) = b(j+d)$. Again this is impossible, because $(a, b)=1$, $0 < i-d < b$. We have thus proved that $ai \not\equiv bj$ (mod $(a+b)$) if $ai > bj$. A similar argument proves that $ai \not\equiv bj$ (mod $(a+b)$) in case $ai < bj$. This completes the proof of the lemma.

Lemma 2.1 can be stated equivalently as follows.

LEMMA 2.1′. *If* $(a, b)=1$, *then the numbers contained in the progressions,*

$$\begin{aligned} &ai + (a+b)t, \quad 0 \leqq i \leqq b-1,\ t \geqq 0 \\ &bj + (a+b)t, \quad 1 \leqq j \leqq a,\ t \geqq 0, \end{aligned} \tag{2.2}$$

are distinct.

We define now the set $S_{a,b}$ to consist of the integers n in (1.6), each of whose exponents, $e_1, \cdots, e_r$, is contained in the set (2.2). To prove the assertion of Section 1 concerning S_k, note that in the case $a=2$, $b=2k+1$, (2.1) becomes the set $0, 2, 4, \cdots, 2\cdot 2k, 2k+1, 2(2k+1)$; the numbers of this set which are not least nonnegative residues (mod $(2k+3)$) are the numbers, $2(k+2)$, $2(k+3), \cdots, 2(2k), 2(2k+1)$. Each of these numbers is $<2(a+b)=4k+6$. It therefore follows that the nonnegative integers not contained in (2.2), in the case $a=2$, $b=2k+1$, are the odd numbers $1, 3, 5, \cdots, 2k-3, 2k-1$. This proves

Remark 2.2.

$$S_{2,2k+1} = S_k.$$

Let the characteristic function of $S_{a,b}$ be denoted $\rho_{a,b}(n)$, that is, $\rho_{a,b}(n)=1$ or 0 according as n is or is not contained in $S_{a,b}$. We also introduce the auxiliary sequence, $S^*_{a,b}$, consisting of the n in (1.6) whose exponents $e_1, \cdots, e_r$ are all contained in the set (2.1). Let the characteristic function of $S^*_{a,b}$ be denoted $\rho^*_{a,b}(n)$. It is understood that both $S_{a,b}$ and $S^*_{a,b}$ contain the number 1.

Next we obtain relations for the generating functions, $F_{a,b}(s)$ and $F^*_{a,b}(s)$, of $\rho_{a,b}(n)$ and $\rho^*_{a,b}(n)$, respectively.

LEMMA 2.2. *If* $(a, b)=1$, *then for* $s>1$,

$$F_{a,b}(s) \equiv \sum_{n=1}^{\infty} \frac{\rho_{a,b}(n)}{n^s} = \frac{\zeta(as)\zeta(bs)}{\zeta(abs)}; \tag{2.3}$$

$$F^*_{a,b}(s) \equiv \sum_{n=1}^{\infty} \frac{\rho^*_{a,b}(n)}{n^s} = \frac{F_{a,b}(s)}{\zeta((a+b)s)}. \tag{2.4}$$

Proof. Evidently, $S_{a,b}$ and $S^*_{a,b}$ are both multiplicative sets, that is, $\rho_{a,b}(n)$ and $\rho^*_{a,b}(n)$ are multiplicative functions of n. By Euler factorization, one obtains then

$$F^*_{a,b}(s) = \prod_p \left(\sum_{m=0}^{\infty} \frac{\rho^*_{a,b}(p^m)}{p^{ms}} \right),$$

and hence by Remark 2.1,

$$F^*_{a,b}(s) = \prod_p \left(1 + \frac{1}{p^{as}} + \frac{1}{p^{2as}} + \cdots + \frac{1}{p^{(b-1)as}} + \frac{1}{p^{bs}} + \frac{1}{p^{2bs}} + \cdots + \frac{1}{p^{abs}}\right) \tag{2.5}$$

where the products are over the primes p. Hence, by Lemma 2.1′ and (2.5), it follows that

$$F_{a,b}(s) = \overset{*}{F}_{a,b}(s) \prod_p \left(1 + \frac{1}{p^{(a+b)s}} + \frac{1}{p^{2(a+b)s}} + \cdots\right),$$

and therefore that

$$F_{a,b}(s) = \overset{*}{F}_{a,b}(s)\zeta((a+b)s), \qquad s > 1. \tag{2.6}$$

It is easily verified by (2.5) that

$$\begin{aligned} \overset{*}{F}_{a,b}(s) &= \prod_p \left(1 - \frac{1}{p^{abs}}\right)\left[\left(1 + \frac{1}{p^{as}} + \frac{1}{p^{2as}} + \cdots\right) + \frac{1}{p^{bs}} + \frac{1}{p^{2bs}} \cdots\right] \\ &= \prod_p \left(1 - \frac{1}{p^{abs}}\right)\left(1 - \frac{1}{p^{as}}\right)^{-1}\left(1 - \frac{1}{p^{bs}}\right)^{-1}\left(1 - \frac{1}{p^{(a+b)s}}\right), \end{aligned}$$

so that $\overset{*}{F}_{a,b}(s) = \zeta(as)\zeta(bs)/\zeta(abs)\zeta((a+b)s)$, $s>1$. The lemma results then by (2.6).

Let us define $\tau_{a,b}(n)$, for arbitrary a, b, to be the number of representations of n in the form, $n=d^a\delta^b$, where d and δ are positive integers. Then by Dirichlet multiplication, Lemma 2.2 can be restated in the form

LEMMA 2.2′. *If* $(a, b)=1$, *then*

$$\rho_{a,b}(n) = \sum_{d^{ab}\delta=n} \mu(d)\tau_{a,b}(\delta), \tag{2.7}$$

$$\overset{*}{\rho}_{a,b}(n) = \sum_{d^{a+b}\delta=n} \mu(d)\rho_{a,b}(\delta). \tag{2.8}$$

3. Asymptotic formulas. We require the following known estimate for the summatory function $T_{a,b}(x)$ of $\tau_{a,b}(n)$.

LEMMA 3.1 [7]. *If* $b>a$, *then*

$$T_{a,b}(x) \equiv \sum_{n\leqq x} \tau_{a,b}(n) = \zeta\left(\frac{b}{a}\right)x^{1/a} + \zeta\left(\frac{a}{b}\right)x^{1/b} + O\left(\frac{1}{x^{a+b}}\right). \tag{3.1}$$

The following simple estimate is also used:

$$\sum_{n\leqq x} \frac{1}{n^s} = \zeta(s) + O(x^{1-s}) \quad \text{if} \quad s > 1.$$

We now prove our main theorem, in particular, the following estimate for the number $S_{a,b}(x)$ of integers $n \leqq x$ contained in $S_{a,b}$.

THEOREM 3.1. *If* $b>a>1$, $(a, b)=1$, *then*

$$S_{a,b}(x) = \left(\frac{\zeta(b/a)}{\zeta(b)}\right)x^{1/a} + \left(\frac{\zeta(a/b)}{\zeta(a)}\right)x^{1/b} + O(x^{1/c}), \qquad c = a+b. \tag{3.2}$$

Remark 3.1. By Remark 2.2, $S_{2,2k+1}(x)=S_k(x)$, and therefore (1.7) results from (3.2).

Proof. By (2.7)

$$S_{a,b}(x) = \sum_{n\leq x} \rho_{a,b}(n) = \sum_{d^{ab}\delta\leq x} \mu(d)\tau_{a,b}(\delta) = \sum_{n\leq x^{1/ab}} \mu(n)T\left(\frac{x}{n^{ab}}\right).$$

Hence by Lemma 3.1, with $h=ab$,

$$\begin{aligned} S_{a,b}(x) &= \zeta\left(\frac{b}{a}\right)x^{1/a}\sum_{n\leq x^{1/h}}\frac{\mu(n)}{n^b} + \zeta\left(\frac{a}{b}\right)x^{1/b}\sum_{n\leq x^{1/h}}\frac{\mu(n)}{n^a} + O\left(x^{1/c}\sum_{n\leq x^{1/h}}\frac{1}{n^{h/c}}\right) \\ &= \zeta\left(\frac{b}{a}\right)x^{1/a}\left(\frac{1}{\zeta(b)} + O(x^{1/h-1/a})\right) + \zeta\left(\frac{a}{b}\right)x^{1/b}\left(\frac{1}{\zeta(a)} + O(x^{1/h-1/b})\right) \\ &\quad + O(x^{1/c}), \end{aligned}$$

from which the theorem follows.

Analogous to $S_{a,b}(x)$, we define $S^*_{a,b}(x)$ to be the number of integers $n\in S^*_{a,b}$, $n\leqq x$. The following estimate for $S^*_{a,b}(x)$ results as a corollary of Theorem 3.1. This estimate was obtained by W. D. Powers [**9**, Theorem 4.1] on the basis of asymptotic results involving "unitary" divisor functions. The proof given here is simpler in its computational details than that of Powers.

THEOREM 3.2. *If $b>a>1$, $(a, b)=1$, then for $x\geqq 2$,*

$$(3.3)\qquad S^*_{a,b}(x) = \left(\frac{\zeta(b/a)}{\zeta(b)\zeta(c/a)}\right)x^{1/a} + \left(\frac{\zeta(a/b)}{\zeta(a)\zeta(c/b)}\right)x^{1/b} + O(x^{1/c}\log x),$$

where $c=a+b$.

Proof. By (2.8)

$$(3.4)\qquad S^*_{a,b}(x) = \sum_{n\leq x} \rho^*_{a,b}(n) = \sum_{d^{a+b}\delta\leq x} \mu(d)\rho_{a,b}(\delta) = \sum_{n\leq x^{1/c}} \mu(n)S_{a,b}\left(\frac{x}{n^c}\right).$$

Applying Theorem 3.1,

$$\begin{aligned} S^*_{a,b}(x) &= \frac{\zeta(b/a)x^{1/a}}{\zeta(b)}\sum_{n\leq x^{1/c}}\frac{\mu(n)}{n^{c/a}} + \frac{\zeta(a/b)x^{1/b}}{\zeta(a)}\sum_{n\leq x^{1/c}}\frac{\mu(n)}{n^{c/b}} + O\left(x^{1/c}\sum_{n\leq x^{1/c}}\frac{1}{n}\right) \\ &= \frac{\zeta(b/a)x^{1/a}}{\zeta(b)}\left(\frac{1}{\zeta(c/a)} + O(x^{1/c-1/a})\right) + \frac{\zeta(a/b)x^{1/b}}{\zeta(a)}\left(\frac{1}{\zeta(c/b)} + O(x^{1/c-1/b})\right) \\ &\quad + O(x^{1/c}\log x), \end{aligned}$$

and (3.3) is proved.

Remark 3.2 It will be observed that $S_{1,b}$, for every b, is the set of all positive integers, while $S^*_{1,b}$ is the set of $(b+1)$-free integers. In the latter case, we note

by (2.4) that $F^*_{1,b}(s)=\zeta(s)/\zeta((b+1)s)$, so that (2.8) may be replaced by the simpler identity,

$$\overset{*}{\rho}_{b+1}(n) \equiv \overset{*}{\rho}_{1,b}(n) = \sum_{d^{b+1}\delta=n} \mu(d).$$

The latter relation leads to the classical result $(k=b+1)$,

$$Q_k(x) \equiv \sum_{n\leq x} \overset{*}{\rho}_k(n) \frac{x}{\zeta(k)} + O(x^{1/k}),$$

for the number $Q_k(x)$ of k-free integers $\leqq x (k\geqq 2)$.

Remark 3.3. For refinements of (3.1) obtainable by analytic methods, the reader is referred to the bibliography of [3]. These refinements imply improved estimates in both Theorems 3.1 and 3.2 (cf. [3, section 3]).

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A REAL NON-DESARGUESIAN PLANE

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It is known that the axioms of alignment of a general projective geometry of a plane do not imply Desargues' Theorem. Two classical examples are known wherein the real Euclidean projective plane is deformed into a non-Desarguesian plane. This paper gives another such example.

The first example, which is due to Hilbert, has for its points all those of the Euclidean projective plane with those on the line at infinity adjoined to them. To define lines in this geometry, we consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(referred to a pair of rectangular axes) and a point $A(\alpha, 0)$, such that no circle through A cuts the ellipse in more than two real points, the condition for which is $\alpha^2 > 2a^2 - b^2$. The lines of this geometry are (i) all lines of the Euclidean projective plane which touch the ellipse or do not meet it in real points (including the line at infinity), (ii) all lines which meet the ellipse in two points P, Q with the segment of the line PQ within the ellipse replaced by the arc of the circle APQ and (iii) all Euclidean lines through A.

The second example, which is due to Moulton, has again all the points of the Euclidean plane, with its lines defined as follows: (i) all the lines of the Euclidean plane parallel to the coordinate axes, which we assume to be rectangular, (ii) lines whose slopes are negative and (iii) lines with positive slopes with their parts above the x-axis replaced by lines with half that slope.

Both these non-Desarguesian geometries are affine, if we define two lines to be parallel provided they do not intersect in a finite point, which will be the case if and only if their Euclidean continuations are parallel. The essential difference between these planes, however, is in the way they are closed at infinity. In Hilbert's plane the line at infinity is just the same as in the underlying Euclidean plane and does not undergo any deformation. But in the latter case, a part of the line at infinity is such that a point of it corresponds to two points of the underlying Euclidean plane (i.e. as if it were folded).

The non-Desarguesian plane which we shall discuss presently, has again all the points of the Euclidean plane together with those on the line at infinity. The lines of this plane (to be called Lines hereafter to distinguish them from Euclidean lines which will be called lines) are all the lines except those cutting off positive intercepts on the co-ordinate axes, which we assume to be rectangular. The portion of such a line intercepted between the axes is replaced by the finite arc of the parabola touching the axes at its points of intersection with the axes. The other lines, namely those parallel to the co-ordinate axes and those not cutting off positive intercepts on both the axes will be Lines. If we define two Lines to be parallel provided they do not meet in a finite point (i.e. their Euclidean portions are parallel), we can show that this is an affine geometry of

Hilbert's type, i.e. a geometry in which the line at infinity is the same as the one in the underlying Euclidean plane.

We can show that the points and Lines defined as above satisfy the alignment postulates, by proving the following theorems. For convenience, we designate as "the arc (h, k)" that portion of the Line replacing the line

$$\frac{x}{h}+\frac{y}{k}=1 \qquad (h, k>0)$$

in the first quadrant, i.e. the arc of the parabola touching the axes at $(h, 0)$ and $(0, k)$.

THEOREM 1. *If the lines*

$$\frac{x}{h_1}+\frac{y}{k_1}=1 \quad \textit{and} \quad \frac{x}{h_2}+\frac{y}{k_2}=1 \qquad (h_1, h_2, k_1, k_2>0)$$

intersect at a point outside the first quadrant or are parallel, the corresponding arcs (h_1, k_1) and (h_2, k_2) do not intersect, but if the lines intersect at a point in the first quadrant then the arcs also intersect in just one point of the first quadrant.

In the first place let $h_2>h_1$ and $k_2>k_1$, so that the lines do not intersect in the first quadrant.

By considering the four points of intersection of the parabolas

$$\begin{aligned} \pm\sqrt{(k_1x)} \pm \sqrt{(h_1y)} &= \sqrt{(h_1k_1)} \\ \pm\sqrt{(k_2x)} \pm \sqrt{(h_2y)} &= \sqrt{(h_2k_2)}, \end{aligned}$$

we can show that if the lines intersect in the second or fourth quadrants (i.e. $h_2k_1>h_1k_2$ or $h_2k_1<h_1k_2$ respectively) then all the points are outside the arc (h_1, k_1).

If the lines are parallel then two of the points are infinite and the other two are outside the arc (h_1, k_1).

If the lines intersect in the first quadrant, let $k_2>k_1$ and $h_1>h_2$ for definiteness. Then only the point

$$\left[\frac{h_1h_2(k_1+k_2-2\sqrt{(k_1k_2)})}{k_1h_2+k_2h_1-2\sqrt{(k_1k_2h_1h_2)}}, \frac{k_1k_2(h_1+h_2-2\sqrt{(h_1h_2)})}{k_1h_2+k_2h_1-2\sqrt{(k_1k_2h_1h_2)}}\right]$$

lies on both the arcs (h_1, k_1) and (h_2, k_2) and the other points are outside both the arcs.

THEOREM 2. *If the line $y=mx+c$ $(m>0)$ or a line parallel to the x or y axis intersects the line*

$$\frac{x}{h}+\frac{y}{k}=1 \qquad (h, k>0)$$

in the first quadrant, it intersects the arc (h, k) in just one point.

Let H be the point $(h, 0)$ and K be $(0, k)$.

If P is the point of the intersection of the two lines and P^1 is the harmonic conjugate of P with respect to HK, then OP^1 is the polar of P with respect to the parabola

$$\pm\sqrt{\frac{x}{h}} \pm \sqrt{\frac{y}{k}} = 1.$$

(P^1 may incidentally be also a point at infinity).

If OP^1 meets the line $y=mx+c$ or a parallel to the x or y axis in Q, we see that the points of intersection of PQ with the parabola are on either side of the line HK, which establishes our result.

THEOREM 3. *There is just one Line joining any two points of the plane.*

We need prove the theorem only for three cases: (i) when both the points are in the first quadrant and the line joining them makes an obtuse angle with the x-axis, i.e. if (x_1, y_1) and (x_2, y_2) are two points with $\{x_1, x_2, y_1, y_2\}>0$, we can take $x_1>x_2$, $y_2>y_1$, for definiteness; (ii) a point in the first quadrant and the other in the second such that the line joining them makes an obtuse angle with the x-axis, i.e. points like (x_1, y_1) and $(-x_2, y_2)$ with $\{x_1, x_2, y_1, y_2\}>0$ and $y_2>y_1$; (iii) a point in the first quadrant and the other in the fourth such that the line joining them makes an obtuse angle with the x-axis, i.e. points like (x_1, y_1) and $(x_2, -y_2)$ such that $\{x_1, x_2, y_1, y_2\}>0$ and $x_2>x_1$.

In case (i), we can show that out of the four parabolas through (x_1, y_1) and (x_2, y_2) of the type

$$\pm\sqrt{\frac{x}{h}} \pm \sqrt{\frac{y}{k}} = 1,$$

there is only one arc (h, k) containing both these points and this is,

$$h = \frac{x_1y_2 + x_2y_1 - 2\sqrt{(x_1x_2y_1y_2)}}{y_1 + y_2 - 2\sqrt{(y_1y_2)}}, \quad k = \frac{x_1y_2 + x_2y_1 - 2\sqrt{(x_1x_2y_1y_2)}}{x_1 + x_2 - 2\sqrt{(x_1x_2)}}.$$

In case (ii), consider the Line joining $(-x_2, y_2)$ and (x_1, y_1), where $\{x_1, x_2, y_1, y_2\}>0$ and $y_2>y_1$. If $(-x_2, y_2)$ lies on

$$\frac{x}{h} + \frac{y}{k} = 1 \qquad (h, k > 0)$$

and (x_1, y_1) lies on

$$\pm\sqrt{\frac{x}{h}} \pm \sqrt{\frac{y}{k}} = 1,$$

then by eliminating k we get

$$h^2(y_2 - y_1)^2 + 2h\{y_1x_2(y_2 - y_1) - x_1y_2(y_1 + y_2)\} + (x_1y_2 - x_2y_1)^2 = 0.$$

If h_1, h_2 are the two roots of the equation both of them are seen to be positive. Let $h_2 > h_1$. The arc (h_2, k_2), where

$$k_2 = \frac{h_2 y_1}{h_2 + x_2}$$

is the only admissible arc containing the point (x_1, y_1). There is thus one Line through the two points.

The proof of case (iii) is similar.

We can produce two triangles having a centre of perspective but no axis of perspective. For let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles whose vertices are all in the second quadrant. Let L be the point of intersection of lines B_1C_1 and B_2C_2, M that of C_1A_1 and C_2A_2, and N that of A_1B_1 and A_2B_2. Let the line LMN cut off positive intercepts on the co-ordinate axes with M, N in the second quadrant and L alone in the first with both the lines B_1C_1 and B_2C_2 making acute angles with the x-axis. We can see that the Line MN will not pass through L.

This geometry is an affine geometry by virtue of the following theorem.

THEOREM 4. *Through any point in the plane there exists just one Line parallel to a given Line, two Lines being defined to be parallel if they do not intersect in any finite point of the plane (or their Euclidean portions are parallel).*

It is enough to prove the theorem only in the case when the given point (x_1, y_1) is in the first quadrant and the Line cuts off positive intercepts on the axes. Let

$$\pm \sqrt{\frac{x}{h_1}} \pm \sqrt{\frac{y}{k_1}} = 1 \qquad (x \leqq h_1,\ y \leqq k_1)$$

be the equation to the arc (h_1, k_1) of the given Line. Let

$$\pm \sqrt{\frac{x}{h_2}} \pm \sqrt{\frac{y}{k_2}} = 1 \qquad (x \leqq h_2,\ y \leqq k_2)$$

be the equation to the arc through the point (x_1, y_1). $h_2/h_1 = k_2/k_1$ is the condition for the Euclidean portions to be parallel. Substituting this we get

$$h_2 = x_1 + \frac{y_1 h_1}{k_1} \pm 2\sqrt{\left(\frac{x_1 y_1 h_1}{k_1}\right)}.$$

If the negative sign be taken, $h_2 > x_1$ and $k_2 > y_1$ give contradictory conditions. Hence the positive sign alone needs to be taken, which proves the theorem.

MATHEMATICAL NOTES

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THE CLOSEST PACKING OF EQUAL SPHERES IN A LARGER SPHERE

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Let U be the unit sphere with center O in an n-dimensional Euclidean space, and let $M(r)$ denote the maximum number of open spheres of radius r with centers in or on U that do not overlap. Obviously, $M(r)=1$ when $r>1$, and $M(r)=2$ when $\frac{1}{2}\sqrt{3}<r\leqq 1$. $M(r)$ is clearly the greatest number of spheres of radius r that can be packed into a sphere of radius $1+r$.

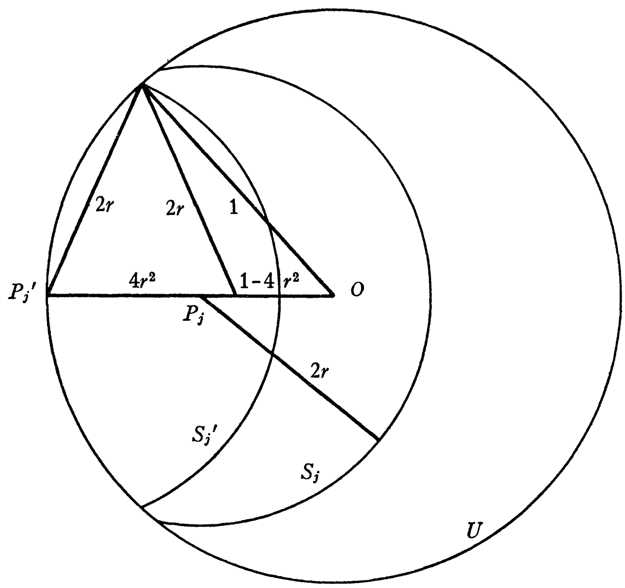

FIG. 1

Theorem 1 gives the value of $M(r)$ for $r\geqq\sqrt{\frac{1}{2}}$, and Theorems 3 and 4 give asymptotic upper and lower bounds for $r<\sqrt{\frac{1}{2}}$ based on known bounds for the greatest number of equal spherical caps that will fit on a sphere without overlapping. We first prove a lemma asserting that none of the centers of the spheres need to lie in a certain spherical shell.

Let $m(r)$ denote a configuration of $M(r)$ nonoverlapping spheres of radius r with centers $P_1, P_2, \cdots, P_M$ in or on U. Then we have

LEMMA. For $r>\frac{1}{2}$, *there exists an* $m(r)$ *with every* P_i *on* U; *for* $r<\frac{1}{2}$, *there exists an* $m(r)$ *with every* P_i *either on* U *or less than* $1-4r^2$ *from* O. *When* $r=\frac{1}{2}$, *one* P_i *is* O *and the rest are on* U.

Proof. Suppose we have an $m(r)$ containing a center P_j inside U and at least $1-4r^2$ from O. Since the sphere about P_j does not overlap the sphere about any P_i $(i\neq j)$, every P_i $(i\neq j)$ must lie outside an open sphere S_j of radius $2r$ about P_j. Let P_j' be the end of a unit radius from O through P_j, and let S_j' be the open sphere of radius $2r$ about P_j'. (See Fig. 1.) Then $S_j'\cap U\in S_j\cap U$. Hence, replacing P_j by P_j' cannot result in any overlap, and it gives a new $m(r)$ with P_j' on U. The foregoing proof fails when $r=\frac{1}{2}$; if P_j is at O, it cannot be moved, and all P_i $(i\neq j)$ must therefore be on U.

Since a sphere of radius r centered on U subtends at O a cone of half angle arcsin r, which cuts U in a cap of this angular radius, provided $r\leqq 1$, it follows that, for $\frac{1}{2}<r\leqq 1$, $M(r)=N(\arcsin r)$, where $N(\phi)$ is the greatest number of open spherical caps of angular radius ϕ that can be placed without overlapping on a sphere in an n-dimensional Euclidean space. Using Rankin's expression [1] for $N(\phi)$, and defining $N(\arcsin r)$ to be 1 for $r>1$, we thus have

THEOREM 1.

$$(1) \qquad M(r) = \begin{cases} N(\arcsin r) & (r > \frac{1}{2}) \\ \left[\dfrac{2r^2}{2r^2-1}\right] & \left(r \geqq \sqrt{\dfrac{n+1}{2n}}\right), \\ n+1 & \left(\sqrt{\tfrac{1}{2}} < r \leqq \sqrt{\dfrac{n+1}{2n}}\right), \\ 2n & (r = \sqrt{\tfrac{1}{2}}). \end{cases}$$

From Rankin [1], we see that, when $r>\sqrt{\frac{1}{2}}$, the P_i can be at the vertices of a regular $(M-1)$-dimensional simplex inscribed in and concentric with U. When $r=\sqrt{\frac{1}{2}}$, the P_i must be the centers of the faces of an n-dimensional cube circumscribing U.

$M(r)$ and $N(\phi)$ are nonincreasing stepfunctions, continuous on the left. Let $M^*(r)$ and $N^*(\phi)$ denote the functions that have the same values, respectively, at points of continuity and are continuous on the right. Thus, $M^*(r)$ is the maximum number of nonoverlapping spheres of radius r with centers interior to U. If we interpret $M^*(1/0)$ to mean 1, we have

THEOREM 2. $N(\arcsin r)\leqq M(r)\leqq N(\arcsin r)+M^*(r/(1-4r^2))$ *for* $r\leqq\frac{1}{2}$.

Proof. The lower bound is the maximum number of nonoverlapping spheres of radius r whose centers lie *on* U. For $r=\frac{1}{2}$, one additional sphere can be introduced, concentric with U. For $r<\frac{1}{2}$, there is no need of having any P_i inside U unless it is less than $1-4r^2$ from O, and there can be no more than $M^*(r/(1-4r^2))$ of the P_i less than $1-4r^2$ from O. The argument $r/(1-4r^2)$ represents a scaling up of the radius r to account for the radius of the sphere in which the P_i are packed, and the * takes account of the fact that this sphere is open.

COROLLARY 1. *$M(r)=N(\arcsin r)+1$ for $(\sqrt{17}-1)/8 \leqq r \leqq \frac{1}{2}$.*

Proof. In this range of r, we have $M^*(r/(1-4r^2))=1$ by Theorem 1, and so the upper bound of Theorem 2 is $N(\arcsin r)+1$. This bound is attained by centering one sphere at O and the rest on U.

By repeated use of Theorem 2, which remains true when M is replaced by M^* and N by N^*, we have

COROLLARY 2. *$M(r) \leqq N(\arcsin r)+N^*(\arcsin r')+\cdots+N^*(\arcsin r^{(k)})$, where $r'=r/(1-4r^2)$, $r''=r'/(1-4r'^2)$, $\cdots$, and $r^{(k)}$ is the first term of this sequence exceeding $\frac{1}{2}$.*

Proof. Although Theorem 2 is applicable only for $r^{(j)} \leqq \frac{1}{2}$, Theorem 1 covers the case of $r^{(j)} > \frac{1}{2}$ and shows that the series terminates therewith.

Introducing the factor $2/e$, as suggested by Coxeter [2], into Rankin's approximate upper bound [1] on $N(\phi)$, we have

$$N(\phi) \leqq \frac{(2\pi n^3 \cos 2\phi)^{1/2} + O(\sqrt{n})}{e(\sqrt{2}\cdot \sin \phi)^{n-1}} \qquad \text{for } \phi < \tfrac{1}{4}\pi.$$

Hence,

$$N(\arcsin r) \leqq \frac{\sqrt{2\pi n^3(1-2r^2)} + O(\sqrt{n})}{e(\sqrt{2}\cdot r)^{n-1}} \qquad \text{for } r < \sqrt{\tfrac{1}{2}}. \tag{2}$$

Since $r^{(j)} \geqq r^{(j-1)}/(1-4r^2)$ and $1-2r^{(j)^2} \leqq 1-2r^2$, we see that for $r<\frac{1}{2}$ the upper bound on $M(r)$ of Corollary 2 is dominated by $2n$ plus a geometric series whose ratio is $(1-4r^2)^{n-1}$ and whose first term is the right-hand side of (2), i.e., by $2n$ plus $1/[1-(1-4r^2)^{n-1}]$ times the right-hand side of (2). The $2n$ is required in order to cover the case where $r^{(k)} \geqq \sqrt{\frac{1}{2}}$, as (2) does not then apply to the term $N^*(\arcsin r^{(k)})$; Theorem 1 applies instead. However, both the $2n$ and the factor $1/[1-(1-4r^2)]$ can be absorbed into the $O(\sqrt{n})$, and we have

THEOREM 3. $M(r) \leqq \dfrac{\sqrt{2\pi n^3(1-2r^2)} + O(\sqrt{n})}{e(\sqrt{2}\cdot r)^{n-1}}$ for $r < \sqrt{\frac{1}{2}}$.

Proof. We have just proven this inequality for $r<\frac{1}{2}$. Theorem 1, along with (2), shows it to be true for $\frac{1}{2}<r<\sqrt{\frac{1}{2}}$. For $r=\frac{1}{2}$, we use Corollary 1 of Theorem 2 and absorb the 1 into the $O(\sqrt{n})$. For large n and any r, the interior of U evidently affords only an increase by a factor very close to 1 in the number of spheres that can be packed beyond the number $N(\arcsin r)$ that can be packed with their centers on the surface.

To obtain an explicit lower bound for $M(r)$, we suppose that the P_i are confined to the surface of U, and we note that the entire surface of U must be contained in the union of the open spheres S_i ($i=1, 2, \cdots, M$) of radius $2r$ centered on the P_i, for, if there were a point of the surface not contained therein, it could serve as the center of an additional nonoverlapping sphere of radius r.

For $r \leqq 1$, each S_i cuts out a spherical cap on the surface of U of angular radius 2 arcsin r. Since the sum of the surface contents of all these caps must be at least the surface content of U, $M(r)$ is at least the ratio of the surface content of U to the surface content of a cap of angular radius 2 arcsin r, which gives us

THEOREM 4. *$M(r) \geqq \sqrt{2\pi n}(1-2r^2)/(2r\sqrt{1-r^2})^{n-1}$ for $r < \sqrt{\tfrac{1}{2}}$.*

For small r, the upper and lower bounds of Theorems 3 and 4 agree with the known bounds for the densest packing of equal spheres in Euclidean space except for a factor $O(\sqrt{n})$. Unfortunately, their ratio is of the order of $n/(2-2r^2)^{(n-1)/2}$, and so $M(r)$ is not at all closely bounded for large n when $r < \sqrt{\tfrac{1}{2}}$.

References

1. R. A. Rankin, The closest packing of spherical caps in n dimensions, Proc. Glasgow Math. Assoc., 2 (1955) 139–144.
2. H. S. M. Coxeter, An upper bound for the number of equal nonoverlapping spheres that can touch another of the same size, Proceedings of a Symposium on Convexity, Amer. Math. Soc., 1962.

ON AREA-BISECTORS OF PLANE CONVEX SETS

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1. Introduction. In [1] Zarankiewicz considers the straight lines which divide a plane convex set into two equal areas. He proves the following theorems:

THEOREM 1. *In each bounded convex set there is a point through which at least three different area-bisectors pass.*

THEOREM 2. *The necessary and sufficient condition that a point be a center of symmetry of a bounded convex set is that the point be the only point through which at least three different area-bisectors pass.*

Zarankiewicz states that it is still an open question whether there exists a plane convex figure on which there is a point through which pass exactly n area-bisectors when n is an even integer greater than 3. This note will give examples to show that this question is answered in the affirmative for all integers.

2. Area-bisectors of regions of regular $(2k+1)$-fold symmetry. For a regular polygon of $2k+1$ sides, or any closed convex curve which has $2k+1$ axes of symmetry, these axes are area-bisectors and they all pass through the center. Other lines through the center are not area-bisectors except when the curve is a circle. Hence, the center is a point of exactly $2k+1$ area-bisectors.

There is only one area-bisector for each orientation. The center of the polygon always lies between a side of the polygon and the area-bisector parallel to

that side. Hence, the envelope of the area-bisectors is a star-shaped curve shown in the middle of the polygon of Figure 1. Like the polygon, it has $2k+1$ axes of symmetry. It has the property that there is exactly one tangent for each orientation. Since two triangles of equal area are formed by two successive area-bisectors and the included segments of the polygon boundary, these area-bisectors must bisect each other. Hence, each area-bisector is tangent to the curve at its midpoint. The envelope may be considered as the locus of the intersection of successive tangents.

The star-shaped curve of Figure 1 is shown as the enlarged curve $APTDSRBQPETSCRQA$ of Figure 2. From every point within the region $PQRST$, exactly $2k+1$ lines can be drawn tangent to the star-shaped curve. Hence, each point of this region lies on exactly $2k+1$ area-bisectors.

Through a point on the boundary of this region, but not a vertex of this region (Figure 3), exactly $2k$ lines can be drawn tangent to the star-shaped curve since the two tangents to this branch of the curve have coalesced into one.

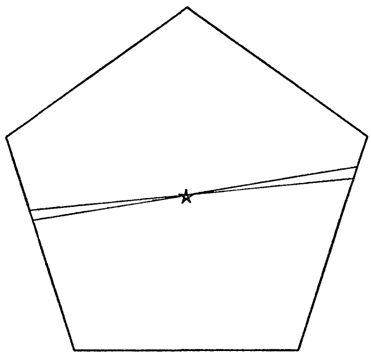

FIG. 1

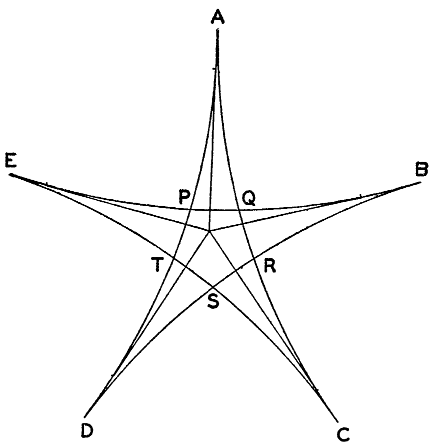

FIG. 2

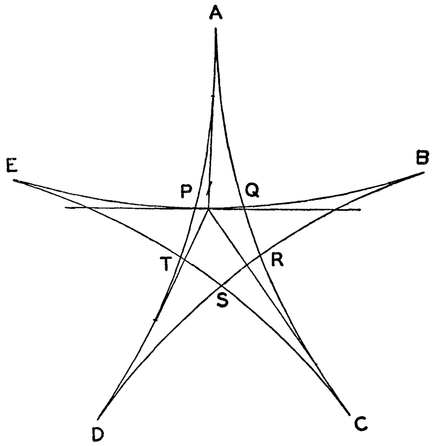

FIG. 3

Hence, there are a linear infinity of points through which pass exactly $2k$ area-bisectors. Since k can be any integer, the question raised in the first section is answered.

Reference

1. K. Zarankiewicz, Division of plane convex sets by lines, (in Polish) Wiadom. Mat., 2 (1959) 228–234, reviewed in Mathematical Reviews 22 (1961) 1200.

A COMBINATORIAL WORD PROBLEM

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Introduction. In a recent article [1] M. T. L. Bizley considers the generalization of a problem which first appeared as E1485 in this MONTHLY and was solved by C. N. Bhaskaranandha [2], the man-woman-dog problem, and solves it using techniques considerably different from those of Bhaskaranandha. Before that time the author had considered and solved a complementary problem, using entirely different techniques, and now would seem a good time to present it.

The man-woman-dog problem and its generalization are concerned only with nonrepetitive cases, i.e., the symbols used may occur once and only once in the sequence and the length of the sequence is in fact determined by the number of distinct symbols available. In the present problem, however, unlimited repetitions of symbols are allowed, and the results are quite different.

The problem is this: Given m distinct symbols partitioned into disjoint sets $A_1, \cdots, A_r, B$, how many distinct sequences of n terms (n-sequences) may be formed using these symbols, subject only to the condition that no two consecutive elements of a given sequence may both be members of the same set A_j for any $j=1, \cdots, r$, and allowing symbols to be repeated indefinitely. Another way

of looking at this problem, and the approach which suggests the title of the article, is to consider these sequences to be "words." The problem would then be to determine how many words of a given length we may form using our given "letters" subject only to the restriction that certain combinations of letters are not allowed.

As an example we might let $n=6$ and have $A_1=\{1, 2, 3\}$, $A_2=\{4\}$, $B=\{5, 6\}$. Then the following would all be permissible 3-sequences and would be distinct:

1–4–6; 3–5–5; 3–6–5; 4–2–4; etc.

but the following would not be permissible:

1–3–5; 4–4–1; 5–3–1; etc.

As we shall see in the next section it is a simple matter to show that there are exactly 124 permissible 3-sequences corresponding to this arrangement.

A calculational procedure. Unfortunately, what the author has been able to obtain is not an explicit formula for $f(n)$, the number of permissible n-sequences, but instead a recursion relation

$$f(n) = a_1 f(n-1) + \cdots + a_d f(n-d),$$

where the a_i and d depend only upon the choice of A_j, B. Such a relation may be solved by finding the roots of a corresponding polynomial of degree d, if such is in fact possible, and using d particular values of $f(n)$ to evaluate some constants. To obtain d particular values of $f(n)$, as well as to illustrate the ideas involved in the derivation of the recursion in a subsequent section, we offer this simple example of a calculation for the case $A_1=\{1, 2, 3\}$, $A_2=\{4\}$, $B=\{5, 6\}$.

element $n=$	1	2	3	4
1 A_1	1	3	17	73
2 A_1	1	3	17	73
3 A_1	1	3	17	73
$T_1(n)$	3	9	51	219
4 A_2	1	5	21	103
$T_2(n)$	1	5	21	103
5 B	1	6	26	124
6 B	1	6	26	124
$T(n)$	2	12	52	248
$f(n)$	6	26	124	570

The explanation of this chart is simple. The number in the nth column by each element $1, \cdots, 6$ is just the number of permissible n-sequences ending in that element. $T_j(n)$ is just the number of permissible n-sequences ending in a member of A_j, with $T(n)$ corresponding to B, and is given as the sum of the entries by its members in the nth column. The entry by an element i in the nth column is just $f(n-1)-T_j(n-1)$ where $i \in A_j$, or if $i \in B$ then the entry is just $f(n-1)$. Also, $f(n)=T_1(n)+T_2(n)+T(n)$.

[Later we shall also obtain a different recursion relation which, although not as generally convenient as the first one, may be used advantageously to calculate the first few values of $f(n)$.]

Notation and remarks. We note that the usage of the integers $1, \cdots, m$ is unimportant to the problem as any m distinct symbols will suffice equally well. In fact, we may represent any arrangement $A_1, \cdots, A_r, B$ by merely listing the number of elements in each set or, more conveniently, by the symbolism

$$(a_1, b_1)(a_2, b_2) \cdots (a_k, b_k)(c),$$

where the a_i are all distinct, the pair (a_j, b_j) meaning that there are b_j A-sets each containing a_j symbols, the number of elements in B being c. For convenience we wish the b_i to be nonzero, although c may equal zero.

Given an arrangement $(a_1, b_1) \cdots (a_k, b_k)(c)$ let $f(n)$ be the number of permissible n-sequences. What we shall obtain, as mentioned before, is not an explicit formula for $f(n)$ but instead a recursion formula for $f(n)$ of the form

$$f(n) = \gamma_1 f(n-1) + \cdots + \gamma_{k+1} f(n-k-1),$$

where the γ_i are all independent of n. We shall not be concerned here with the question of the solution of this recursion relation but just with the derivation of the γ_i.

Let n be fixed, $n \geqq k+1$. For convenience the following notation has been adopted:

$$g(r) = f(n-r) \qquad \text{for } r = 0, 1, \cdots, k+1.$$

$$u_r = \sum_{i=1}^{k} (a_i)^r b_i.$$

$N_p(s)$ is the number of permissible $(n-s)$ sequences ending in an element belonging to an A-set with exactly a_p elements, $p=1, \cdots, k$.

$N_{k+1}(s)$ is the number of permissible $(n-s)$ sequences ending in elements belonging to the B set.

Derivation of the recursion relation. From the definitions adopted in the last section Lemmas 1–3 follow immediately.

LEMMA 1.

$$g(r) = \sum_{j=1}^{k+1} N_j(r) \qquad \textit{for } r = 0, 1, \cdots, k+1.$$

LEMMA 2.

$N_p(s) = a_p b_p g(s+1) - a_p N_p(s+1)$ *for* $s = 0, \cdots, k$ *and* $p = 1, \cdots, k$.

LEMMA 3.

$N_{k+1}(s) = cg(s+1)$.

From Lemma 2, with the help of a simple induction step, we may prove

$$(1) \qquad N_r(k-s) = a_r b_r \left(\sum_{j=1}^{s} (-a_r)^{j-1} g(k-s+j) \right) + (-a_r)^s N_r(k)$$

for $s = 0_1 \cdots, k$ and $r = 1, \cdots, k$.

(To digress a moment, we observe that from these preliminary results we may obtain the following relation without any difficulty:

$$f(n) = (c+u_1)f(n-1) - u_2 f(n-2) + u_3 f(n-3) - \cdots + (-1)^{n-1} u_n f(0),$$

where $f(0) = 1$. Although this relation may be convenient if only a few values of $f(n)$ are required, it is not as convenient for other purposes, however, as the one involving only the values of the preceding $k+1$ terms.)

Now, by Lemmas 1 and 3 we get

$$(2) \qquad f(n) = g(0) = \sum_{j=1}^{k} N_j(0) + cg(1),$$

and from (1), with $s = k$, we get

$$(3) \qquad N_r(0) = a_r b_r \sum_{j=1}^{k} (-a_r)^{j-1} g(j) + (-a_r)^k N_r(k) \qquad (r = 1, \cdots, k).$$

Hence

$$(4) \qquad f(n) = cg(1) + \sum_{j=1}^{k} \left[\left(a_j b_j \sum_{t=1}^{k} (-a_j)^{t-1} g(t) \right) + (-a_j)^k N_j(k) \right].$$

From Lemmas 1 and 3 we also get

$$(5) \qquad \sum_{r=1}^{k} N_r(s) = g(s) - N_{k+1}(s) = g(s) - cg(s+1).$$

Now, changing the variable s in (1) to $k-s$, rearranging, summing over r, and using (5) we obtain

$$(6) \qquad \begin{aligned} &(-a_1)^{k-s} N_1(k) + \cdots + (-a_k)^{k-s} N_k(k) \\ &\qquad = g(s) - cg(s+1) - \sum_{r=1}^{k} a_r b_r \left(\sum_{j=1}^{k-s} (-a_r)^{j-1} g(j+s) \right). \end{aligned}$$

We are interested in expressing $\sum_{j=1}^{k} (-a_j)^k N_j(k)$ in terms of $g(1), \cdots,$

$g(k+1)$. We may obtain such an expression from (6) if we use the following elementary fact:

$$a_i^k = \sum_{j=1}^{k} \delta_j(-1)^{j+1} a_i^{k-j} \tag{*}$$

where $\delta_j = \sum \Pi_j a_i$ (the elementary symmetric function of order j on k letters), the sum being taken over all possible products of distinct elements taken j at a time.

Now let $\beta_s = g(s) - cg(s+1) - \sum_{r=1}^{k} a_r b_r \sum_{j=1}^{k-s} (-a_r)^{j-1} g(j+s)$. Then, using (6) and (*) we obtain

$$\sum_{i=1}^{k} (-a_i)^k N_i(k) = -\sum_{i=1}^{k} \delta_i B_i. \tag{7}$$

Now, finally, substituting (7) back into (4), and remembering the definition of u_r, we get our result:

THEOREM. *Let* $\gamma_s = u_s(-1)^{s-1} - \delta_s + c\delta_{s-1} + \sum_{i=1}^{s-1} \delta_i u_{s-i}(-1)^{s-i-1}$, *where* $\delta_0 = 1$ *and* $\delta_{k+1} = 0$. *Then we have*

$$f(n) = \sum_{s=1}^{k+1} \gamma_s g(s) = \gamma_1 f(n-1) + \cdots + \gamma_{k+1} f(n-k-1).$$

(Note that, by (*), $\gamma_{k+1} = c\delta_k$, and that all the γ_s are integers.)

Conclusion. There are many possible avenues of generalization, such as introducing new categories of sets C, elements of which may not be repeated at all in any one sequence. (This problem can, the author believes, be reduced to the original one.) Other generalizations might be obtained by thinking of these sequences as words and introducing some sort of order-dependent conditions. These problems might be interesting for future research and, aside from their own intrinsic mathematical value, might have applications to information theory and other related fields.

References

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A COMMON FIXED POINT THEOREM FOR COMMUTING MAPPINGS

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The old question of whether two commuting continuous mappings of $[0, 1]$ into itself have a common fixed point remains unanswered except in certain special cases. This note presents a new special case involving Lipschitz constants for both functions. We prove the following: if f and g are mappings of

$[0, 1]$ into itself such that $f(g(x))=g(f(x))$ for all $x\in[0, 1]$ and $|f(x)-f(y)| \leqq \alpha|x-y|$ and $|g(x)-g(y)| \leqq \beta|x-y|$ for all $x, y \in [0, 1]$, where $\beta<(\alpha+1)/(\alpha-1)$, then there exists a common fixed point for both f and g. (We shall assume, since $\beta>0$ and $\alpha>0$, that $\alpha>1$.)

Proof. Let N be the set of fixed points for g. It follows immediately from the commutativity of f and g that $f(N)\subset N$ and since g is continuous, $N\neq\emptyset$ and is closed. We shall now prove by contradiction that f and g have a common fixed point. Let a be the smallest point in N and b the largest in N. Since $f(N)\subset N$, we see that $a<f(a)$ and $f(b)<b$ (the strict inequality follows from our assumption that f and g do not have a common fixed point). We may now pick x_0, $x_1\in N$ such that $x_0<f(x_0)$, $f(x_1)<x_1$, $x_0<x_1$, and, furthermore, such that x_1-x_0 cannot be made smaller without violating these three inequalities. This possibility follows from the fact that N is compact and f is continuous.

Let us set $y_0=f(x_0)$ and $y_1=f(x_1)$. Let us show that $y_0\geqq x_1$ and $y_1\leqq x_0$. Since $f(N)\subset N$, we have y_0, $y_1\in N$. Now if $z\in N$ and $x_0<z<x_1$, then $f(z)<z$ or $z<f(z)$. For example, if $f(z)<z$, then $x_0<f(x_0)$, $f(z)<z$, $x_0<z$, but $z-x_0<x_1-x_0$, which contradicts the fact that x_1-x_0 is minimum under the conditions expressed by these three inequalities. Similar arguments will show that if $z\in N$, then either $z\leqq x_0$ or $x_1\leqq z$. Since $x_0<y_0\in N$ and $x_1>y_1\in N$, we must have $x_1\leqq y_0$ and $y_1\leqq x_0$.

We also showed above that either $g(x)\leqq x$ or $x\leqq g(x)$ for all $x\in I_0=[x_0, x_1]$. Otherwise, g would have a fixed point in the interior of I_0, which we just showed to be impossible. We shall assume that $x\leqq g(x)$ for all $x\in I_0$ since the other case can be handled by similar reasoning.

Since $x_0<f(x_0)$ and $f(x_1)<x_1$, it follows that f has a fixed point in I_0. Let s be the largest fixed point for f in I_0 and let $t=g(s)$. Since t is a fixed point for f and $s<t$, we must have $x_1<t$.

We shall now derive additional inequalities from the Lipschitz conditions on f and g. In doing so, however, we shall leave off the absolute value signs since we will know which quantities are nonnegative from those inequalities already derived. We assume that $\alpha>1$.

$$t - x_0 \leqq f(t) - f(x_1) \leqq \alpha(t - x_1). \tag{1}$$

$$t - x_0 = g(s) - g(x_0) \leqq \beta(s - x_0). \tag{2}$$

$$s - x_0 \leqq f(s) - f(x_1) \leqq \alpha(x_1 - s). \tag{3}$$

From (1) we obtain

$$\alpha x_1 - x_0 \leqq (\alpha - 1)t. \tag{4}$$

From (2) and (4) we obtain:

$$\alpha x_1 - x_0 \leqq (\alpha - 1)[\beta(s - x_0) + x_0];$$

hence, $\alpha x_1\leqq\beta(\alpha-1)(s-x_0)+\alpha x_0$, from which we obtain

$$\alpha(x_1 - x_0) \leqq \beta(\alpha - 1)(s - x_0). \tag{5}$$

From (3) we obtain $s-x_0 \leqq \alpha[(x_1-x_0)-(s-x_0)]$ and, hence,

$$(\alpha+1)(s-x_0) \leqq \alpha(x_1-x_0). \tag{6}$$

From (5) and (6) we obtain $\alpha(\alpha+1)(x_1-x_0) \leqq \alpha\beta(\alpha-1)(x_1-x_0)$, or

$$(\alpha+1)/(\alpha-1) \leqq \beta. \tag{7}$$

Thus, inequality (7) contradicts our assumption that $\beta<(\alpha+1)/(\alpha-1)$. Therefore, f and g have a common fixed point. Q.E.D.

In the proof above we assumed that $\alpha>1$. If $\alpha \leqq 1$, then a different approach may be used to show that f and g have a common fixed point (in this case g need only be continuous). Thus, if $\alpha \leqq 1$, then $|f(x)-f(y)| \leqq |x-y|$. If c is the smallest fixed point for f and d the largest and $c \leqq x \leqq d$, then $f(x)-c \leqq |f(x)-c| \leqq |f(x)-f(c)| \leqq x-c$ (which means $f(x) \leqq x$) and $d-f(x) \leqq |d-f(x)| \leqq |f(d)-f(x)| \leqq d-x$ (which means $x \leqq f(x)$). Hence, $f(x)=x$. Thus, $H=\{x: c \leqq x \leqq d\}$ is the fixed point set for f. Since f and g commute, $g(H) \subset H$. Since g is continuous, g must have a fixed point in H. Hence, f and g have a common fixed point.

SUMS OF NORMAL SEMI-ENDOMORPHISMS

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1. Introduction. It is known (see Heerema [1]) that the normal endomorphisms and normal anti-endomorphisms of a group generate a ring. Since endomorphisms and anti-endomorphisms are special semi-endomorphisms, it is of interest to examine also the system generated by the normal semi-endomorphisms of a group. In the present note, we establish a sufficient condition for the normal semi-endomorphisms of a group to generate a ring.

2. Definitions. Let G be a multiplicatively written group and let e denote the identity element of G.

Definition 1. *A single-valued mapping ϕ of G into G is a normal semi-endomorphism of G if*

$$\phi(xyx) = \phi(x)\phi(y)\phi(x), \qquad \text{all } x, y \in G \tag{1}$$

$$\phi(e) = e \tag{2}$$

$$\phi(x^{-1}yx) = x^{-1}\phi(y)x, \qquad \text{all } x, y \in G. \tag{3}$$

Definition 2. *If α and β are single-valued mappings of G into G, define $\alpha+\beta$ and $\alpha\beta$ by $(\alpha+\beta)(x)=\alpha(x)\beta(x)$ and $(\alpha\beta)(x)=\alpha(\beta(x))$ for all $x\in G$. In accordance with this definition, define O and I by $O(x)=e$ and $I(x)=x$ for all $x\in G$.*

Definition 3. *A group G is said to have Property P if, for each normal semi-endomorphism $\phi\neq 0$ of G, C_ϕ has no elements of order 2, where $C_\phi=\{\text{all } x\in G \mid x\phi(y)=\phi(y)x, \text{ all } y\in G\}$.*

3. Lemmas. Lemmas 1 and 2 are consequences of Definition 2 and the normality condition (3) of Definition 1. We shall supply a proof of Lemma 2.

LEMMA 1. *If α, β, γ are single-valued mappings of G into G, then* $(\alpha+\beta)\gamma=\alpha\gamma+\beta\gamma$.

LEMMA 2. *If α, β are normal semi-endomorphisms of G, then* $\alpha+\beta=\beta+\alpha$.

Proof. For all $x\in G$, $\beta(x)=\beta(x^{-1}xx)=x^{-1}\beta(x)x$ or $x\beta(x)=\beta(x)x$. Then $\alpha(x)=\alpha(\beta(x)^{-1}x\beta(x))=\beta(x)^{-1}\alpha(x)\beta(x)$. Thus, $\alpha(x)\beta(x)=\beta(x)\alpha(x)$ for all $x\in G$ and $\alpha+\beta=\beta+\alpha$.

In the following lemmas, it is notationally convenient to employ the commutator $[x, y]=x^{-1}y^{-1}xy$, for $x, y\in G$.

LEMMA 3. *Let G have Property P and let ϕ be a normal semi-endomorphism of G. If $[x, y]=e$, then $\phi(xy)=\phi(x)\phi(y)$.*

Proof. If $\phi=0$, the result is trivial. Suppose $\phi\neq 0$. Then $\phi(x)=\phi(x[x, y])=\phi(y^{-1}xy)=y^{-1}\phi(x)y$. Thus, $[x, y]=e$ implies that $[\phi(x), y]=e$ or $[y, \phi(x)]=e$. This, in turn, implies that $[\phi(x), \phi(y)]=e$.

For all $z\in G$, $\phi(xyzyx)=\phi(x)\phi(y)\phi(z)\phi(x)\phi(y)$. Also $\phi(xyzyx)=\phi(xyzxy)=\phi(xy)\phi(z)\phi(xy)$. Let $a=\phi(x)\phi(y)$ and let $b=\phi(xy)$. Then $a\phi(z)a=b\phi(z)b$, for all $z\in G$. In particular, for $z=e$, we obtain $a^2=b^2$ and $b^{-1}a=ba^{-1}$. Then, for $z=xy$, we obtain $ab=ba$. Thus, $b^{-1}a\in C_\phi$ and $(b^{-1}a)^2=e$. Since G has Property P, $a=b$ and the desired conclusion is reached.

LEMMA 4. *If $\alpha, \beta_1, \cdots, \beta_n$ are normal semi-endomorphisms of G and if G has Property P, then* $\alpha(\beta_1+\cdots+\beta_n)=\alpha\beta_1+\cdots+\alpha\beta_n$.

Proof. We shall use induction on n, the result being trivial for $n=1$. Suppose that for $k\geqq 1$,

$$\alpha(\beta_1+\cdots+\beta_k)=\alpha\beta_1+\cdots+\alpha\beta_k.$$

By applications of Lemma 2, $(\beta_1+\cdots+\beta_k)+\beta_{k+1}=\beta_{k+1}+(\beta_1+\cdots+\beta_k)$, i.e., $[(\beta_1+\cdots+\beta_k)(x), \beta_{k+1}(x)]=e$, all $x\in G$. By Lemma 3,

$$\alpha((\beta_1+\cdots+\beta_k)(x)\beta_{k+1}(x))=\alpha(\beta_1+\cdots+\beta_k)(x)\alpha\beta_{k+1}(x).$$

The desired result now follows from the induction hypothesis.

4. Main results. Let S be the collection of all finite sums of normal semi-endomorphisms of group G. In general, the system $\{S, +, \cdot\}$ is not a ring. For example, if G were the direct product of three cyclic groups each of order 2, the system $\{S, +, \cdot\}$ would not be left distributive. Sufficient conditions for $\{S, +, \cdot\}$ to be a ring are afforded, however, by the following theorems.

THEOREM 1. *If group G has Property P, then $\{S, +, \cdot\}$ is a ring with identity.*

Proof. By Lemma 2 and obvious calculations, $\{S, +\}$ is an Abelian group. Let $\alpha, \beta\in S$, i.e., $\alpha=\alpha_1+\cdots+\alpha_m$ and $\beta=\beta_1+\cdots+\beta_n$ where the α_i, β_j are

normal semi-endomorphisms of G. Using Lemmas 1 and 4,

$$\alpha\beta = \alpha_1\beta_1 + \cdots + \alpha_1\beta_n + \cdots + \alpha_m\beta_1 + \cdots + \alpha_m\beta_n.$$

Since the $\alpha_i\beta_j$ are also normal semi-endomorphisms of G, $\alpha\beta \in S$. I is the identity of $\{S, +, \cdot\}$. By the preceding lemmas, both distributive laws hold.

THEOREM 2. *If G is a finite group of odd order or if G is a non-Abelian simple group, then $\{S, +, \cdot\}$ is a ring with identity.*

Proof. If G is finite of odd order then, by Lagrange's Theorem, G has Property P and the desired result follows from Theorem 1.

Now let G be a non-Abelian simple group. Let $\phi \neq 0$ be a normal semi-endomorphism of G, let $x \in C_\phi$, and let $y \in G$. Then, for all $z \in G$,

$$y^{-1}xy\phi(z)y^{-1}x^{-1}y = y^{-1}x\phi(yzy^{-1})x^{-1}y = y^{-1}\phi(yzy^{-1})y = \phi(y^{-1}yzy^{-1}y) = \phi(z).$$

Hence, $y^{-1}xy\phi(z) = \phi(z)y^{-1}xy$, for all $z \in G$. Thus, $y^{-1}xy \in C_\phi$, for all $x \in C_\phi$ and all $y \in G$. Therefore, C_ϕ is a normal subgroup of G. Since $\phi \neq 0$, $C_\phi = (e)$ and G has Property P. The desired conclusion follows from Theorem 1.

Reference

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ACKNOWLEDGMENT

The authors of the paper "On the number of partitionings of a set of n distinct objects" (this MONTHLY 69 (1962) 782–785) acknowledge, as pointed out by John Riordan, that their method is due to A. C. Aitken (1933) and is described in Riordan's book "An Introduction to Combinatorial Analysis."

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University and
A. L. SHIELDS, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, University of Michigan, Ann Arbor, Mich.

THE DEFINITION OF $dy{:}dx$

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We start by considering the following problem, which is conveniently solved by using differentials.

Find the locus of a point P which moves in the xy-plane, starting at $(1, 1)$, in such a way that the direction of motion is always perpendicular to OP, where O is the origin.

A solution: The direction-ratio of the locus is $dx:dy$, the condition for perpendicularity is $x\cdot dx+y\cdot dy=0$, and the solution through $(1, 1)$ of this equation is $x^2+y^2=2$.

My main contention is that with the usual definition of differential this solution is not sound; but that with a good definition (such as one which I shall suggest) it will be sound.

Many elementary texts define the differentials dx and dy under the assumption that y is an explicit function of x, say $y=\phi(x)$, and lead to the result that:

If $y=\phi(x)$, then $dy=\phi'(x)\cdot dx$ where $\phi'(x)$ exists, and dy is undefined elsewhere.

Such a definition is of no use for solving problems like ours, because it can never allow the solution to have vertical tangents. Indeed there is really nothing which we can do with differentials, on this definition, which we cannot do equally easily without them. In particular, if we really are relying on this definition of differential, we cannot say that $x^2+y^2=2$ is a solution of the equation $x\cdot dx+y\cdot dy=0$, because we have not defined dy at the point $(\sqrt{2}, 0)$.

We therefore look for a better definition. The first improvement is to start from an implicit, not an explicit, relation between x and y.

Secondly, the fundamental fact that only the *ratio* of dx to dy matters should be an obvious consequence of the definition, not something which appears mysteriously in the course of working. A good way to ensure this is to define only this ratio and not even to pretend to define dx and dy individually.

These considerations suggest the following definition, to be made after the implicit-function theorem has been proved and the formula for differentiating inverse functions has been derived.

DEFINITION. *The value of $dx:dy$ under the relation $\phi(x, y)=0$ is $1:\eta'(x)$, where the relation has a local solution $y=\eta(x)$ and η is differentiable; it is $\xi'(y):1$, where the relation has a local solution $x=\xi(y)$ and ξ is differentiable.*

Notes.

1) Where both $\eta'(x)$ and $\xi'(y)$ exist, η and ξ are locally inverse functions and so the two ratios $1:\eta'(x)$ and $\xi'(y):1$ are equal. Thus the value of $dx:dy$ is uniquely defined.

2) Where neither $\eta'(x)$ nor $\xi'(y)$ exists, the value of $dx:dy$ is not defined.

3) The value of $dx:dy$ is easily seen to be $-\phi_2(x, y):\phi_1(x, y)$ wherever this ratio exists and ϕ_1 or ϕ_2 is continuous.

4) If the given relation between x and y is $y=\eta(x)$, then the value of $dx:dy$ is $1:\eta'(x)$ wherever $\eta'(x)$ exists, whence $dy=\eta'(x)\cdot dx$. Thus our definition yields the standard result in the explicit case, and so everything which follows from the standard definition follows from ours.

5) The main reason why we did not simply define $dx:dy$ to be $-\phi_2(x, y)$: $\phi_1(x, y)$ where this ratio exists and to be undefined elsewhere is that this would not work for cases like $\phi(x, y)=x^3-y^3$ at $(0, 0)$.

6) The definition can be made more formal if desired. We are defining a binary function whose values are ratios, and the definition could be so framed. It is for traditional reasons that we denote the value at (x, y) of this function by $dx:dy$.

7) We started with an implicit relation between x and y. A parametric relation $x=\xi(t)$, $y=\eta(t)$ can be treated similarly. We define the differential only for values of t where one or other of the two equations is locally solvable for t; and we easily find that the value of $dx:dy$ is $\xi'(t):\eta'(t)$ wherever this ratio exists.

DISTINCT ELEMENTS IN NON-COMMUTATIVE GROUPS AND LOOPS

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The principal purpose of this note is to present a simple proof, using only the axioms of group theory, of the following result:

THEOREM 1. *If a and b are noncommuting elements of a group G, then the five elements a, b, ab, ba, and e (identity) of G are distinct. Moreover, one of a^2 and aba is distinct from all of these five.*

As an immediate corollary, we obtain

THEOREM 2. *A noncommutative group has at least six elements.*

That is, if G is a noncommutative group, it has at least one pair of noncommuting elements a and b, whence by Theorem 1, at least six distinct elements can be exhibited.

This proof of Theorem 2 has the advantage of requiring no structure theory or homomorphism laws for groups whatever, and can serve as an exercise for students who are just learning the group axioms.

For the proof of Theorem 1, the following weakened form of the group axioms will be used:

(i) For every pair of elements x and y in G, the product xy is a uniquely defined element of G.

(ii) There is an element e in G with $ex=xe=x$ for all x in G.

(iii) If $xy=xz$ then $y=z$, and if $yx=zx$ then $y=z$, for all x, y, z in G.

(iv) $x(yx)=(xy)x$ and $x(xy)=(xx)y$ for all x and y in G.

Since (iv) is significantly weaker than the associative law, Theorems 1 and 2 hold not only for groups, but also for a large class of *loops*, including diassociative loops [1], and in particular all Moufang loops [1]. Basing a proof directly on the axioms greatly simplifies the task of determining the generality with which the theorem holds. (Any system satisfying the axioms (i), (ii), (iii) is a

semi-loop with unit. Any *finite* system satisfying these axioms is a *finite loop*, and conversely.)

Proof of Theorem 1. We will first examine the members of the set $\{e, a, b, ab, ba\}$ two at a time, and show that each of the ten possible equalities leads to a contradiction of $ab \neq ba$.

1. $e=a \Rightarrow ab=eb=b=be=ba$.
2. $e=b \Rightarrow ab=ae=a=ea=ba$.
3. $e=ab \Rightarrow a=ea=(ab)a=a(ba) \Rightarrow e=ba \Rightarrow ab=ba$.
4. $e=ba \Rightarrow a=ae=a(ba)=(ab)a \Rightarrow e=ab \Rightarrow ab=ba$.
5. $a=b \Rightarrow ab=aa=ba$.
6. $a=ab \Rightarrow e=b$, reducing to case 2.
7. $a=ba \Rightarrow e=b$, reducing to case 2.
8. $b=ab \Rightarrow e=a$, reducing to case 1.
9. $b=ba \Rightarrow e=a$, reducing to case 1.
10. $ab=ba$ contradicts $ab \neq ba$.

Thus, $\{e, a, b, ab, ba\}$ are all distinct. Next, we show that a^2 is distinct from each of $\{a, b, ab, ba\}$.

11. $a^2=a \Rightarrow a=e$, reducing to case 1.
12. $a^2=b \Rightarrow ab=a(a^2)=a(aa)=(aa)a=a^2a=ba$.
13. $a^2=ab \Rightarrow a=b$, reducing to case 5.
14. $a^2=ba \Rightarrow a=b$, reducing to case 5.

Thus, either a^2 is a sixth element of G, or possibly $a^2=e$. In the latter case, we show that aba (well-defined by axiom (iv), which assures that $(ab)a=a(ba)$) is distinct from each of $\{e, a, b, ab, ba\}$. As a preliminary observation, we note that $a(a(ba))=(aa)(ba)=e(ba)=ba$.

15. $aba=e \Rightarrow ba=a(aba)=ae=a$, reducing to case 7.
16. $aba=a \Rightarrow ab=e$, reducing to case 3.
17. $aba=b \Rightarrow ab=a(aba)=ba$.
18. $aba=ab \Rightarrow (ab)a=ab \Rightarrow a=e$, reducing to case 1.
19. $aba=ba \Rightarrow a(ba)=ba \Rightarrow a=e$, reducing to case 1.

This completes the proof.

×	E	A	B	C	D	F
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

FIG. 1

Examples.

1. In the multiplication table for the noncommutative group of six elements (Figure 1), we may choose $a=A$, $b=B$. In this case, $e=E$, $a=A$, $b=B$, $ab=D$, $ba=F$, and $a^2=E$ but $aba=C$.

However, choosing $a=D$ and $b=B$, we find $e=E$, $a=D$, $b=B$, $ab=A$, $ba=C$, and $a^2=F$, while $aba=B$. Thus, it is not possible to specify abstractly whether a^2 or aba will be the sixth element.

×	E	A	B	C	D
E	E	A	B	C	D
A	A	D	E	B	C
B	B	C	D	A	E
C	C	E	A	D	B
D	D	B	C	E	A

FIG. 2

×	E	A	B	C	D
E	E	A	B	C	D
A	A	E	C	D	B
B	B	D	E	A	C
C	C	B	D	E	A
D	D	C	A	B	E

FIG. 3

2. In the noncommutative loop of order five (Figure 2), neither half of the weak associative law (iv) holds. If we pick the noncommuting elements $a=A$, $b=B$, we find that only four of $\{e, a, b, ab, ba\}$ are distinct, because $AB=E$, although $BA=C$. With the choice $a=A$, $b=D$, it happens that all five of $e=E$, $a=A$, $b=D$, $ab=C$, $ba=B$ are distinct. However, $a^2=b$ (i.e. $A^2=D$); while both $(ab)a=CA=E$ and $a(ba)=AB=E$.

Example 2 cannot be improved to only four distinct elements, in view of the following result.

THEOREM 3. *If a and b are noncommuting elements of a loop L, then among the eight elements e, a, b, ab, ba, a^2, $a(ba)$, $(ab)a$, at least five are distinct.*

In particular, if e, a, b, ab, ba are not all distinct, then one of ab and ba (say ba) must equal e. If a^2 is not distinct from each of e, a, b, ab then $a^2=b$. (Indeed, $a^2=e \Rightarrow a^2=e=ba \Rightarrow a=b \Rightarrow ab=ba$.) Also, if $(ab)a$ is not distinct from each of e, a, b, ab then $(ab)a=b$. (For example, $(ab)a=e \Rightarrow (ab)a=e=(b)a \Rightarrow ab=b \Rightarrow a=e \Rightarrow ab=b=ba$.) However, if both $a^2=b$ and $(ab)a=b$ then $(a)a=a^2=(ab)a$, whence $a=ab$, which is inconsistent with $ab \neq ba$.

Again, this furnishes a painless proof of the corollary:

THEOREM 4. *A noncommutative loop has at least five elements.*

Finally, one may determine the separate roles of the *flexible law*

(iv–a) $\qquad x(yx) = (xy)x \qquad$ for all x, y in G,

and the *left alternative law*

(iv–b) $\qquad x(xy) = (xx)y \qquad$ for all x, y in G,

as they enter into Theorem 1. Specifically,

THEOREM 5. *If a and b are noncommuting elements of a "flexible loop" L, then the five elements e, a, b, ab, ba are distinct. However, L need not contain any further elements.*

The proof of distinctness of e, a, b, ab, ba in Theorem 1 clearly holds for any flexible loop. On the other hand, Figure 3 exhibits a noncommutative flexible loop with only five elements.

THEOREM 6. *If a and b are noncommuting elements of a "left alternative loop" L, then L contains at least six distinct elements, including a, b, ab, ba, and at least two of e, a^2, $a(ba)$, $(ab)a$, a^3.*

The method of proof is fully analogous to Theorems 1 and 3, and may safely be left as an exercise.

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A SIMPLE PROOF OF A BASIC THEOREM OF THE CALCULUS

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The purpose of this note is to provide an elementary proof of the theorem: *If $f'(x)=0$ for every x belonging to an interval $[a, b]$, then $f(x)\equiv c$ for all x belonging to $[a, b]$.* The usual proof makes use of the following theorems: A real-valued function continuous over a closed interval takes on its maximum and minimum in that interval, Rolle's Theorem, the Mean Value Theorem, etc. The following proof depends on none of these theorems.

Proof. Assume the contrary, namely, that there exist x_1 and x_2 of $[a, b]$, with $x_1<x_2$ and $f(x_1)\neq f(x_2)$ and, for definiteness, that $f(x_1)<f(x_2)$ (in case $f(x_1)>f(x_2)$, consider the function $g=-f$). Let $\epsilon=[f(x_2)-f(x_1)]/(x_2-x_1)\,(\epsilon>0)$, so that $f(x_2)-f(x_1)=\epsilon(x_2-x_1)$.

For this ϵ, let $A=\{x \mid x_1<x\leqq b, f(x)-f(x_1)\geqq\epsilon(x-x_1)\}$. Then A is nonempty ($x_2\in A$) and bounded from below. Let x^* be its greatest lower bound: $x^*=\inf A$. Since $f'(x_1)=0$, there exists a positive $\delta_1<x_2-x_1$ such that $x_1<x<x_1+\delta_1$ implies $f(x)-f(x_1)<\epsilon(x-x_1)$, and hence $x^*>x_1$. On the other hand, since $f'(x^*)=0$, there exists a positive δ_2 less than x^*-x_1 such that $x^*-\delta_2<x<x^*$ implies $f(x^*)-f(x)<\epsilon(x^*-x)$.

Let x_3 be any point in $(x^*-\delta_2, x^*)$, so that $x_1<x_3<x^*$, and simultaneously:

$$f(x_3)-f(x_1)<\epsilon(x_3-x_1)\quad\text{(by definition of } x^*\text{)}$$

$$f(x^*)-f(x_3)<\epsilon(x^*-x_3)\quad\text{(by the preceding implication),}$$

and consequently, by addition, $f(x^*)-f(x_1)<\epsilon(x^*-x_1)$.

This inequality implies that $x^*<x_2$. Finally, $f'(x^*)=0$ implies the existence of a positive $\delta_3<x_2-x^*$ such that $x^*<x<x^*+\delta_3$ implies $f(x)-f(x^*)<\epsilon(x-x^*)$ and hence (in conjunction with the last preceding inequality) $f(x)-f(x_1)<\epsilon(x-x_1)$ in violation of the definition of x^* as the *greatest* lower bound of A.

CYCLIC DISLOCATIONS IN DETERMINANTS AND A NEW EXPANSION OF A DETERMINANT

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In a recent paper [1], P. R. Vein proves and uses a lemma on cyclic dislocations in determinants. Many other relations exist among determinants of this type. In this note, Vein's Lemma and a new lemma are stated, then a theorem is given which follows from the two lemmas. Lemma 2 and Theorem 1 below give relations involving the product of the sum of the elements of a column by the sum of the corresponding cofactors. Theorem 1 also gives a new way of expanding a determinant which the author finds interesting because of the symmetry of the expansion. The relations are given for determinants of order three, but they can easily be extended to determinants of any order. The proofs are easy and are not given.

Consider a determinant of order three

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = | C_1C_2C_3 |,$$

where the C's represent the columns of Δ. Let

$$\delta C_i = \begin{vmatrix} a_{3i} \\ a_{1i} \\ a_{2i} \end{vmatrix} \quad \text{and} \quad \delta^2 C_i = \begin{vmatrix} a_{2i} \\ a_{3i} \\ a_{1i} \end{vmatrix} \qquad i = 1, 2, 3.$$

As usual, A_{ij} represents the cofactor of a_{ij}.

LEMMA 1. (P. R. VEIN.) $|\delta C_1C_2C_3| + |C_1\delta C_2C_3| + |C_1C_2\delta C_3| = 0$ *and*

$$|\delta^2C_1C_2C_3| + |C_1\delta^2C_2C_3| + |C_1C_2\delta^2C_3| = 0.$$

LEMMA 2.

$$|C_1C_2C_3| + |\delta C_1C_2C_3| + |\delta^2C_1C_2C_3| = (a_{11} + a_{21} + a_{31})(A_{11} + A_{21} + A_{31})$$

and two similar equations formed by operating on one column at a time.

THEOREM 1.

$$3|C_1C_2C_3| = (a_{11} + a_{21} + a_{31})(A_{11} + A_{21} + A_{31})$$
$$+(a_{12} + a_{22} + a_{32})(A_{12} + A_{22} + A_{32}) + (a_{13} + a_{23} + a_{33})(A_{13} + A_{23} + A_{33}).$$

Reference

1. P. R. Vein, A lemma on cyclic dislocations in determinants and an application in the verification of an identity, this MONTHLY, 69 (1962) 120–124.

THE PROOF OF THE TRIANGLE INEQUALITY

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The writer has been perhaps overexposed to proofs of the triangle inequality which are based on the discriminant of a quadratic or other lengthy calculations. Some of these exposures were sponsored by the Mathematical Association of America. In addition to a concise proof, a discussion of this inequality should include a clear statement of the case of equality. (Cf. K. Urbanik and F. B. Wright, Absolute-valued algebras, Proc. Amer. Math. Soc., 11 (1960) 861–866; especially their proof of Lemma 3.) The purpose of this note is to provide a geometric proof of the triangle inequality. Our proof leads at once to an equally simple proof of the quadrilateral inequality and its consequent polygonal inequalities. These matters we propose, however, to discuss in a separate publication.

It is a fact that if z and w are complex numbers then

$$(1) \qquad |z+w| \leqq |z| + |w| \textit{ and equality holds iff } |z|w = |w|z.$$

The fact (1) is called the triangle inequality.

A logical proof of (1) is not only possible but almost canonical. We have

$$\begin{aligned} |z+w| \leqq |z| + |w| &\text{ iff } (z+w)(\bar{z}+\bar{w}) \leqq z\bar{z} + 2|z|\,|w| + w\bar{w} \\ &\text{ iff } z\bar{w} + w\bar{z} \leqq 2|z|\,|w| \\ &\text{ iff } z\bar{w}|z|\,|w| + w\bar{z}|z|\,|w| \leqq 2|z|^2|w|^2 \\ &\text{ iff } 0 \leqq (|w|z - |z|w)(|w|\bar{z} - |z|\bar{w}). \end{aligned}$$

Observing that all of these implications hold if " $\leqq$ " is everywhere replaced by " $=$ ", we have proved (1).

The statement (1) remains a fact if we require only that z and w are vectors in a vector space X over the complex field which is equipped with a sesquilinear inner product which is positive definite, i.e., there is a map of $X \times X$ into the complex field denoted by (z, w) such that $(w, z) = \overline{(z, w)}$, $(z_1+z_2, w) = (z_1, w)+(z_2, w)$, $(\alpha z, w) = \alpha(z, w)$ for all z, z_1, z_2, w in X and complex α; and $(z, z) > 0$ unless $z = 0$. In this situation we have $|z| = (z, z)^{1/2}$ and

$$\begin{aligned} &|z+w| \leqq |z| + |w| \\ &\text{iff } (w, z) + (z, w) \leqq 2|z|\,|w| \quad (\text{or } \operatorname{Re}(z, w) \leqq |z|\,|w|) \\ &\text{iff } (|z|w, |w|z) + (|w|z, |z|w) \leqq 2|z|^2|w|^2 \\ &\text{iff } 0 \leqq (|w|z - |z|w, |w|z - |z|w). \end{aligned}$$

We have not only proved (1), but our first statement yields

$$(2) \qquad |(z, w)| \leqq |z|\,|w| \textit{ and equality holds iff } z, w \textit{ are linearly dependent.}$$

We obtain (2) by replacing z by $e^{-i\theta}z$ where $(z, w) = |(z, w)|e^{i\theta}$ (θ real) when we realize that $|w|ze^{-i\theta} = |z|w$ iff z and w are linearly dependent. The fact (2) is often labeled "the inequality of Schwarz."

Moral: What geometry hath joined together, let neither algebra nor analysis put asunder.

UNIQUE FACTORIZATION

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1. In proving the fundamental theorem of arithmetic, the usual method is to show that (a) every natural number greater than one is a product of primes, and that (b) if a prime divides a product of integers, it must divide one of the factors. The final deduction of the uniqueness is, of course, a simple matter of induction. There are, however, direct proofs by induction of the fundamental theorem, such as the one in Hardy and Wright and the others to which they refer. These have the virtue of avoiding step (b) and hence do not make use of the Euclidean Algorithm: but some students find these short proofs tricky, though to a mathematician they seem clear and pleasing. Hence, we shall give here a proof by induction of (b) which does not use the Euclidean Algorithm. It is very straightforward, and in its most naive formulation it uses all the defining properties of the natural numbers. This matter will be discussed after we prove (b).

2. *Proof of* (*b*). We consider only natural numbers and prove that

$$p \mid mn \rightarrow p \mid m \text{ or } p \mid n,$$

where p always denotes a prime. Assuming that there are primes which do not have this property, we let p be the least such and then let m be the least natural number such that there is an n with $p|mn$, but neither m nor n is a multiple of p. Now, if m were larger than p we could replace it by $m-p$ and contradict the minimality of m. Hence we have, say:

$$p\cdot k = m\cdot n; \qquad 1 < m < p; \qquad p\nmid n.$$

Since we know by (a) that m is divisible by a prime, say q, and we now know that q is less than p, this prime must have property (b). Thus, since q divides $p\cdot k$ and does not divide p, it must divide k. Cancellation of q from both sides of the equation now contradicts the minimality of m.

3. In examining the assumptions used in proving (b), we shall begin with those used in (a). There it is required that we have a system with a binary operation and a mapping v from the system to a well-ordered set, such that for all nonunits x and y we have $v(x\cdot y) > \text{Max}\{v(x), v(y)\}$. Cases where v is not the identity map are:

(i) the nonconstant polynomials in indeterminates over a ring without zero-divisors, with $v(f) = \deg(f)$;

(ii) the matrices over the integers of given order, with

$$v(A) = |\det(A)| > 1.$$

The properties of the natural numbers used in (b) are the trichotomy law and the fact that $x > y > 0$ implies that $x = y + z$ for some z with $x > z > 0$. These lead to the Peano properties, no matter how one tries to re-word these properties in terms of a mapping v. Thus, the simplest way of completing the proof of (b) is also the least general.

To consider departures from this level of simplicity, we consider an integral domain D with a mapping v from D^* to the natural numbers, such that $v(x) = v(y)$ iff x and y differ only in a unit factor, with $v(1) = 1$, and with the "multiplicative" property as before. Now, when we consider the possibility of having $v(m) > v(p)$, we try replacing m by $m + px$ rather than by $m - p$, for some element x of D. That is to say, we can prove (b) if we have the additional property that $v(x) > v(y)$ implies $v(x + ky) < v(x)$ for some k. This property, however, is equivalent to the fact that whenever $v(x) > v(y)$, we can find k with $v(x + ky) \leqq v(y)$. This is not really weaker than the Euclidean Algorithm, since the case $v(x + ky) = v(y)$ would require x to be a multiple of y.

Finally, the proof can also be completed even in a wider context. In place of $m + p \cdot x$ we can use $m \cdot y + p \cdot x$, as long as the latter is nonzero. This yields the known fact that a sufficient extra condition to ensure uniqueness of factorization is the Dedekind-Hasse condition that, whenever $v(x) > v(y)$ and y does not divide x, there exist elements a and b such that $a \cdot x + b \cdot y$ is nonzero and that $v(a \cdot x + b \cdot y) < v(y)$.

4. In conclusion, I wish to express my thanks to Professors Straus and Gordon for useful discussions.

IDEALS AND ADEALS OF A RING

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Following Willcox (see [2]), we define a subset M of a ring R to be an adeal if (i) $x \in M$ and $y \in M$ implies $x + y \in M$; (ii) $x \in M$, $r \in R$ implies $xr \in M$ and $rx \in M$. An ideal is then an adeal which is also a subring.

The problem of showing that an adeal of R need not be a subring of R has appeared as Problem E 1472 [1]. The weakest condition on R found by Willcox to ensure that every adeal be an ideal, is that for each $a \in R$ there exists a nonzero integer k such that $ka \in aR + Ra$, where ka has the usual meaning. That it is an interesting problem to try to find still weaker conditions has been pointed out by the editor of "Elementary Problems and Solutions," this MONTHLY [2].

For this purpose it is sufficient to find conditions on the ring R which ensure

that if a is an element of an adeal M, $-a$ must also belong to M. Or indeed, conditions under which $k(-a)$ belongs to the adeal M for some positive integer k, since if $k=1$, it is the previous case, while if $k>1$, $(k-1)$ is a positive integer, and $-a=k(-a)+(k-1)a$ will belong to M by the closure of M under addition. It is easy to give conditions under which this is true:

(i) $1\in R$;

(ii) for every $a\in R$, there is an integer $k>0$ such that $k(-a)\in Ra$;

(iii) for every $a\in R$, there is an integer $k>0$ such that $k(-a)\in aR$;

(iv) for every $a\in R$, there is an integer $k>0$ such that $k(-a)\in RaR$;

(v) for every $a\in R$, there is an integer $k\neq 0$ such that $ka\in Ra$;

(vi) for every $a\in R$, there is an integer $k\neq 0$ such that $ka\in aR$;

(vii) for every $a\in R$, there is an integer $k\neq 0$ such that $ka\in RaR$, where

$$Ra = \{ra \mid r \in R\}, \qquad aR = \{ar \mid r \in R\},$$

and

$$RaR = \{r_1ar_2 \mid r_1, r_2 \in R\}.$$

The condition (i) is sufficient, since if $1\in R$, $-1\in R$, and so $-a=a(-1)\in M$ for every $a\in M$, by the multiplicative property.

Each of the next three conditions is proved by using essentially the same fact, namely that if $a\in M$, then $Ra\subseteq M$, $aR\subseteq M$, and $RaR\subseteq M$.

The last three are not essentially stronger, since $(-k)a=k(-a)$ by common agreement, while if $ka\in Ra$, so that $ka=ra$, then $k(-a)=(-r)a\in Ra$ also, and similarly for aR and RaR.

We can of course make weaker conditions from these by conjunctions. For example:

(viii) for every $a\in R$, there is an integer $k\neq 0$ such that $ka\in aR+Ra$, where $aR+Ra=\{ar_1+r_2a \mid r_1, r_2\in R\}$. This is the condition reported by Willcox as the weakest his class had found. That it is weaker than conditions (v) or (vi) is evident, since if $ka=ra$, then $ka=a\cdot 0+ra\in aR+Ra$, so that if a k can be found for (v), it can surely also be found for (viii).

It is possible to give a condition which is not only sufficient (as all the above conditions are), but also necessary, that every adeal of a ring R be also an ideal. Before giving this condition we may note that if a (two-sided) ideal I contains an element a, then it must also contain $\{\cdots, -2a, -a, 0, a, 2a, 3a, \cdots\}$, by the additive group properties, and it must also contain aR and Ra, by closure under multiplication by elements of R on the right or the left. It would also have to contain the elements of RaR, as say left multiples of elements of aR. It would also have to contain sums of these elements (and indeed differences, but, as before, the minus sign may be put with a factor from R). If we define $(RaR)^*$ to be the collection of all finite sums of elements from RaR, we can say that the ideal I must contain:

$$\{ka \mid k \text{ an integer}\} + Ra + aR + (RaR)^*$$

which includes all possible combinations, since 0 belongs to each of the four collections. (We do not need $(Ra)^*$ because of the distributive property: $r_1a+r_2a=(r_1+r_2)a\in Ra$). Indeed, this is the smallest ideal of R which contains the element a.

An adeal A, on the other hand, would have to contain, by similar arguments,

$$\{ka \mid k \geqq 0,\ k \text{ an integer}\} + Ra + aR + (RaR)^*$$

which is the smallest adeal containing the element a.

Our main result is the following theorem.

THEOREM. *Every adeal A of a ring R will be an ideal if and only if for every $a\in R$, there is an integer $k\neq 0$ such that $ka\in aR+Ra+(RaR)^*$.*

Proof. Necessity. For any $a\in R$,

$$A = \{ka \mid k \geqq 0,\ k \text{ an integer}\} + Ra + aR + (RaR)^*$$

is an adeal of R and hence an ideal of R, so $-a\in A$. Thus $-a\in ma+Ra+aR+(RaR)^*$, or $-(m+1)a\in Ra+aR+(RaR)^*$ for some integer $m\geqq 0$.

Sufficiency. If A is an adeal of R and $a\in A$, then there exists an integer $k\neq 0$ such that $ka\in aR+Ra+(RaR)^*$. It follows that $-ka\in aR+Ra+(RaR)^*\subseteq A$. Therefore, A is an ideal.

It should be noted that if the ring R is commutative, then the condition given reduces to (v) or (vi) or (vii), these being the same in this case, and the same as (viii) also. Therefore, we have

COROLLARY. *If R is a commutative ring, then Willcox's condition is a necessary and sufficient condition on R to insure that every adeal be an ideal.*

It would be desirable to give an example of a ring, every adeal of which is an ideal, but which does not satisfy the sufficient condition of Willcox. This requires a noncommutative ring without identity in which every adeal is an ideal. However no example of such a ring is known to the author.

I wish to thank the referee for several suggestions.

References

1. This MONTHLY, 68 (1961) 573.
2. This MONTHLY, 69 (1962) 167–168.

REPEATED RIEMANN INTEGRATION

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In order to determine when the order of integration may be changed in a repeated integral, it is usually necessary to associate the repeated integral with the corresponding multiple integral. (For an exception, see Apostol [1].) This association necessitates the discussion of multiple integration, product sets and

product measures, and the final statement is a theorem like that of Fubini. The theorem below avoids this discussion and is nontrivial in the sense that Riemann integration cannot be replaced throughout by integration over general measure.

THEOREM. *Let (Y, B, μ) be a measure space. If $f(x, y)$ is Riemann-integrable as a function of x on $a \leqq x \leqq b$ for almost all y in Y, is measurable as a function of y for all x in $[a, b]$, and $|f(x, y)| \leqq F(y)$, where $\int_Y F(y)d\mu < \infty$, then $H(x) = \int_Y f(x, y)d\mu$ is Riemann-integrable on $[a, b]$ and $\int_a^b H(x)dx = \int_Y G(y)d\mu$, where $G(y) = \int_a^b f(x, y)dx$.*

The proof is based on the following lemma.

LEMMA. *Let $g(x)$ be bounded on $[a, b]$, and let*

$$S(P_n) = \sum_{k=1}^{n} g(\xi_k)\Delta x_k,$$

where P_n is the following partition of $[a, b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b, \quad x_{k-1} \leqq \xi_k \leqq x_k, \quad \Delta x_k = x_k - x_{k-1}$$

and

$$\|P_n\| = \max_{1 \leqq k \leqq n} \Delta x_k.$$

If $\lim_{n\to\infty} S(P_n)$ exists for every sequence $\{P_n\}$ of partitions for which $\lim_{n\to\infty} \|P_n\| = 0$, then g is Riemann-integrable on $[a, b]$.

Proof. The upper and lower Riemann integrals exist since g is bounded. Suppose they are not equal. Then there is a sequence of partitions $\{P_n\}$, $\|P_n\| \to 0$ and a sequence of partitions $\{Q_n\}$, $\|Q_n\| \to 0$, such that

$$\lim_{n\to\infty} S(P_n) = \overline{\int_a^b} gdx, \qquad \lim_{n\to\infty} S(Q_n) = \underline{\int_a^b} gdx.$$

If $\{R_n\}$ is the sequence of partitions defined by $R_{2k+1} = P_{2k+1}$, $R_{2k} = Q_{2k}$, then $\|R_n\| \to 0$ and $\lim_{n\to\infty} S(R_n)$ does not exist, contrary to the hypothesis of the lemma. Thus the upper and lower integrals are equal and g is Riemann-integrable.

In order to prove the main theorem let $\{P_n\}$ be a sequence of partitions such that $\|P_n\| \to 0$ as $n \to \infty$ and form the sum

$$S(P_n, y) = \sum_{k=1}^{n} f(\xi_k, y)\Delta x_k.$$

Then, for almost all y in Y,

$$\lim_{n\to\infty} S(P_n, y) = \int_a^b f(x, y)dx = G(y)$$

and $|S(P_n, y)| \leqq (b-a)F(y)$. Now

$$\int_Y S(P_n, y)d\mu = \sum_{k=1}^{n} \int_Y f(\xi_k, y)d\mu\Delta x_k = \sum_{k=1}^{n} H(\xi_k)\Delta x_k.$$

By Lebesgue's convergence theorem

$$\lim_{n\to\infty} \sum_{k=1}^{n} H(\xi_k)\Delta x_k = \lim_{n\to\infty} \int_Y S(P_n, y)d\mu = \int_Y G(y)d\mu,$$

and this is true for every sequence $\{P_n\}$ for which $\|P_n\| \to 0$. By the lemma, $H(x)$ is therefore Riemann-integrable and the above equality asserts

$$\int_a^b H(x)dx = \int_Y G(y)d\mu.$$

From the theorem we have the following

COROLLARY. *If $f(x, y)$ is bounded, is Riemann-integrable in x on $[a, b]$ for each y in $[\alpha, \beta]$, and is Riemann-integrable in y on $[\alpha, \beta]$ for each x in $[a, b]$, then*

$$G(y) = \int_a^b f(x, y)dx, \qquad H(x) = \int_\alpha^\beta f(x, y)dy$$

are Riemann-integrable in their respective intervals and $\int_\alpha^\beta G(y)dy = \int_a^b H(x)dx$.

The proof follows as a consequence of the fact that a Riemann-integrable function is Lebesgue measurable.

The theorem can be extended to functions of several variables, e.g., if $f(x, y, z)$ is Riemann-integrable in x and z separately for almost all y in Y, if $f(x, y, z)$ is measurable in y for every x and z in $[a, b]$, and if $|f(x, y, z)| \leqq F(y)$, where $F(y)$ is μ-integrable on Y, then in the repeated integral

$$\int_x \int_Y \int_z f(x, y, z)dzd\mu dx$$

the integration may be taken in any order. For, by the theorem, Y and z integration may be interchanged. To show that integration in any order is valid it is necessary to show that x and Y integration can be interchanged. Now

$$h(x, y) = \int_z f(x, y, z)dz$$

is integrable over Y from the theorem with x treated as a parameter, and is Riemann-integrable in x from the corollary with y treated as a parameter. The result now follows from the application of the theorem to $h(x, y)$.

Reference

1. Thomas M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1957, p. 221.

SOME REMARKS ON THE CLOSURE OPERATOR IN TOPOLOGICAL SPACES

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In [1], the following exercise occurs: In a topological space X, $c(O)=c(O\cap E)$ for every O open in X and every E dense in X, c denoting the closure operator. It is the intent of this note to give two characterization theorems, both of which contain the above exercise as a special case. Int will denote the interior operator and $\mathcal{C}$ the complement operator.

DEFINITION. *A subset A of a topological space X will be termed semi-open iff there exists an open set O such that $O\subset A\subset c(O)$.*

It is easy to see that A is semi-open iff $A\subset c(\text{Int } A)$. An open set is trivially semi-open.

THEOREM 1. *Let E be a subset of a topological space X. Then E is dense iff for every O open in X, $c(O)=c(O\cap E)$.*

Proof. The necessity is the exercise mentioned above. To show the sufficiency, suppose that E is not dense. Then there exists a nonempty open set O^* such that $O^*\cap E=\emptyset$. Then $\emptyset\neq c(O^*)=c(O^*\cap E)=c(\emptyset)=\emptyset$, a contradiction.

THEOREM 2. *Let A be a subset of a topological space X. Then A is semi-open iff for every set E dense in X, $c(A)=c(A\cap E)$.*

Proof. Necessity. Let A be semi-open and E dense in X. Then $A\subset c(\text{Int } A)$ and hence $c(A)\subset c(\text{Int } A)=c(E\cap\text{Int } A)$ (Theorem 1) $\subset c(E\cap A)\subset c(A)$. It follows then that $c(A)=c(E\cap A)$. *Sufficiency.* Let $E^*=\text{Int } A\cup\{\mathcal{C}c(\text{Int } A)\}\cap\mathcal{C}A$. Assume for the moment that E^* is dense. Then $A\subset c(A)=c(A\cap E^*)$ $=c(A\cap[\text{Int } A\cup\{\mathcal{C}c(\text{Int } A)\}\cap\mathcal{C}A])=c(\text{Int } A)$ and thus $A\subset c(\text{Int } A)$. It suffices then to show that E^* is dense in X. Deny. There exists then an open set $O^*\neq\emptyset$ such that $O^*\cap E^*=\emptyset$. Then (1) $O^*\cap\text{Int } A=\emptyset$ and (2) $O^*\cap\{\mathcal{C}c(\text{Int } A)\}$ $\cap\mathcal{C}A=\emptyset$. But (1) implies that $O^*\cap c(\text{Int } A)=\emptyset$ and thus $O^*\subset\mathcal{C}c(\text{Int } A)$. (2) then implies that $\emptyset=O^*\cap\{\mathcal{C}c(\text{Int } A)\}\cap\mathcal{C}A=O^*\cap\mathcal{C}A$. Thus $\emptyset\neq O^*\subset A$ and $\emptyset\neq O^*=O^*\cap\text{Int } A$ which contradicts (1).

COROLLARY. *Let E be a subset of a topological space X. Then E is dense iff for every semi-open set A in X, $c(A)=c(A\cap E)$.*

Proof. The sufficiency follows from Theorem 1 and the fact that an open set is semi-open. The necessity follows from Theorem 2.

Reference

1. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955, p. 57.

and he is waiting for the chance to use them. I am further convinced that teaching procedures that help a student into mathematics in the way described above are more likely to exhibit mathematics in all its beauty and that students respond to and appreciate this beauty. Finally, I am convinced that this is the way to produce for both the student and the teacher the real enjoyment that mathematics can afford, that people hear so much about.

I have set before you a task that cannot be accomplished overnight. Indeed I am fairly certain that it cannot be accomplished completely in 10 years and by that time who knows what else will have been discovered. But at least it is a task that can be accomplished and that is worth the doing, and each new effort made toward it seems likely to produce enough rewards to make the effort worthwhile.

As time passes our business man finds that his role in the house is changing. He no longer helps with dishes—the maid has a machine for that. The tool box in the basement (he has always been handy with tools) gathers dust, for the butler is a handy man and has new tools that repair magically the things that fall apart. He has more time, and gives some to his business (which does improve rapidly) for he soon finds new markets for his goods. The rest he spends in designing and building homes for his children who will be needing them soon to raise families of their own. He tries to build them so that they will not have to experience the shock, the divorce from reality, that has just victimized him.

THE THIRD INTERNATIONAL MATHEMATICAL OLYMPIAD FOR STUDENTS OF EUROPEAN COMMUNIST COUNTRIES

IZAAK WIRSZUP, University of Chicago

Teams from Bulgaria, Czechoslovakia, East Germany, Hungary, Poland, and Rumania participated in the Third International Mathematical Olympiad [1, 2, 4], held in Hungary from July 8 to July 16, 1961. The Soviet Union, which had sent no delegation to the Second International Mathematical Olympiad [3], was not represented in this Olympiad either. Poland returned this year, after having missed the previous year's contest. Albania has not been represented at any of the Olympiads.

Each team was made up of eight secondary school students who had been winners in their country's national mathematical Olympiad, and was led by two mathematics educators.

This Olympiad, sponsored by the Hungarian Ministry of Culture, Education, and Higher Schools, was conducted by the Janos Bolyai Mathematical Society of Hungary. Professor Janos Surányi, Secretary General of the Society, was chairman of the International Committee of the Olympiad, which was the official body for this international competition and which comprised one leader from each team.

Several weeks before the Olympiad opened, each country submitted from seven to ten problems. The sponsors of the Olympiad made a preliminary selec-

tion of the most suitable problems, and then, two days before the teams arrived in Budapest, the International Committee of the Olympiad met and selected the actual problems to be used, formulated the official texts of these problems, translated them into six languages, and set the rules for grading the solutions. As there were six problems needed for the contest and six participating countries, the Committee decided to choose one problem from each country. The first day's contest consisted of three problems: one in algebra, one in arithmetic, and one in trigonometry; three geometry problems constituted the second day's contest. The maximum score, attainable by solving all problems correctly, was 40. The use of handbooks, tables, notes, or any other aid materials was forbidden in the contest.

The problems of the Olympiad are given below. The country that submitted the problem and the maximum score for its solution appear in parentheses.

First day of the Olympiad. (Algebra, arithmetic, trigonometry)

1. Solve the system of equations

$$\begin{aligned} x + y + z &= a, \\ x^2 + y^2 + z^2 &= b^2, \\ xy &= z^2, \end{aligned}$$

where a and b are given numbers.

Determine the conditions the numbers a and b must satisfy so that each of the numbers x, y, z, constituting the solutions of the system, is positive, and x, y, and z are distinct. (Hungary, 6 points)

2. Given the lengths a, b, c of the sides of a triangle. Let S be the area of that triangle. Prove that

$$a^2 + b^2 + c^2 \geqq 4S\sqrt{3}.$$

Under what conditions does equality hold? (Poland, 7 points)

3. Solve the equation

$$\cos^n x - \sin^n x = 1,$$

where n denotes a given arbitrary natural number. (Bulgaria, 7 points)

Second day of the Olympiad. (Geometry)

1. Given the triangle $P_1P_2P_3$ and an arbitrary point P in the interior of the triangle. The straight lines P_1P, P_2P, P_3P intersect the opposite sides of the triangle at the points Q_1, Q_2, Q_3, respectively.

Prove that among the ratios

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

there is at least one not greater than 2, and at least one not less than 2.

(East Germany, 6 points)

2. Construct the triangle ABC, given the sides $AC=b$, $AB=c$, and the angle $AMB=\omega$, where M is the midpoint of the side BC, and ω is an angle less than 90°.

Prove that the problem is solvable if and only if

$$b \tan \frac{\omega}{2} \leqq c < b.$$

When does equality hold? (Czechoslovakia, 7 points)

3. Given the plane E and three noncollinear points A, B, C situated on the same side of the plane E so that the plane determined by the points A, B, C is not parallel to the plane E. Let A', B', C' be three arbitrary points of the plane E; let L, M, N denote the midpoints of the segments AA', BB', CC', respectively, and let S denote the center of gravity of the triangle LMN. (Positions of the points A', B', C' on the plane E, for which the points L, M, N are not the vertices of a triangle, are not considered here.)

Find the locus of the points S when the points A', B', C' move in the plane E independently of each other. (Rumania, 7 points)

The contests took place on July 10 and 11 at the Chemical University in Veszprém near Lake Balaton. Most of the contestants used the full four hours allotted for each day's problems. Both team leaders corrected the solutions submitted by the contestants of their own country, and proposed point-scores. Then, at a meeting lasting several hours, the International Committee discussed the scores proposed by the team leaders, voted on the final point-scores, and selected prize-winners and students deserving honorable mention. All decisions were reported as unanimous [4].

At a banquet held on July 14 in Budapest the well-known Hungarian mathematician G. Alexits distributed the prizes and certificates to the successful contestants.

RESULTS OF THE OLYMPIAD

Country	Prizes			Honorable Mention
	I 40–37 points	II 36–34 points	III 33–30 points	29–20 points
Hungary	2	3	1	2
Poland	1			6
Rumania		1	1	4
Czechoslovakia			1	3
East Germany			1	3
Bulgaria				1

Of the 48 Olympiad participants, one Hungarian student (who also received one of the first prizes at the Second International Mathematical Olympiad [4]) achieved a perfect score of 40 points, a Polish student scored 39, and another Hungarian scored 37.

Hungary, whose team won in the Olympiad in 1960, again demonstrated her superiority over the teams of the other communist countries in the 1961 competitions. The Polish team, which placed sixth in the First Olympiad, stood second in the Third. Czechoslovakia did rather poorly, and Bulgaria was last.

East Germany, which was last in the first two International Olympiads [3], improved considerably in the 1961 Olympiad after taking measures to prepare systematically and intensively. For the first time the East Germans organized elimination competitions for students of grade 12. These were held in connection with the First (East) Berlin Mathematical Olympiad in 1961. The German team in the Third Olympiad consisted of the top eight contestants in these competitions. Several weeks before their departure, the eight students were told that they were to be delegates to Budapest and received suggestions on how to prepare for the Olympiad. From July 3 through July 7, 1961, the team met in Berlin for special training through lectures and exercises. They discussed problems in detail, with particular attention to the problems of previous International Olympiads, and reviewed topics that had been useful in the competitions. The lectures and exercises were conducted by Professor Lilly Görke of the Pedagogical Department of Humboldt University and by several other experienced mathematics teachers.

The results achieved by the East German team at the Third International Olympiad indicate that they have approached the level of performance of the other participating countries.

This table, comparing the results of the East Germans at the last two Olympiads, shows the improvement:

Olympiad	Number of prizes	Number of certificates	Top score achieved	Total number of points scored by team
Second	—	1	16 (out of 45)	38
Third	1	3	31 (out of 40)	146

Nevertheless, the German delegation leaders, who compare the International Mathematical Olympiads to sports competitions, in which championships are achieved only after prolonged special training, asserted [1] that their team spent too short a time in preparation. They stated that their contestants often saw the main ideas of the problems, but could not formulate their

thoughts on paper with the rigor, completeness, and accuracy required at the competitions. The German team leaders believe that these qualities can be acquired with intense training and exercise but cannot be sufficiently mastered during the regular school instruction program. They proposed expanding and organizing the various extracurricular activities in mathematics, with more extra training for the members of the German delegation to the next International Olympiad.

The Fourth International Mathematical Olympiad was scheduled for July, 1962, in Czechoslovakia, as part of the centennial celebration of that country's Society of Mathematicians and Physicists (JČMF).

This is part of a Survey of Recent East European Literature in Elementary, Secondary, and College Mathematics, a project conducted by A. L. Putnam and I. Wirszup, Department of Mathematics, The University of Chicago, under a grant from the National Science Foundation.

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LETTERS FOR STREWING (A MODEST BUT URGENT PROPOSAL)

DAVID SALSBURG, Trinity College, Hartford

At the eighth annual meeting of the Northeastern Section of the Association, William Duren of the University of Virginia presented a paper on the preparation of college teachers of mathematics. In it, he noted that preliminary estimates indicate

(a) almost two thirds of the full-time faculty members in mathematics hired by American colleges last year had no degrees beyond their B.A., and that

(b) only 5 per cent of those hired consider themselves to be actively pursuing an advanced degree.

I am a member of the intersection of these two sets and am thus concerned with the implications of his talk. His figures would seem to imply that while the respectable mathematical community has been arguing over the lessening of standards involved in the development of lesser degrees, the enemy has slipped into the camp. An academic law of supply and demand is taking hold of the American college system, and for want of formally qualified people, various

If Professor Duren's preliminary figures prove correct, there is an urgency involved. Something must be done in the next year or two, or standards will be so muddled by a flood of incompetence that it may take years for the American college system to recover from it. This proposal is offered because I feel that an exam and certification is something that can be quickly organized and accepted with a minimum of hesitation by most mathematicians. Even if, in the beginning, it fails to stem this flood, the very act of establishing national standards can have only a beneficial effect. If alternative degrees are then offered by institutions, they can be measured in terms of such standards.

CREATIVE TEXTBOOKS

WILLIAM EDWARD CHRISTILLES, St. Mary's University, San Antonio, Texas

A definite aid in the teaching of mathematics would be an increase in the number of available textbooks that demonstrate the actual manner in which the subject matter was developed. We have too long been satisfied with the traditional development standardized by the majority of our textbook writers.

If we are to produce creative, productive mathematicians we must teach them in a manner that clearly illustrates the means by which mathematics is discovered. Unfortunately, it is easier to write a well-organized textbook emphasizing the so-called natural development of the subject matter. This natural development rarely follows the actual order in which the mathematics was developed and, more significantly, usually de-emphasizes the initial inductive means used in obtaining the proofs of the theorems contained therein.

Writing new textbooks that stress the creative nature of mathematics is not a simple task. Those mathematicians responsible for the majority of the new developments in mathematics are not always willing, unfortunately, to take the time to write the high quality textbooks needed, yet they are best qualified to do so. Many times, in the presentations of their work to the mathematical community, research mathematicians present only the final conclusions of their work which yield little insight into the means by which they obtained their results. This makes it almost impossible for anyone except those men themselves and those closely associated with them to write a really effective textbook clearly indicating the creative nature of their work.

Great stress must be laid, nevertheless, upon the writing of more and better textbooks that emphasize the creative nature of mathematics. We simply do not have enough really productive mathematicians to fill the needs for teaching our many creative students. The development and use of adequate textbooks to aid in supplementing this lack of manpower is one realistic alternative.

Already new and improved means of presenting mathematics to our students in the classroom and in textbooks through clearer and more modern techniques are proving successful. We must not allow ourselves, however, to be content with the production of mathematicians who are only vast storehouses of knowledge, yet lack the creative ability necessary for a fully productive career. We must

continue to develop well-rounded mathematicians and at the same time embody in their professional development strong emphasis upon productivity.

The time is past when mathematicians can afford the luxury of pet ideals and isolationism. We must unite and work together for the betterment of our profession. The oustanding creative mathematicians must aid or at least help supervise the writing of better textbooks that stress the true nature of mathematics. With the help of such men we will be able to produce more highly qualified creative mathematicians. The writing of textbooks emphasizing creativity, although only one suggested avenue of improvement, is necessary if we are to preserve the high standards set by our profession.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
Collaborating Editor, C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine.

PROBLEMS FOR SOLUTION

E 1591. *Proposed by Frank Hawthorne, Education Department, Albany, N. Y.*

By how many different paths following the king's move in chess can the sentence "Able was I ere I saw Elba" be traced in the accompanying diagram?

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
A	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A
A	B	L	L	L	L	L	L	L	L	L	L	L	L	L	L	L	B	A
A	B	L	E	E	E	E	E	E	E	E	E	E	E	E	E	L	B	A
A	B	L	E	W	W	W	W	W	W	W	W	W	W	W	E	L	B	A
A	B	L	E	W	A	A	A	A	A	A	A	A	A	W	E	L	B	A
A	B	L	E	W	A	S	S	S	S	S	S	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	I	I	I	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	E	E	E	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	E	R	E	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	E	E	E	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	I	I	I	I	I	S	A	W	E	L	B	A
A	B	L	E	W	A	S	S	S	S	S	S	S	A	W	E	L	B	A
A	B	L	E	W	A	A	A	A	A	A	A	A	A	W	E	L	B	A
A	B	L	E	W	W	W	W	W	W	W	W	W	W	W	E	L	B	A
A	B	L	E	E	E	E	E	E	E	E	E	E	E	E	E	L	B	A
A	B	L	L	L	L	L	L	L	L	L	L	L	L	L	L	L	B	A
A	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A
A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

E 1592. *Proposed by Mira Bhargava, McGill University*

Prove that the union of two subgroups of a group is itself a subgroup if and only if one contains the other.

E 1593. *Proposed by Reuben Hersh, Stanford University*

If $P(x)$ and $Q(x)$ are arbitrary polynomials, k an arbitrary constant, and D the operator d/dx, prove that

$$P(D)[e^{kx}Q(x)]\,|_{x=0} = Q(D)[P(x)]\,|_{x=k}.$$

E 1594. *Proposed by Walter Penney, Navy Department, Washington, D. C.*

Triangle ABC is a scalene triangle with three acute angles, integral sides, and $\sin A = 5/13$. What is the minimum possible perimeter?

E 1595. *Proposed by R. V. Moody, Toronto, Ontario*

Let ABC be any (nondegenerate) triangle and let the lines parallel to its sides and one-fourth the way from each side to the opposite vertex determine the triangle DEF. Let $A'B'C'$ be any triangle circumscribed about triangle DEF. Prove that at least one vertex of triangle $A'B'C'$ must lie on or within triangle ABC.

E 1596. *Proposed by Carl Evans, Cornell Aeronautical Laboratory, Inc., Buffalo, New York*

Let P_n be a convex polygon of n sides with no four vertices concyclic. Call a triangle ABC whose vertices are among those of P_n a *covering triangle* of P_n if the circumcircle of ABC contains the remaining $n-3$ vertices of P_n in its interior. Show that there are exactly $n-2$ covering triangles of P_n and that they constitute a dissection of P_n by $n-3$ diagonals.

E 1597. *Proposed by Ralph Greenberg, University of Pennsylvania*

Let $E_1(x_1) = x_1$, $E_2(x_1, x_2) = [E_1(x_1)]^{x_2}, \cdots, E_n(x_1, x_2, \cdots, x_n) = [E_{n-1}(x_1, x_2, \cdots, x_{n-1})]^{x_n}$, and let $a_n > a_{n-1} > \cdots > a_1 > e$. Which permutation of the a_i's maximizes $E_n(x_1, x_2, \cdots, x_n)$?

E 1598. *Proposed by John Rainwater, University of British Columbia*

Let m be a positive integer and define the real polynomials $f(x)$ and $g(x)$ by $(1+ix)^m = f(x)+ig(x)$. Prove that for arbitrary real numbers a and b, the polynomial $af(x)+bg(x)$ has only real roots.

E 1599. *Proposed by David Friedman, University of California at Berkeley*

Let

$$f_0(x) = c^x, \qquad f_n(x) = c^x \prod_{i=0}^{n-1} (ax + bi) \qquad \text{for } n = 1, 2, \cdots,$$

where $c>0$ and a, b are arbitrary real numbers. Show that

$$f_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} f_i(x) f_{n-i}(y).$$

E 1600. *Proposed by J. E. Schneider, Franklin and Marshall College*

Evaluate the determinant

$$\begin{vmatrix} \binom{0}{0} & \binom{1}{1} & \cdots & \binom{n}{n} \\ \binom{1}{0} & \binom{2}{1} & \cdots & \binom{n+1}{n} \\ \cdot & \cdot & \cdots & \cdot \\ \binom{n}{0} & \binom{n+1}{1} & \cdots & \binom{2n}{0} \end{vmatrix}.$$

SOLUTIONS

Six Discs

E 1533 [1962, 808]. *Proposed by D. J. Newman and W. E. Weissblum, Yeshiva University*

Six circular areas are given in the plane with the property that none contains the center of any other. Prove that they have no point in common.

Solution by E. L. Magnuson, HRB-Singer, Inc., State College, Pa. Assume, on the contrary, that the circular areas have a point in common. Consider the six spokes connecting this point with each center. Two spokes have an included angle not exceeding 60°. The center associated with the shorter (or equal length) spoke then lies within the circular area whose center is associated with the other spoke. This contradicts our initial assumption.

Also solved by J. W. Baldwin, David Benfield and R. A. Derrig (jointly), Walter Bluger, F. P. Callahan, Jr., H. L. Chow, D. I. A. Cohen, Dennis Couzin, Michael Goldberg, Ralph Greenberg, S. H. Greene, Kit Hanes, Ned Harrell, R. A. Jacobson, John Jean, Jr., C. F. Marion, D. C. B. Marsh, R. W. Moses, Jr., P. N. Muller, C. S. Ogilvy, S. J. Ryan, J. V. Ryff, E. L. Spitznagel, Jr., B. R. Toskey, Andy Vince, R. J. Wisner, Dale Woods, and the proposers.

Ryff located the problem in R. Nevanlinna, *Eindeutige Analytische Functionen* (Springer, Zweite Auflage), Sec. 5, Chap. 5, p. 145.

Hopeless or Immaterial

E 1534 [1962, 808]. *Proposed by J. H. Edmonston, Federal Power Commission, Washington, D. C.*

A circular park is planned, with center O and radius r. Construction of radial and concentric paths is planned. Heavy pedestrian traffic is anticipated joining

two points A and B on the circumference. It is desired to construct radial paths AO, BO, and a concentric path with radius r', meeting AO and BO at A' and B', respectively, such that the composite path

$$p = \overline{AA'} + \widehat{A'B'} + \overline{B'B}$$

is minimized (so as partly to eliminate motivation for walking on the grass). Determine r'.

Solution by J. E. Wilkins, Jr., General Dynamics Corporation, San Diego, California. Let θ be the angle subtended at O by the arc AB. Then

$$p = 2(r - r') + r'\theta = 2r + (\theta - 2)r'.$$

For minimum p, we therefore should select r' as r if $\theta < 2$ and as 0 if $\theta > 2$, while when $\theta = 2$ the choice of r' is immaterial.

Also solved by J. W. Baldwin, Donald Barnes, Merrill Barnebey, Joseph Beer, Ralph Bennett, Walter Bluger, Robert Bowen, Brother L. F. Zirkel, R. E. Cassidy, D. I. A. Cohen, Gus Di Antonio, L. F. Epstein, Jane Evans, Michael Goldberg, S. H. Greene, Ned Harrell, H. A. Heckart, H. W. Hickey, A. R. Hyde, R. A. Jacobson, John Jean, Jr., M. S. Klamkin, D. C. B. Marsh, P. N. Muller, J. C. Nichols, C. S. Ogilvy, Stanton Philipp, J. L. Pietenpol, S. J. Ryan, C. M. Sandwick, Sr., Perry Scheinok, E. M. Scheuer, A. L. Schreiber, Wayne Shepherdson, E. L. Spitznagel, Jr., B. R. Toskey, Dennis Travis, Gary Venter, Andy Vince, Julius Vogel, K. S. Williams, B. B. Winter, David Zeitlin, and the proposer. Many of these solutions were incomplete.

Sum of the Elements of a Matrix

E 1535 [1962, 809]. *Proposed by S. W. Golomb and A. W. Hales, Jet Propulsion Laboratory, California Institute of Technology*

An $n \times n$ array (matrix) of nonnegative integers has the property that for any zero entry, the sum of the row plus the sum of the column containing that entry is at least n. Show that the sum of all elements of the array is at least $n^2/2$.

Solution by Bruce Blum, Applied Physics Laboratory, Silver Spring, Maryland. We employ induction. The case for $n = 1$ is obvious. Assume the desired condition is true for $n = k - 1$ and consider a $k \times k$ matrix. If there are no zero entries the condition is satisfied. If $a_{ij} = 0$, then the sum of row i and column j is at least k, and the sum of the elements of the $(k-1)\times(k-1)$ submatrix obtained by deleting row i and column j is at least $(k-1)^2/2$, giving a total sum of at least $(k^2+1)/2$.

Also solved by Joseph Beer, L. W. Beineke, D. G. Beverage, Brother L. F. Zirkel, H. L. Chow, D. I. A. Cohen, Jane Evans, J. C. Hennessey, C. F. Marion, D. C. B. Marsh, P. N. Muller, S. J. Ryan, H. J. de St. Germain, B. R. Toskey, Andy Vince, and the proposers. Several of these solutions were faulty.

Rectangular Hyperbolas

E 1536 [1962, 809]. *Proposed by R. L. Syverson, University of Minnesota*

Find functions $y=f(x)\in C'$ such that $x,\ y>0$, $y'\neq 0$, and $dY/dX=dy/dx$, where X and Y exist at each point satisfying $y=f(x)$ and are respectively the x- and y-intercepts of the tangent to the function.

Solution by R. J. Cormier, University of Missouri. Since $X=x-y/y'$ and $Y=y-y'x$, we have (assuming $y''\neq 0$) $dY/dX=-xy'^2/y$. Setting this equal to y', we get $xy'+y=0$, the solutions of which are $xy=k$, $k>0$.

Also solved by Joseph Beer, Michael Goldberg, Ralph Greenberg, Erwin Just and Norman Schaumberger (jointly), D. C. B. Marsh, Stanton Philipp, J. L. Pietenpol, Perry Scheinok, B. R. Toskey, Andy Vince, Charles Wexler, J. E. Wilkins, Jr., and the proposer.

Editorial Note. We have $dY/dX=dy/dx=-Y/X$, whence $XY=C$. But it is known that the curves having this property are the rectangular hyperbolas $xy=k$. To satisfy the conditions of the problem we must take $k>0$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Collaborating Editors: L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; and A. WILANSKY, Lehigh University.

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editors. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5101. *Proposed by K. Mahler, The University, Manchester, England*

If m and n are algebraic numbers satisfying $m^n=n^m$, $mn(n-1)(m-1)(m-n)\neq 0$, prove that two coprime rational integers h and k exist such that

$$m=(h/k)^{k/(h-k)},\qquad n=(k/h)^{h/(k-h)},\qquad hk(h-k)\neq 0.$$

If, in addition, m and n are algebraic integers, show that h or k may be chosen equal to 1.

5102. *Proposed by R. P. Boas, Jr., and W. R. Mann, Northwestern University*

The equation $\sin x=x$ clearly has no real roots other than $x=0$. Does it have any complex roots?

5103. *Proposed by D. S. Mitrinovic, Belgrade, Yugoslavia*

Determine integers a, b, c, d such that the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^k = \begin{pmatrix} A(k) & B(k) \\ C(k) & D(k) \end{pmatrix}$$

holds for all positive integral k if A, B, C, D are polynomials in k. Can the result be generalized for matrices of order n?

5104. *Proposed by Paul Sally, Jr., Boston College*

Consider the lattice L of points in E_n having integer valued coordinates. We define a *line* in L to be a line in E_n which contains points of L and is parallel to one of the coordinate axes. A set $S \subset L$ is called *admissible* if it is bounded, and if every line which intersects S contains at least two points of S. A set $N \subset L$ is called *null* if every line which intersects N contains an even number of points of N. Clearly every null set is an admissible set.

Prove (or disprove): Every nonempty admissible set contains a nonempty subset which is a null set.

5105. *Proposed by Harley Flanders, Purdue University*

Let A be a ring with unit element and M a unital A-module. Suppose that for each submodule N satisfying $0 < N$, there is a submodule P such that $M = N + P$, $P < M$. Is M then semi-simple? (See Bourbaki, *Algèbre*, Ch. 8, p. 36, exercise 4.)

5106. *Proposed by Fred Suvorov, Princeton, N. J.*

Let S be the projective plane over the complex numbers. Show that any finite set of points in S can be obtained as the complete intersection of two (possibly reducible) algebraic curves.

5107. *Proposed by Andrew Zachariou, Athens, Greece*

Find the conditions under which the group of automorphisms of a cyclic group of finite order is itself a cyclic group.

5108. *Proposed by Albert Wilansky, Lehigh University*

Give an example of two Banach spaces X, Y such that the set $X \cap Y$ contains a sequence B which is a basis for each space but not a basis for $X \cap Y$ as a normed space with the norm $p_X + p_Y$; p_X, p_Y being the norms of X, Y.

5109. *Proposed by M. S. Klamkin, State University of New York at Buffalo, and A. L. Tritter, Data Processing, Inc.*

Given the infinite permutation

$$P \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \cdots \\ 1 & 2 & 4 & 3 & 5 & 7 & 6 & 8 & 10 & 12 \cdots \end{pmatrix}$$

where the second row is formed by taking in order from the natural numbers, 1 odd, 2 even, 3 odd, $\cdots$, $2n$ even, $2n+1$ odd, $\cdots$. What is the cycle structure of this permutation?

5110. *Proposed by L. Carlitz, Duke University*

Show that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(\frac{\cos\theta}{x\cos\theta+i\sin\theta}\right)^{\nu}d\theta=\begin{cases}\pi(x+1)^{-\nu} & (|1-x|<1)\\ \pi(x-1)^{-\nu} & (|1+x|<1),\end{cases}$$

where $R(\nu)>-1$.

SOLUTIONS

An Uncovered Irrational

5019 [1962, 316]. *Proposed by W. M. McKeeman, Stanford University*

Consider the interval [0, 1] and a denumerable ordering of the rationals in it. Cover the nth rational with an interval of length 2^{-n-1} with center at the rational. Then the total length covered is less than or equal to $\frac{1}{2}$ and there are numbers not covered. Find one.

Solution by J. H. van Lint, Technical University, Eindhoven, Netherlands. We give an example in the case of the familiar ordering by increasing denominators: $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \cdots$, with reducible fractions omitted. If p/q is the nth rational in this ordering and x is in the corresponding covering interval, then $|x-p/q| < 2^{-n-2}$. We have $n>1+\sum_{k<q}\phi(k)=3q^2/\pi^2+O(q\log q)$, in fact

$$n+2>q^2/4 \tag{1}$$

for all q (Hardy & Wright, *Theory of Numbers*, Th. 330). We now show that $\alpha=2^{-1/2}$ is not covered.

For every rational p/q we have the result, due to Liouville,

$$\left|\alpha-\frac{p}{q}\right|>1/4q^2, \tag{2}$$

(see Leveque, *Topics in Number Theory*, v. II, Th. 4.1.) As $1/4q^2>2^{-q^2/4}$ for $q\geqq 6$ we see, using (1) and (2), that α is not covered by any interval corresponding to a rational with denominator larger than 5. Of the remaining numbers only $\frac{2}{3}$ and $\frac{3}{4}$ are of interest ($\alpha=0.707\cdots$). It is easily checked that the corresponding intervals do not cover α, which completes the proof.

Also solved by L. K. Arnold, Robert Bowen, Robert Breusch, E. J. Burr, A. E. Dent, W. B. Easton, R. Hines, R. A. Jacobson, Erwin Just, John McGuire, J. G. Mauldon, Robert Mitchell, Jim Morrow, D. B. Parker, Stephen Plafker, Dennis Travis, A. Zabrodsky, and the proposer.

Editorial Note. The proposer shows that $\sqrt{2}-1$ is uncovered by all the intervals associated with the same ordering of the rationals. Other solvers either construct a special ordering which leaves a specified irrational uncovered, or give a constructive method by which an uncovered irrational can be determined, usually involving infinite processes, for an arbitrary ordering.

A Diophantine Equation

5020 [1962, 316]. *Proposed by J. A. H. Hunter, Toronto, Canada*

(1) Show that the Diophantine equation $(x^2-e)(z^2-e)=(y^2-e)^2$ has infinitely many integral solutions for each value of e such that $p^2-2q^2=e$ has integral solutions.

(2) The quartic equation in (1) possesses also other solutions (e.g., $x=2$, $y=8$, $z=22$, $e=-6$). Determine the set $\{e\}$ for which solutions exist.

Solution by R. Venkatchalam Iyer, Trivandrum, India. Let

$$p^2 - 2q^2 = e.$$

Multiplying this repeatedly by $(1^2-2\cdot 1^2)=-1$ we have

$$(p+2q)^2 - 2(p+q)^2 = -e, \qquad (3p+4q)^2 - 2(2p+3q)^2 = e,$$

and so on. Therefore

$$\begin{aligned}(p^2-e)\{(3p+4q)^2-e\} &= 2q^2\cdot 2(2p+3q)^2 = \{2q(2p+3q)\}^2 \\ &= \{(p+2q)^2-(p^2-2q^2)\}^2 = \{(p+2q)^2-e\}^2.\end{aligned}$$

Hence, each solution of $p^2-2q^2=e$ provides infinitely many integral solutions of the given equation as required in (1).

Also solved (to the same extent) by the proposer.

Editorial Note. Iyer goes on to obtain certain classes of values of e for which x, y, z exist (e.g. $e=a^2+2a-1$, $-3a^2-2a\pm 1$, $2-y^2$) but no general characteristics of the set $\{e\}$ result. Of course we can choose tentatively x, y, z at our pleasure and then, from the original equation, we have $e=(y^4-x^2z^2)/(2y^2-x^2-z^2)$. Now observe that if $(x, y, z; e)$ is a solution, so also is $(kx, ky, kz; k^2e)$. If e is not an integer at first, the solution can be transformed so that the new value of e is integral. The first values observed are: $e=0$, ± 1, ± 2, ± 4, -5, -6, ± 7, ± 9, ± 11, 13, ± 14, -15, ± 17. No rule appears for distinguishing satisfactory values of e beforehand, nor is there any general relation, as in (1) by which further solutions can be obtained from a given solution with a fixed e. Thus the question is still open and comments will be welcome.

Property of a Banach Space

5021 [1962, 317]. *Proposed by W. A. J. Luxemburg and A. C. Zaanen, California Institute of Technology*

If B is a Banach space (i.e., a complete normed linear space) and B^* its adjoint space, then it is a known theorem that a subset of B^* is weak*-compact (i.e., $\sigma(B^*, B)$-compact) if and only if the subset is weak*-closed and bounded in norm.

(1) Show by means of an example that this theorem becomes false if the hypothesis that B is a Banach space is replaced by the hypothesis that B is a normed linear space.

(2) Show that if B is an arbitrary normed linear space, then a weak*-compact subset of B^* is bounded in norm if and only if the weak*-closure of its convex hull is weak*-compact.

Solution by A. K. Snyder and A. Wilansky, Lehigh University.

(1) Let B be the space of all finite sequences, with $\|x\| = \sup |x_n|$. Let $f_n(x) = n|x_n|$. Then $\{f_n\} \cup \{0\}$ is compact since $f_n \to 0$ in the weak* topology. But $\|f_n\| = n$.

(2) THEOREM. *Let B be a normed space and S a weak*-compact subset of B^* which is either bounded in norm or convex. Then S^0 is a neighborhood of* 0 *in B.*

In the first case, the assertion is trivial, using only norm-boundedness of S.

Now in the case that S is convex, S^0 is a $\tau(B, B^*)$ neighborhood of 0 by definition, hence a norm neighborhood by [1]. Q.E.D.

To solve the problem, let S be weak*-compact and norm bounded. Then by (2) and the Tychonoff-Alaoglu theorem S^{00} is weak*-compact. But S^{00} is the convex weak*-closure of S.

Conversely, let S be weak*-compact and convex, without loss of generality. By (2), S^0 is a neighborhood of 0 in B. Thus S^{00} is a disc in B^*. But $S^{00} = S$.

Reference

1. N. Bourbaki, *Espaces Vectoriels Topologiques*, Chapter 4, p. 71, Proposition 6.

Also solved by S. T. M. Ackermans, and the proposers.

Sum and Product Related to the Gamma Function

5022 [1962, 317]. *Proposed by T. S. Nanjundiah, Central College, Bangalore, India*

Let

$$H_n = \sum_{\nu=1}^{n} \frac{1}{x+\nu-1}, \qquad G_n = \prod_{\nu=1}^{n}\left(1+\frac{x-1}{\nu}\right), \qquad n = 1, 2, \cdots.$$

Prove:

$$-\frac{\Gamma'(x)}{\Gamma(x)} < H_p + H_q - H_{pq} \leqq 1/x \qquad \text{for } x \geqq 1/2. \tag{1}$$

$$\frac{1}{\Gamma(x)} \begin{matrix} > \\ < \end{matrix} \frac{G_pG_q}{G_{pq}} \begin{matrix} \geqq \\ \leqq \end{matrix} x \qquad \text{for } \begin{cases} 0 < x < 1 \\ x > 1. \end{cases} \tag{2}$$

(Problem E 819 [1949, 109] is the special case $x = 1$.)

Solution by the proposer. Write $(p, q) = H_p + H_q - H_{pq}$, $[p, q] = G_pG_q/G_{pq}$, and consider the functions:

$$h(x) = (p, q) - (p, q+1), \quad g(x) = [p, q]/[p, q+1], \quad p > 1, \quad q \geqq 1.$$

One finds

$$h(x) = \sum_{\nu=1}^{p} \frac{1}{x+pq+\nu-1} - \frac{1}{x+q},$$

$$g(x) = \prod_{\nu=1}^{p} \left(1 + \frac{x-1}{pq+\nu}\right) \Big/ \left(1 + \frac{x-1}{q+1}\right),$$

from which we have $g'(x)/g(x) = h(x)$. The roots of $h(x)=0$ are all real [1]. In fact, $p-1$ of the roots are negative being situated in the intervals $-(pq+\nu) < x < -(pq+\nu-1)$, $1 \leqq \nu < p$; the only remaining root $\xi = \xi_{p,q}$ lies in $0 < x < \frac{1}{2}$, for

$$h(0) < \frac{p}{pq} - \frac{1}{q} = 0, \qquad h(\tfrac{1}{2}) > \frac{p^2}{\sum_{\nu=1}^{p} (pq + \nu - \frac{1}{2})} - \frac{1}{q + \frac{1}{2}} = 0.$$

Using these facts and observing that $g(x) > 0$ $(x \geqq 0)$, we infer:

$$(*) \qquad \operatorname{sgn} g'(x) = \operatorname{sgn} h(x) = \operatorname{sgn}(x - \xi), \qquad x \geqq 0.$$

We go on to complete the proofs of (1) and (2).

Proof (1). (*) implies $h(x) > 0$ $(x \geqq \frac{1}{2})$. This leads to

$$\text{(1a)} \qquad (p, q) \leqq (p, 1) = H_1 = 1/x, \qquad x \geqq \tfrac{1}{2},$$

with equality only for $p=1$ or $q=1$. If we set $h_n = H_n - \log n$, we can write

$$(p, q) = h_p + h_q - h_{pq}.$$

Now we have $h_n - h_{n+1} = \log(1+1/n) - 1/(x+n)$, and it is known [2] that $(1+1/n)^{x+n} \downarrow e$ if (and only if) $x \geqq \frac{1}{2}$, so that $h_n > h_{n+1}$ $(x \geqq \frac{1}{2})$. Also $\lim_{n\to\infty} h_n = -\Gamma'(x)/\Gamma(x)$, [3]. Hence it follows that

$$\text{(1b)} \qquad (p, q) \geqq h_p > -\Gamma'(x)/\Gamma(x), \qquad x \geqq \tfrac{1}{2}.$$

Proof (2). (*) implies that $g(x)$ is decreasing in $0 \leqq x \leqq \xi$ and increasing in $x \geqq \xi$. Consequently, since $g(0) = g(1) = 1$, we have $g(x) < 1$ $(0 < x < 1)$ and $g(x) > 1$ $(x > 1)$. This leads to the inequality

$$\text{(2a)} \qquad [p, q] \underset{\leqq}{\overset{\geqq}{}} [p, 1] = G_1 = x \qquad \text{for } \begin{cases} 0 < x < 1 \\ x > 1, \end{cases}$$

with equality, in either case, only for $p=1$ or $q=1$.

Finally, setting $g_n = n^{1-x} G_n$, we have $[p, q] = g_p g_q / g_{pq}$. Now $g_n > 0$ $(x > 0)$ and we have $g_n/g_{n+1} = (1+1/n)^x/(1+x/n)$, so that a known inequality yields $g_n < g_{n+1}$ $(0 < x < 1)$ and $g_n > g_{n+1}$ $(x > 1)$. Also $\lim_{n\to\infty} g_n = 1/\Gamma(x)$. Hence it follows that

$$\text{(2b)} \qquad [p, q] \underset{\geqq}{\overset{\leqq}{}} g_p \underset{>}{\overset{<}{}} \frac{1}{\Gamma(x)} \qquad \text{for } \begin{cases} 0 < x < 1 \\ x > 1. \end{cases}$$

References

1. Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis* II, 1925, p. 41.
2. ——— I, 1925, p. 30.
3. Whittaker and Watson, *Modern Analysis*, pp. 235, 247.

Also solved by Robert Breusch, J. B. Linder, G. B. Parrish, and B. E. Rhoades.

A Set Having Precisely Two Points on Every Line

5024 [1962, 317]. *Proposed by D. A. Moran and J. J. Rotman, University of Illinois*

Prove that there exists a subset of the plane which meets every straight line in precisely two points.

Prove that such a set is 0-dimensional.

I. *Solution by W. B. Easton, Princeton University.* We show the existence of such a set by transfinite induction. Let c be the cardinal number of the continuum and let α be the smallest ordinal number with cardinal c. (Greek letters will denote ordinal numbers.) Let f be a one-one function from the set of ordinals less than α onto the set of lines in the plane (which has cardinality c).

Let $\beta<\alpha$, and suppose that a set E_η has been chosen for all $\eta<\beta$ such that E_η contains at most two points on any line and such that for $\mu<\eta$, $E_\mu\subseteq E_\eta$. We define E_β as follows: (1) If $\beta=\gamma+1$, let E_γ have n ($n=0, 1, 2$) points on the line $f(\gamma)$. Then, choose $2-n$ points on $f(\gamma)$ which are not on any line which already has two points in E_γ. This is possible since there are c points on $f(\gamma)$ and there are fewer than c pairs of points in E_γ. We now let E_β be the set obtained by adding the chosen points to E_γ. (2) If β is a limit ordinal we set $E_\beta=\bigcup_{\eta<\beta} E_\eta$.

The set E_α defined by this process has the required property, since any line $f(\gamma)$ has two points in $E_{\gamma+1}$ and hence in E_α.

To show that such a set E is totally disconnected (0-dimensional), we suppose that, on the contrary, there is a connected subset of E, say S_1, containing at least two points.

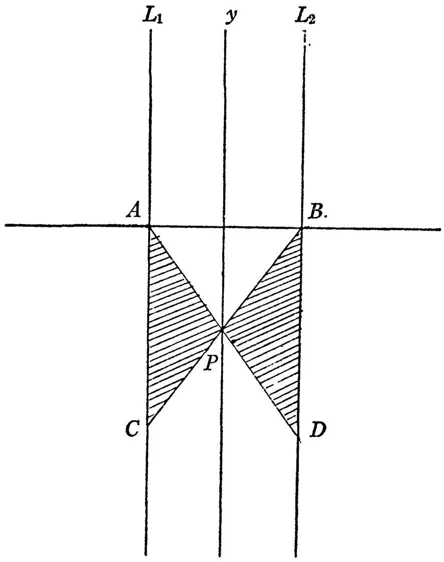
Then there are two distinct parallel lines L_1 and L_2, each of which intersects S_1. Then there is a connected subset S of S_1 such that S lies between L_1 and L_2 except for unique points A and B on L_1 and L_2, respectively.

We may assume that S lies on only one side of the line through A and B, for it is certainly the union of two such sets, at least one of which is connected.

By an affine transformation, which preserves the given property of E, we may assume that $A=(-1, 0)$, $B=(1, 0)$, L_1 is the line $x=-1$, L_2 is the line $x=1$, and S lies below the x-axis.

Since S is connected, it must intersect the y-axis. Let P be a point of S on the y-axis. We may assume that the removal of P from S would disconnect A and B. If not, we may redefine S to be the component of $S-\{P\}$ containing A and B and start over; on the second try there will be only one point of S on the y-axis, so its removal must disconnect A and B.

Now we construct the lines connecting P with A and B, as in the figure. Let the components of $S-\{P\}$ containing A and B be C_A and C_B, respectively. Any line through A intersecting the segment PB must intersect $C_A \cup \{P\}$, since the latter is connected. Hence, it cannot intersect C_B except at A. Hence, except for point A, C_A must lie outside triangle ABD. Similarly, C_B lies outside triangle ABC, except for point B. The line through B and P cannot intersect C_A, which must therefore lie inside the closed triangle APC. Similarly, C_B must lie inside the closed triangle BPD.



We have shown that S is contained in the closure of the shaded area in the figure and that, in particular, S is bounded and contains only one point on the y-axis.

Let P' be a point of C_B. Since the line through A and P' can contain no other point of C_B, removing P' would disconnect A and P. By the construction of the preceding paragraph, with obvious modifications, P' is the only point of S on the vertical line through it. The same result holds for points of C_A.

Let ϕ be the function defined on the interval $[-1, 1]$ by: $\phi(x)=y$ if $(x, y) \in S$. Then ϕ is a convex function, as may be seen through a simple indirect argument. Hence we can easily show that ϕ is continuous.

Hence ϕ attains its minimum, say at x_0. There are either one or two points of S on the line $y=\phi(x_0)$. There cannot be two since ϕ is convex. Hence there is a point P_0 of E on $y=\phi(x_0)$ which is not in S. But then a line L through P_0 can be drawn such that $(x_0, \phi(x_0))$ is on one side of it and both A and B are

on the other. Since the removal of $(x_0, \phi(x_0))$ from S would disconnect A from B we can easily see that L contains two points of S and hence three points of E. But this is impossible, so E must be 0-dimensional.

II. *Note by R. L. Wilder, University of Michigan and The Florida State University.* The present problem was solved by S. Mazurkiewicz in Comptes Rendus Soc. Sc. et Lettres de Varsovie, 7 (1914) 322–383, especially 382–383. This reference is supplied in W. Sierpinski, *Cardinal and Ordinal Numbers*, Warsaw, 1958, pp. 446–447.

Also solved by J. B. Linder.

Union of Ideals

5025 [1961, 438]. *Proposed by David Mumford, Cambridge, Mass.*

Given a commutative ring R with unity element. Prove or disprove the statement: If an ideal A is contained in the (set-theoretic) union of a finite set of ideals, then A is contained in one of these ideals.

I. *Solution by J. L. Pietenpol, Columbia University.* The statement is false. Consider the ring $\{0, 1, a_1, a_2, \cdots, a_{14}\}$ of 16 elements generated by $1, a_1, a_2, a_3$, where $1+1=0$, $a_i+a_i=0$, and where every product $a_ia_j=0$. Then the union of the four ideals $\{0, a_1, a_2, a_1+a_2\}$, $\{0, a_1, a_3, a_1+a_3\}$, $\{0, a_2, a_3, a_2+a_3\}$, $\{0, a_1+a_2+a_3\}$ is itself an ideal and is not contained in any of the four.

II. *Solution by Melvin Henriksen, Purdue University.* Let Z_2' denote the integers modulo 2 with the trivial multiplication, and let $A=Z_2' \oplus Z_2'$. Then

$$A = \{(0, 0), (1, 0)\} \cup \{(0, 0), (0, 1)\} \cup \{(0, 0), (1, 1)\}.$$

Since each of the sets in brackets is an ideal, a negative answer to the original question may be obtained by imbedding A as an ideal in a ring with unity element in the usual way.

This example is a minor modification of one due to R. E. Johnson, given in N. H. McCoy, *A note on finite unions of ideals and subgroups*, Proc. Amer. Math. Soc., 8 (1957) 633–637, where it is shown that if an ideal in any ring is contained in a finite union of ideals, then some power of it is contained in one of them.

III. *Solution by Leo Sauve, St. Patrick's College, Ottawa.* The statement is of course known to be true if the finite set consists entirely of prime ideals. (See, e.g., D. G. Northcutt, *Ideal Theory*, 1953, p. 13). But it is false if the ideals are not prime. In *Commutative Algebra* (vol. I, 1958, p. 215), O. Zariski and P. Samuel give the following counterexample: If k is a finite field and R is the residue class ring $k[X, Y]/(X^2, XY, Y^2)=k[x, y]$ (where x and y are the residues of X and Y) then the finite union $\bigcup_{a,b\in k} (ax+by)$ is the ideal (x, y).

Also solved by G. M. Bergman, A. Brumer and P. Sally, P. S. Landweber, Joe Lipman, N. H. McCoy, B. L. Osofsky, Azriel Rosenfeld, M. H. Schultz, B. R. Toskey, Dennis Travis, Seth Warner, W. C. Waterhouse, and the proposer.

Switching Circuit

5026 [1962, 438]. *Proposed by A. A. Mullin, University of Illinois*

Consider the following Boolean function of four variables

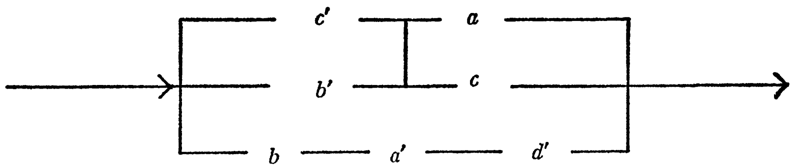
$$f(a, b, c, d) = b'(a + c) + b(ac' + a'd').$$

Show that every switching circuit realization of f contains at least seven contacts and that a circuit exists which contains precisely seven contacts.

Solution by D. K. Hodson, University of Maine. Considering the given function to be in alternational normal form, the method of W. V. Quine can be readily applied. (See Quine, *A Way to Simplify Truth Functions*, this MONTHLY, 62 (1955) 627–631.) Using the alternational symbol $\vee$ for $+$, and considering juxtaposition as conjunction, there evolves the developed normal form

$$ab'cd \vee ab'cd' \vee ab'c'd \vee ab'c'd' \vee a'b'cd \vee a'b'cd'$$
$$\vee\; abc'd \vee abc'd' \vee a'bcd' \vee a'bc'd',$$

and the prime implicants $ab' \vee b'c \vee ac' \vee a'bd' \vee a'cd \vee bc'd'$. Surveying for the least number of prime implicants which are subsumed by the developed normal form, one finds that the two switching functions (1) $ac' \vee ab' \vee a'bd'$ and (2) $b'c \vee ac' \vee a'bd'$ are the simplest equivalents of the original function. Since each contains seven contacts, and since there exist no simple equivalent switching functions (guaranteed by Quine's algorithm) every other equivalent switching function will contain at least seven variables. The graph of (1) is given below.



Also solved by Daniel Ashler, D. A. Breault, R. A. Hughes and D. L. Hobde, W. J. Wayne, and the proposer.

Editorial Note. Breault and the proposer follow the method detailed in S. H. Caldwell, *Switching Circuits and Logical Design*, Wiley, New York, 1958, 158–165.

Maximal Nonsingular Subspaces

5027 [1962, 438]. *Proposed by A. J. Goldman, National Bureau of Standards.*

Let $M_n(F)$ be the set of $n \times n$ matrices over the field F, considered as an n^2-dimensional vector space over F. Call a vector subspace of $M_n(F)$ nonsingular if all its nonzero members are nonsingular matrices. Find maximal nonsingular subspaces of $M_n(F)$.

Solution by Imanuel Marx, Purdue University. Each nonsingular subspace of $M_n(F)$ is one-dimensional, comprising the scalar multiples kA of a single

nonsingular matrix A. For suppose A and B are any two linearly independent nonsingular matrices: then AB^{-1} has at least one characteristic value λ, for which $|A-\lambda B| = |AB^{-1}-\lambda I|\ |B| = 0$. Thus each two-dimensional linear subspace generated by two linearly independent members of $M_n(F)$ contains at least one singular matrix and, a fortiori, higher-dimensional subspaces have the same property.

Also solved by the proposer.

RECENT PUBLICATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College, and E. P. VANCE, Oberlin College

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any other of the editors or officers of the Association.

Analytic Function Theory, Volume II. By Einar Hille. Ginn, Boston, 1962. 483 pp. $9.00.

This volume, a very worthy successor to the author's Volume I, contains a natural extension of its material into more advanced topics. It is not bound by style or references, however, to selected material in the first volume, and it could well be used as an independent text for initial graduate studies in the field of function theory, and closely related graduate studies in applied fields.

Chapter 10 (the first chapter of Volume II) surveys the more theoretical general methods of analytic continuation, with the power-series element basis, and with special emphasis on the algebraic permanence of functional relations. In the next chapter the methods are specialized to representation theory, in terms of infinite series and integrals. For example Leau's work is complemented by including applications of the Hausdorff and other summability methods. Applications of conformal mapping to analytic representation encourage the interested reader to survey many such methods.

The treatment of algebraic functions on Riemann surfaces extends the subject further in Chapter 12. An application of the simpler results here provides the background for the Liouville and Jacobi theory of elliptic functions in Chapter 13. The topics are covered thoroughly, briefly, and in an interesting fashion; a careful treatment of the elliptic modular function is included.

A text in the English language that discusses the Nevanlinna theory of meromorphic functions nearly as clearly and thoroughly as this one, probably does not exist. Chapter 14 is splendidly organized for this material, and for the introduction to potential theory amplified in Chapter 16. Chapter 15 treats Montel's theory of normal families, making use of the properties of the spherical metric brought out in the previous chapter. "Lemniscates," Chapter 16, pro-

vides a variety of topics that the author's skill helps to relate to previous ones in interesting ways, such as Frostman's potential theory and the polynomial approximation theory of Runge and Walsh.

An introduction to theory of conformal mapping, after the discussion of the Fejér and F. Riesz extremal, and Bergman's kernel function, includes a continuation of the study of Fekete's transfinite diameter begun in Chapter 16. There is a brief survey of a few extremal properties of the univalent power series and an appealingly simple approach to the boundary value problem. The Schwarz-Christoffel formula is proved and used. An early form of Landau's application of Bloch's theorem to the Picard theorem (completely covered in Chapter 14) ends the chapter.

The emphasis shifts in Chapter 18 from power-series representations to representations in the half-plane. Regarding the standard Phragmén-Lindelöf theory, as well as the form of Schwarz's Lemma, the reviewer expected a treatment that was more elementary in outline, and possibly clearer, in line with the overall spirit of the text. But coverage of modern topics is very complete.

An extension into various topics concerned with half-plane representation completes the work. In this last chapter, convexity properties should perhaps appear to be a little simpler than they would seem here to the general reader.

Exercise lists, always including interesting and challenging problems on nearly all topics presented, are very well organized. This is a good student's text. References, documented very clearly at the end of each chapter, are essentially complete, again for the topics covered. The care taken to prepare these itself deserves commendation.

Printing errors, of which the reviewer has a list, are few. Only one seemed serious, in its implication regarding the properties of the function: on page 360 of the text, in the middle of the page, the mean value of the square of the difference, viz. $M|f-F|^2$, should read $\sum_n |a_n|^2(1-r^n)^2$.

E. L. WHITNEY
University of Alberta

Basic Matrix Theory. By Leonard E. Fuller. Prentice-Hall, Englewood Cliffs, N. J., 1962, 245 pp. $7.00.

Addressed to those using matrices as a tool, this book would be better titled: *Basic Matrix Computation*. Presentation of the theory is often weak: there are few real proofs. Some of the definitions are faulty or vague; many of them are superfluous. (There are 47 definitions in the first 24 pages!). One needed result is left for the reader to guess in an exercise. A brief chapter catalogs results on vector spaces, but fails to make significant contact with the rest of the book.

The language is usually clear, though marred by occasional grammatical errors and by talking equations, theorems, etc. ("This result says . . . "). Specific numerical examples or arguments on small matrices are used to illustrate most of the results, the general case then being stated in English. This strikes me as sound pedagogy at the level for which the book is written and has the additional

exercises, many requiring only substitution in formulas. The level is suitable for a terminal freshman college course or for a high school course where advanced college placement is not an objective.

The book is translated from the ninth edition and is, unfortunately, dated. For example, after defining variables and constants, infinitesimals are defined as follows: "A variable α is called an infinitesimal if, in the course of its variation, its absolute value becomes and then remains less than any previously assigned positive number ϵ, however small." Limits are then defined in terms of infinitesimals and finally functions are defined as follows: "If there is a connection between two variables such that the value of one is defined by the value of the other, we say that they are connected by a functional relationship." These examples, and many others, remind me of the text I used as an undergraduate, but, in terms of present practice, would make the subject very confusing for anyone continuing in mathematics.

It should be mentioned that there is a wide discrepancy between the sterling and dollar price.

H. M. MacNeille
Case Institute of Technology

Intuitive Concepts in Elementary Topology. By B. H. Arnold. Prentice-Hall, N. J. 1962. 182 pp. $6.95.

In the preface the author states "the student is introduced to a few selected topics so that he can get some feeling for the types of results and the methods of proof in the discipline." He also emphasizes that intuition will be heavily relied upon. It is most unlikely that a study of this little book will give very much feeling for the results and methods of modern topology, but it could serve a useful purpose in giving a future student of topology some idea of the origins of the subject.

The book includes the usual historical material on networks (via the Königsberg bridges), the four-color problem for maps on the 2-sphere and generalizations, the Jordan curve theorem (proved for polygons only) and so on. Most mathematicians become aware of this material at some stage of their development, although it is seldom included in a formal course in topology.

The author has used this material in courses at the sophomore/junior level. I agree with his judgement that the material presented could not be handled rigorously at this level, but I found many places where the level of clarity and rigor could have been improved without adding anything in difficulty. Certainly any student who is familiar with one of the more modern treatments of school mathematics (e.g. SMSG or UICSM) will be struck by this deficiency.

Within the limited aims of the book, the earlier chapters are quite successful. But by the time one gets to a so-called intuitive proof of the fundamental theorem of algebra I question whether many sophomores will find the arguments very convincing in view of the number of steps omitted from the argument, some of which are not even made plausible. In particular, the "proof" has the follow-

ing limitations: (i) No definition or explanation is given of the important concept of continuous deformation; (ii) Although *continuous transformation* is defined, no attempt is made (plausible or otherwise) to show the continuity of such key transformations as that induced by the complex polynomial $f(z)$ (page 133). (The word "therefore" in the fourth line, page 117, will mystify many many students—it should be omitted.)

A. L. BLAKERS
University of Western Australia

Introduction to the Theory of Queues. By Lajos Takacs. Oxford, New York, 1962. viii+268 pp. $7.50.

The title of this book is misleading in that it gives a fairly thorough description of selected topics, particularly those to which the author has made significant contributions. About half of the book is devoted to the single server queue with independent interarrival times and service times, emphasizing the transient behavior. Solutions are usually formal in the sense that a generating function or transform is expressed in terms of solutions of some transcendental equations. The second half of the book deals less thoroughly with many server queueing processes, telephone traffic, servicing of machines and particle counters. The book is aimed at readers with a good training in mathematics including probability theory. Because of this and its limited scope, the book is not likely to have a wide appeal despite the fact that it is clearly written and very precise.

G. F. NEWELL
Brown University

Principles of Neurodynamics. By Frank Rosenblatt. Spartan Books, Washington, D. C., 1962. xvi+616 pp. $6.50.

"Neurodynamics" is an American answer to the later works of Academician I. P. Pavlov (1849–1936). And how refreshing to have a neologism that isn't an acronym!

Specifically "neurodynamics" deals with physico-mathematical models for the psychological functioning of the brain as expressed in terms of known principles of neuroanatomy and physiology. Although the brain models arrived at by the author, whose formal training is in the behavioral sciences, are seriously open to questions regarding biological accuracy, he has been quite successful in constructing a number of computer-oriented realizations of them. In that sense the subject matter of the book is based, in part, upon experimental evidence. Except for digressions, this thoroughly documented work, which summarizes four years of inquiry, proceeds on an axiomatic basis. Using principally the methods of discrete mathematics (e.g., matrix algebra, combinatorial analysis and probability theory) the author arrives at a number of technical theorems too complicated to be given here. For the most part he manages to avoid the indiscreet mathematics that marred his earlier technical reports on this subject.

The work has an extensive table of contents but no index. At present "neuro-

dynamics" has its Sun on the horizon. More time is needed to tell whether it is rising or it is setting.

A. A. MULLIN
University of Illinois

Charles Babbage and his Calculating Engines: Selected Writings by Charles Babbage and Others. Edited and with an Introduction by Philip Morrison and Emily Morrison. Dover, New York, 1961. xxxviii+400 pp. $2.00.

In view of the present day interest in computers this fascinating book which deals with the life and work of Charles Babbage, a pioneer in the field of computing machines, is a timely release.

The editors have written an introduction which serves as a coordinating preamble to some chapters from Babbage's *Passages from the Life of a Philosopher* and selections from *Babbage's Calculating Engines* along with other miscellaneous papers, blending everything so as to produce a book which can be read with the ease of a novel and is certainly no less entertaining.

For a large part of his life Babbage was chiefly interested in the calculating engine and it is in this connection that he is usually remembered today. From his writings it is apparent that in addition to understanding the principles of the construction of computers, he had a clear insight into their potential applications and the way to use them. It is perhaps less well known that he was also a pioneer in the field of operations research, and that he had interests in an amazingly broad range of subjects, leading to his investigation of what appears to be a weird collection of problems. He was even one of the first people to obtain a government grant in support of research!

Gregarious by nature, he was acquainted with many of the scientists of his day: Laplace, Fourier, Biot, and Sir Humphry Davy, to mention a few. Never one to avoid a conflict if he had to compromise a principle to do so, he sometimes found himself in opposition to people of considerable authority, with the result that his life was seldom plagued with ennui.

Whether the book is read because of Babbage's connection with the history of computers, or as the story of an unusually talented man, it cannot fail to be of interest.

W. FRASER
Defence Research Board
University of Toronto

Calculus, Volume II. Calculus of several variables with applications to probability and vector analysis. By Tom M. Apostol. Blaisdell, New York, 1962. 525 pp. $9.50.

There is no need to praise again the good writing, shrewd organization, and lively adult tone of this fine text (see my review of Vol. 1, this MONTHLY, 69 (1962) 449–451). The second volume if anything surpasses the first in excellence of workmanship, and occasions for detailed criticism are not worth mentioning.

The coverage of calculus is extensive, including such "advanced" topics as change of variables in multiple integrals, Green's and Stokes' theorems on line and surface integrals, nth order linear differential equations, and Picard's existence theorem. There are also some 88 pages of probability, and a sizable chapter of theorems on numerical analysis, especially approximations to functions by polynomials. Entanglement in the more delicate issues of analysis and topology (uniform continuity, orientation) is, however, carefully avoided. When necessary the facts are stated intuitively, or rigorously but without proof, so as to show how the full story transcends the reach of elementary arguments. Vector notation and language prevail to such an extent that the basic operation of the differential calculus is differentiation with respect to an arbitrary vector. Such advanced notions as Boolean σ-ring, countably additive set function, function space, and seminorm come up spontaneously as the natural way to describe some of the concrete situations under study. They are used as alternative points of view with illuminating or unifying value, and the resulting blend should give the student, in anticipation of more advanced courses, an appreciation of the general and the special as complementary sides of mathematics.

While this book expounds useful technique, and the theory behind it, and while the contact with physics is increased in Volume II, it is still not a course in applied mathematics. It makes no attempt to teach the art of formulating a crude problem into a mathematical problem, and it works out relatively few applications outside mathematics. Without other courses in science where mathematics is systematically used, the student would be left with an excellent idea of the spirit of mathematics, but with only a rather thin idea, based on hearsay rather than experience, of its power.

F. CUNNINGHAM, JR.
Bryn Mawr College

The Mathematical Theory of Linear Systems. By B. M. Brown. Wiley, New York, 1961. 267 pp. $8.00.

Much of the current theoretical effort in the areas of communication and control engineering centers around the discussion of systems which may be described by linear differential equations, linear difference equations, or linear differential-difference equations. The aim of this book is to acquaint the reader with the various analytical techniques which are available for the analysis and synthesis of such systems. The author is remarkably successful.

The author first considers linear differential equations. These are treated by operational and transform methods, including the bilateral Laplace transform. Stability considerations predominate, and the Routh-Hurwitz and Nyquist criteria are presented. Some applications to circuit theory and feedback control theory are described, and an introduction to the statistical design theory of N. Wiener is presented.

Linear difference equations are treated using the z-transform. Stability re-

ing of the t's. The book grew out of a course successfully taught at a Summer Institute for Teachers. The reviewer fears that, in less skillful hands, the material might well discourage teachers with an insufficient appetite for complete logical clarity. Unfortunately, such an appetite is not necessarily associated with great interest in mathematics. This book will have a great influence but its use should be tempered with the common sense which one should retain even in the most abstract pursuits.

D. E. RICHMOND
Williams College

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor B. W. Jones, University of Colorado, represented the Association at the dedication of the Boettcher Center of Science, Engineering and Research of the University of Denver on February 18 to 20, 1963.

Dr. Bruce L. Reinhart of the University of Maryland and RIAS has been chosen by the Washington Academy of Sciences to receive its award for outstanding achievement during 1962 in the field of mathematics.

U. S. Naval Postgraduate School: Professor W. W. Gutzman, University of South Dakota, has been appointed Professor; Professor B. J. Lockhart has been appointed Dean of Academic Administration.

Professor R. H. Breusch, Amherst College, has been appointed Chairman of the Department of Mathematics.

Mr. K. L. Brinkman, Hughes Aircraft, Culver City, California, has been promoted to Senior Staff Physicist.

Mr. Victor Chew, U.S. Naval Weapons Laboratory, Dahlgren, Virginia, has accepted a position as Senior Engineer with R. C. A., Patrick Air Force Base, Florida.

Mr. Keewhan Choi, Massachusetts Institute of Technology, has been appointed Assistant Professor of Biological Statistics at Cornell University.

Associate Professor G. J. Corley, Northwestern State College of Louisiana, has been appointed Professor at East Texas State College.

Mr. E. J. Creager, Jr., Philco Corporation, Fort Bliss, Texas, has accepted a position as Senior Technical Writer with Sanders Associates, Nashua, New Hampshire.

Assistant Professor C. R. Deeter, University of Kansas, has been appointed Assistant Professor at Texas Christian University.

Dr. G. A. Garrett, Union Carbide Nuclear Company, Oak Ridge, Tennessee, has accepted a position as Associate Director of Engineering with the Lockheed Missiles and Space Company, Palo Alto, California.

Mr. J. W. Haake, Convair, San Diego, California, has accepted a position as Operations Research Scientist with the System Development Corporation, Santa Monica, California.

Professor Theodore Hailperin, Lehigh University, will spend the academic year 1963–64 as a visiting staff member and consultant in mathematical logic with the computer and numerical analysis division of the Sandia Corporation, Albuquerque, New Mexico.

Dr. Carl Hammer, R. C. A., Washington, D. C., has accepted a position as Director of Scientific Computer Marketing with the UNIVAC division of the Sperry Rand Corporation, Washington, D. C.

Associate Professor J. R. Hanna, University of Wyoming, has been appointed Associate Professor at the University of Colorado.

Assistant Professor G. W. Kimble, Long Beach State College, has been appointed Associate Professor and Director of the Computing Center of Montana State University.

Mr. G. E. Martin, Naval Avionics Facility, Indianapolis, Indiana, has accepted a position as Research Staff Engineer with Litton Systems, Woodland Hills, California.

Assistant Professor A. J. Mortola, Manhattan College, has been promoted to Associate Professor.

Dr. P. B. Norman, Aerospace Corporation, El Segundo, California, has accepted a position in the Autonetics division of North American Aviation, Anaheim, California.

Mr. J. B. Scott, Western Computing Consultants, Tucson, Arizona, has accepted a position as Assistant Supervisor with the Armour Research Foundation, Annapolis, Maryland.

Mr. L. R. Stidham, Telecomputing Corporation, White Sands Proving Ground, New Mexico, has accepted a position as Research Specialist with North American Aviation, Downey, California.

Dr. J. W. Weihe, Sandia Corporation, Albuquerque, New Mexico, has been promoted to Manager of the Mathematical Research Department.

Professor E. M. Wright, University of Aberdeen, Scotland, has been appointed Principal and Vice-Chancellor.

Professor Emil Artin, Mathematisches Seminar, Hamburg, Germany, died on December 20, 1962. He was a member of the Association for 25 years.

PRIZE AWARDS OF THE SOCIETY OF ACTUARIES

The winners of the prize awards offered by the Society of Actuaries to the five undergraduates ranking highest on the score of the General Mathematics Examination of the November 1962 Preliminary Actuarial Examinations are as follows:

FIRST PRIZE OF $200

Simon, Barry M.—Harvard University

ADDITIONAL PRIZES OF $100 EACH

Goldberg, Robert P.—Massachusetts Institute of Technology
Gross, Michael R.—Carnegie Institute of Technology
Kalbfleisch, James G.—University of Toronto
Martineau, Steven F.—University of British Columbia

The Society of Actuaries has authorized a similar set of five prizes for the May 1963 General Mathematics Examination, which will be jointly sponsored by the Casualty Actuarial Society. Further information concerning these examinations and the one to be given on November 13, 1963, may be obtained from the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

UNIVERSITY OF MONTREAL—SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES

Under the sponsorship of the North Atlantic Treaty Organization (NATO) and the Canadian Mathematical Congress, the second session of the University of Montreal

SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES will be held from July 2 to August 9, 1963.

There will be four main lecturers, each giving a series of thirty lectures. Each lecturer will conduct a weekly discussion period. In addition, certain other mathematicians will be invited to deliver special lectures on chosen topics. The main invited lecturers are: Professor Jean-Pierre Kahane, University of Paris, Séries de Fourier aléatoires; Professor Charles Pisot, University of Paris, Quelques aspects de la théorie des entiers algébriques; Professor Aubert Daigneault, University of Montreal, Théorie des modèles en logique mathématique; Professor Anatole Joffe, University of Montreal, Promenades aléatoires et mouvement brownien.

This Seminar is conceived particularly for young mathematicians and students engaged in study and research at the master's and doctorate level. Mimeographed notes will be distributed to those attending the Seminar. Financial assistance is available. For information and registration forms, write to: Department of Mathematics, University of Montreal, P. O. Box 6128, Montreal, Quebec, Canada.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Summer Meeting at the University of Colorado, Boulder, Colorado, on August 27, 28, and 29, 1963. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

There is no charge for registration, either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants upon request and upon payment of $3.00 to defray the cost involved in handling anonymous listings.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms or for position description forms. These forms must be completed and returned to Providence not later than July 30, 1963, in order to be included in the listings at the Summer Meeting in Boulder. Position Description forms which arrive after this closing date, but before August 10, will be included in the register at the meeting for a late registration fee of $3.00. The printed listings will be available for distribution both during and after the meeting.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Howard University, District of

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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York 53, N. Y. Temporary address from June 15 to August 15, 1963: Dept. of Math., University of California, Berkeley 4, California.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$5.00. For nonmembers the subscription price is \$8.00.

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ON LEARNING, TEACHING, AND LEARNING TEACHING

GEORGE POLYA, Stanford University

"What you have been obliged to discover by yourself leaves a path in your mind which you can use again when the need arises." (G. C. Lichtenberg: *Aphorismen.*)

"Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas." (I. Kant: *Critique of Pure Reason*, translated by J. M. D. Meiklejohn, 1878, p. 429.)

"I [planned to] write so that the learner may always see the inner ground of the things he learns, even so that the source of the invention may appear, and therefore in such a way that the learner may understand everything as if he had invented it by himself." (G. W. von Leibnitz: *Mathematische Schriften*, edited by Gerhardt, vol. VII, p. 9.)

1. Teaching is not a science. I shall tell you some of my opinions on the process of learning, on the art of teaching, and on teacher training.

My opinions are the result of a long experience. Still, such personal opinions may be irrelevant and I would not dare to waste your time by telling them if teaching could be fully regulated by scientific facts and theories. This, however, is not the case. Teaching is, in my opinion, not just a branch of applied psychology—at any rate, it is not yet that for the present.

Teaching is correlated with learning. The experimental and theoretical study of learning is an extensively and intensively cultivated branch of psychology. Yet there is a difference. We are principally concerned here with complex learning situations, such as learning algebra or learning teaching, and their long-term educational effects. The psychologists, however, devote most of their attention to, and do their best work about, simplified short-term situations. Thus, the psychology of learning may give us interesting hints, but it can not pretend to pass ultimate judgment upon problems of teaching (cf. [1]).

2. The aim of teaching. We can not judge the teacher's performance if we do not know the teacher's aim. We can not meaningfully discuss teaching, if we do not agree to some extent about the aim of teaching.

Let me be specific. I am concerned here with mathematics in the high school curriculum and I have an old fashioned idea about its aim: first and foremost, it should teach those young people to THINK.

This is my firm conviction; you may not go along with it all the way, but I assume that you agree with it to some extent. If you do not regard "teaching to think" as a primary aim, you may regard it as a secondary aim—and then we have enough common ground for the following discussion.

"Teaching to think" means that the mathematics teacher should not merely impart information, but should try also to develop the ability of the students to use the information imparted: he should stress know-how, useful attitudes, desirable habits of mind. This aim may need fuller explanation (my whole printed

work on teaching may be regarded as a fuller explanation) but here it will be enough to emphasize only two points.

First, the thinking with which we are concerned here is not day-dreaming but "thinking for a purpose" or "voluntary thinking" (William James) or "productive thinking" (Max Wertheimer). Such "thinking" may be identified here, at least in first approximation, with "problem solving." At any rate, in my opinion, one of the principal aims of the high school mathematics curriculum is to develop the students' ability to solve problems.

Second, mathematical thinking is not purely "formal"; it is not concerned only with axioms, definitions, and strict proofs, but many other things belong to it: generalizing from observed cases, inductive arguments, arguments from analogy, recognizing a mathematical concept in, or extracting it from, a concrete situation. The mathematics teacher has an excellent opportunity to acquaint his students with these highly important "informal" thought processes, and I mean that he should use this opportunity better, and much better, than he does today. Stated incompletely but concisely: Let us teach proving by all means, but let us also teach guessing.

3. Teaching is an art. Teaching is not a science, but an art. This opinion has been expressed by so many people so many times that I feel a little embarrassed repeating it. If, however, we leave a somewhat hackneyed generality and get down to appropriate particulars, we may see a few tricks of our trade in an instructive sidelight.

Teaching obviously has much in common with the theatrical art. For instance, you have to present to your class a proof which you know thoroughly having presented it already so many times in former years in the same course. You really can not be excited about this proof—but, please, do not show that to your class: if you appear bored, the whole class will be bored. Pretend to be excited about the proof when you start it, pretend to have bright ideas when you proceed, pretend to be surprised and elated when the proof ends. You should do a little acting for the sake of your students who may learn, occasionally, more from your attitudes than from the subject matter presented.

I must confess that I take pleasure in a little acting, especially now that I am old and very seldom find something new in mathematics: I may find a little satisfaction in re-enacting how I discovered this or that little point in the past.

Less obviously, teaching has something in common also with music. You know, of course, that the teacher should not say things just once or twice, but three or four or more times. Yet, repeating the same sentence several times without pause and change may be terribly boring and defeat its own purpose. Well, you can learn from the composers how to do it better. One of the principal art forms of music is "air with variations." Transposing this art form from music into teaching you begin by saying your sentence in its simplest form; then you repeat it with a little change; then you repeat it again with a little more color, and so on; you may wind up by returning to the original simple formulation.

Another musical art form is the "rondo." Transposing the rondo from music into teaching, you repeat the same essential sentence several times with little or no change, but you insert between two repetitions some appropriately contrasting illustrative material. I hope that when you listen the next time to a theme with variations by Beethoven or to a rondo by Mozart you will give a little thought to improving your teaching.

Now and then, teaching may approach poetry, and now and then it may approach profanity. May I tell you a little story about the great Einstein? I listened once to Einstein as he talked to a group of physicists in a party. "Why have all the electrons the same charge?" said he. "Well, why are all the little balls in the goat dung of the same size?" Why did Einstein say such things? Just to make some snobs to raise their eyebrows? He was not disinclined to do so, I think. Yet, probably, it went deeper. I do not think that the overheard remark of Einstein was quite casual. At any rate, I learnt something from it: Abstractions are important; use all means to make them more tangible. Nothing is too good or too bad, too poetical or too trivial to clarify your abstractions. As Montaigne put it: The truth is such a great thing that we should not disdain any means that could lead to it. Therefore, if the spirit moves you to be a little poetical, or a little profane, in your class, do not have the wrong kind of inhibition.

4. Three principles of learning. Teaching is a trade that has innumerable little tricks. Each good teacher has his pet devices and each good teacher is different from any other good teacher.

Any efficient teaching device must be correlated somehow with the nature of the learning process. We do not know too much about the learning process, but even a rough outline of some of its more obvious features may shed some welcome light upon the tricks of our trade. Let me state such a rough outline in the form of three "principles" of learning. Their formulation and combination is of my choice, but the "principles" themselves are by no means new; they have been stated and restated in various forms, they are derived from the experience of the ages, endorsed by the judgment of great men, and also suggested by the psychological study of learning.

These "principles of learning" can be also taken for "principles of teaching," and this is the chief reason for considering them here—but about this later.

(1) *Active learning.* It has been said by many people in many ways that learning should be active, not merely passive or receptive: merely by reading books or listening to lectures or looking at moving pictures without adding some action of your own mind you can hardly learn anything and certainly you can not learn much.

There is another often expressed (and closely related) opinion: *The best way to learn anything is to discover it by yourself.* Lichtenberg (an eighteenth century German physicist, better known as a writer of aphorisms) adds an interesting point: *What you have been obliged to discover by yourself leaves a path in your mind*

which you can use again when the need arises. Less colorful is the following statement, but it may be more widely applicable: *For efficient learning, the learner should discover by himself as large a fraction of the material to be learnt as feasible under the given circumstances.*

This is the *principle of active learning* (Arbeitsprinzip). It is a very old principle: it underlies the idea of "Socratic method."

(2) *Best motivation.* Learning should be active, we have said. Yet the learner will not act if he has no motive to act. He must be induced to act by some stimulus, by the hope of some reward, for instance. The interest of the material to be learnt should be the best stimulus to learning and the pleasure of intensive mental activity should be the best reward for such activity. Yet, where we cannot obtain the best we should try to get the second best, or the third best, and less intrinsic motives of learning should not be forgotten.

For efficient learning, the learner should be interested in the material to be learnt and find pleasure in the activity of learning. Yet, beside these best motives for learning, there are other motives too, some of them desirable. (Punishment for not learning may be the least desirable motive.)

Let us call this statement the *principle of best motivation.*

(3) *Consecutive phases.* Let us start from an often quoted sentence of Kant: *Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas.* The English translation uses the terms "cognition, intuition, idea." I am not able (who is able?) to tell in what exact sense Kant intended to use these terms. Yet I beg your permission to present my reading of Kant's dictum:

Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits.

To begin with, please, take the terms of this sentence in some sense that you can illustrate concretely on the basis of your own experience. (To induce you to think about your personal experience is one of the desired effects.) "Learning" should remind you of a classroom with yourself in it as student or teacher. "Action and perception" should suggest manipulating and seeing concrete things such as pebbles, or apples, or Cuisenaire rods; or ruler and compasses; or instruments in a laboratory; and so on.

Such concrete interpretation of the terms may come more easily and more naturally when we think of some simple elementary material. Yet after a while we may perceive similar phases in the work spent on mastering more complex, more advanced material. Let us distinguish three phases: the phases of *exploration, formalization,* and *assimilation.*

A first *exploratory* phase is closer to action and perception and moves on a more intuitive, more heuristic level.

A second *formalizing* phase ascends to a more conceptual level, introducing terminology, definitions, proofs.

The phase of *assimilation* comes last: there should be an attempt to perceive the "inner ground" of things, the material learnt should be mentally digested,

absorbed into the system of knowledge, into the whole mental outlook of the learner; this phase paves the way to applications on one hand, to higher generalizations on the other.

Let us summarize: *For efficient learning, an exploratory phase should precede the phase of verbalization and concept formation and, eventually, the material learnt should be merged in, and contribute to, the integral mental attitude of the learner.*

This is the *principle of consecutive phases.*

5. Three principles of teaching. The teacher should know about the ways of learning. He should avoid inefficient ways and take advantage of the efficient ways of learning. Thus, he can make good use of the three principles we have just surveyed, the principle of active learning, the principle of best motivation, and the principle of consecutive phases: these principles of learning are also principles of teaching. There is, however, a condition: to avail himself of such a principle, the teacher should not merely know it from hearsay, but he should understand it intimately on the basis of his own well-considered personal experience.

(1) *Active learning.* What the teacher says in the classroom is not unimportant, but what the students think is a thousand times more important. The ideas should be born in the students' mind and the teacher should act only as midwife.

This is a classical Socratic precept and the form of teaching best adapted to it is the Socratic dialogue. It is a definite advantage of the high school teacher over the college instructor that in the high school one can use the dialogue form much more extensively than in the college. Unfortunately, even in the high school, time is limited and there is a prescribed material to cover so that all business cannot be transacted in dialogue form. Yet the principle is: Let the students *discover by themselves as much as feasible* under the given circumstances.

Much more is feasible than is usually done, I am sure. Let me recommend you here just one little practical trick: Let the students *actively contribute to the formulation* of the problem that they have to solve afterwards. If the students have had a share in proposing the problem they will work at it much more actively afterwards.

In fact, in the work of the scientist, formulating the problem may be the better part of a discovery, the solution often needs less insight and originality than the formulation. Thus, letting your students have a share in the formulation, you not only motivate them to work harder, but you teach them a desirable attitude of mind.

(2) *Best motivation.* The teacher should regard himself as a salesman: he wants to sell some mathematics to the youngsters. Now, if the salesman meets with sales resistance and his prospective customers refuse to buy, he should not lay the whole blame on them. Remember, the customer is always right in principle, and sometimes right in practice. The lad who refuses to learn mathematics may be right: he may be neither lazy nor stupid, just more interested in

something else—there are so many interesting things in the world around us. It is your duty as a teacher, as a salesman of knowledge, to convince the student that mathematics *is* interesting, that the point just under discussion is interesting, that the problem he is supposed to do deserves his effort.

Therefore, the teacher should pay attention to the choice, the formulation, and a suitable presentation of the problem he proposes. The problem should be related, if possible, to the everyday experience of the students, and it should be introduced, if possible, by a little joke or a little paradox. Or the problem should start from some very familiar knowledge; it should have, if possible, some point of general interest or eventual practical use. If we wish to stimulate the student to a genuine effort, we must give him some reason to suspect that his task deserves his effort.

The best motivation is the student's interest in his task. Yet there are other motivations which should not be neglected. Let me recommend here just one little practical trick. Before the students do a problem, let them *guess the result*, or a part of the result. The boy who expresses an opinion commits himself; his prestige and self-esteem depend a little on the outcome, he is impatient to know whether his guess will turn out right or not, and so he will be actively interested in his task and in the work of the class—he will not fall asleep or misbehave.

In fact, in the work of the scientist, the guess almost always precedes the proof. Thus, in letting your students guess the result, you not only motivate them to work harder, but you teach them a desirable attitude of mind.

(3) *Consecutive phases.* The trouble with the usual problem material of the high school textbooks is that they contain almost exclusively merely routine examples. A routine example is a short range example; it illustrates, and offers practice in the application of, just one isolated rule. Such routine examples may be useful and even necessary, I do not deny it, but they miss two important phases of learning: the exploratory phase and the phase of assimilation. Both phases seek to connect the problem in hand with the world around us and with other knowledge, the first before, the last after, the formal solution. Yet the routine problem is obviously connected with the rule it illustrates and it is scarcely connected with anything else, so that there is little profit in seeking further connections. In contrast with such routine problems, the high school should present more challenging problems at least now and then, problems with a rich background that deserves further exploration, and problems which can give a foretaste of the scientist's work.

Here is a practical hint: if the problem you want to discuss with your class is suitable, let your students do some preliminary exploration: it may whet their appetite for the formal solution. And reserve some time for a retrospective discussion of the finished solution; it may help in the solution of later problems.

(4) After this much too incomplete discussion, I must stop explaining the three principles of active learning, best motivation, and consecutive phases. I think that these principles can penetrate the details of the teacher's daily work and make him a better teacher. I think too that these principles should also

penetrate the planning of the whole curriculum, the planning of each course of the curriculum, and the planning of each chapter of each course.

Yet it is far from me to say that you must accept these principles. These principles proceed from a certain general outlook, from a certain philosophy, and you may have a different philosophy. Now, in teaching as in several other things, it does not matter much what your philosophy is or is not. It matters more whether you have a philosophy or not. And it matters very much whether you try to live up to your philosophy or not. The only principles of teaching which I thoroughly dislike are those to which people pay only lip service.

6. Examples. Examples are better than precepts; let me get down to examples—I much prefer examples to general talk. I am here concerned principally with teaching on the high school level and I shall present you a few examples on that level. I often find satisfaction in treating examples at the high school level, and I can tell you why: I attempt to treat them so that they recall in one respect or the other my own mathematical experience; I am re-enacting my past work on a reduced scale.

(1) *A seventh grade problem.* The fundamental art form of teaching is the Socratic dialogue. In a junior high school class, perhaps in the seventh grade, the teacher may start the dialogue so:

"What is the time at noon in San Francisco?"

'But, teacher everybody knows that' may say a lively youngster, or even 'But teacher, you are silly: twelve o'clock.'

"And what is the time at noon in Sacramento?"

'Twelve o'clock—of course, not twelve o'clock midnight.'

"And what is the time at noon in New York?"

'Twelve o'clock.'

"But I thought that San Francisco and New York do not have noon at the same time, and you say that both have noon at twelve o'clock!"

'Well, San Francisco has noon at twelve o'clock Western Standard Time and New York at twelve o'clock Eastern Standard Time.'

"And on what kind of standard time is Sacramento, Eastern or Western?"

'Western, of course.'

"Have the people in San Francisco and Sacramento noon at the same moment?"

"You do not know the answer? Well, try to guess it: does noon come sooner to San Francisco, or to Sacramento, or does it arrive exactly at the same instant at both places?"

How do you like my idea of Socrates talking to seventh grade kids? At any rate, you can imagine the rest. By appropriate questions the teacher, imitating Socrates, should extract several points from the students:

(a) We have to distinguish between "astronomical" noon and conventional or "legal" noon.

(b) Definitions for the two noons.

(c) Understanding "standard time": how and why is the globe's surface subdivided into time zones?

(d) Formulation of the problem: "At what o'clock Western Standard Time is the astronomical noon in San Francisco?"

(e) The only specific datum needed to solve the problem is the longitude of San Francisco (in an approximation sufficient for the seventh grade).

The problem is not too easy. I tried it on two classes; in both classes the participants were high school teachers. One class spent about 25 minutes on the solution, the other 35 minutes.

(2) I must say that this little seventh grade problem has various advantages. Its main advantage may be that it emphasizes an essential mental operation which is sadly neglected by the usual problem material of the textbooks: *recognizing the essential mathematical concept in a concrete situation.* To solve the problem, the students must recognize a *proportionality*: the time of the highest position of the sun in a locality on the globe's surface changes *proportionally* to the longitude of the locality.

In fact, in comparison with the many painfully artificial problems of the high school textbooks, our problem is a perfectly natural, a "real" problem. In the serious problems of applied mathematics, the appropriate *formulation* of the problem is always a major task, and often the most important task; our little problem which can be proposed to an average seventh grade class possesses just this feature. Again, the serious problems of applied mathematics may lead to practical action, for instance, to adopting a better manufacturing process; our little problem can explain to seventh graders why the system of 24 time zones, each with a uniform standard time, was adopted. On the whole, I think that this problem, if handled with a little skill by the teacher, could help a future scientist or engineer to discover his vocation, and it could also contribute to the intellectual maturity of those students who will not use mathematics professionally.

Observe also that this problem illustrates several little tricks mentioned in the foregoing: The students actively contribute to the formulation of the problem (cf. Sect. 5(1)). In fact, the exploratory phase which leads to the formulation of the problem is prominently important (cf. Sect. 5(3)). Then, the students are invited to guess an essential point of the solution (cf. Sect. 5(2)).

(3) *A tenth grade problem.* Let us consider another example. Let us start from what is probably the most familiar problem of geometric construction: *Construct a triangle, being given its three sides.* As analogy is such a fertile source of invention, it is natural to ask: What is the analogous problem in solid geometry? An average student, who has a little knowledge of solid geometry, may be led to formulate the problem: *Construct a tetrahedron, being given its six edges.*

It may be mentioned here parenthetically that this problem of the tetrahedron comes as close as it can on the usual high school level to practical problems solved by "mechanical drawing." Engineers and designers use well executed drawings to give precise information about the details of three dimensional figures of machines or structures to be built: we intend to build a tetrahedron with specified edges. We might wish, for example, to carve it out of wood.

This leads to asking that the problem should be solved precisely, by straightedge and compasses, and to discussing the question: which details of the tetrahedron should be constructed? Eventually, from a well conducted class discussion, the following definitive formulation of the problem may emerge:

Of the tetrahedron ABCD, we are given the lengths of its six edges

$$AB,\ BC,\ CA,\ AD,\ BD,\ CD.$$

Regard $\triangle ABC$ as the base of the tetrahedron and construct with ruler and compasses the angles that the base includes with the other three faces.

The knowledge of these angles is required for cutting out of wood the desired solid. Yet other elements of the tetrahedron may turn up in the discussion such as

(a) the altitude drawn from the vertex D opposite the base,
(b) the foot F of this altitude in the plane of the base;

(a) and (b) would contribute to the knowledge of the solid, they may possibly help to find the required angles, and so we may try to construct them too.

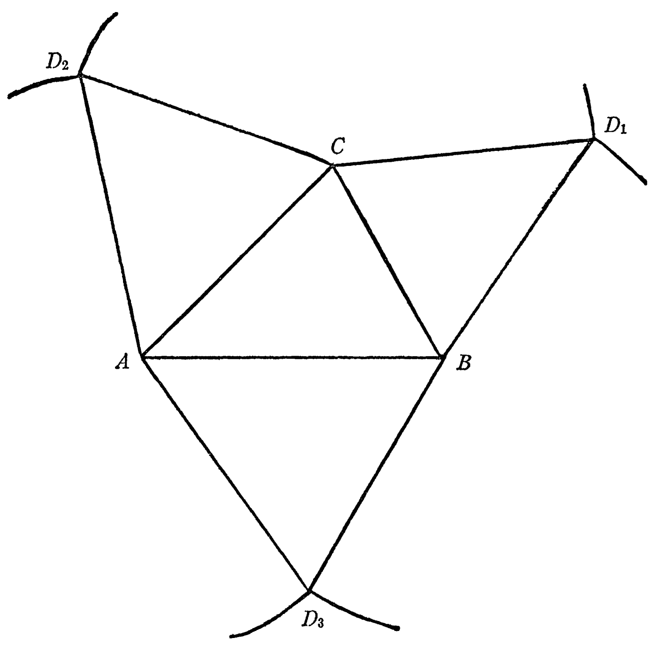

FIG. 1. Tetrahedron from six edges.

(4) We can, of course, construct the four triangular faces which are assembled in Fig. 1. (Short portions of some circles used in the construction are preserved to indicate that $AD_2 = AD_3$, $BD_3 = BD_1$, $CD_1 = CD_2$.) If Fig. 1 is copied

on cardboard, we can add three flaps, cut out the pattern, fold it along three lines, and paste down the flaps; we obtain in this way a solid model on which we can measure roughly the altitude and the angles in question. Such work with cardboard is quite suggestive, but it is not what we are required to do: we should construct the altitude, its foot, and the angles in question with ruler and compasses.

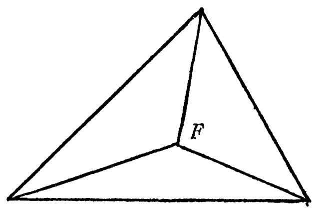

FIG. 2. An aspect of the finished product.

(5) It may help to take the problem, or some part of it, "as solved." Let us visualize how Fig. 1 will look when the three lateral faces, after having been rotated each about a side of the base, will be lifted into their proper position. Fig. 2 shows the orthogonal projection of the tetrahedron onto the plane of its base, $\triangle ABC$. The point F is the projection of the vertex D: it is the foot of the altitude drawn from D.

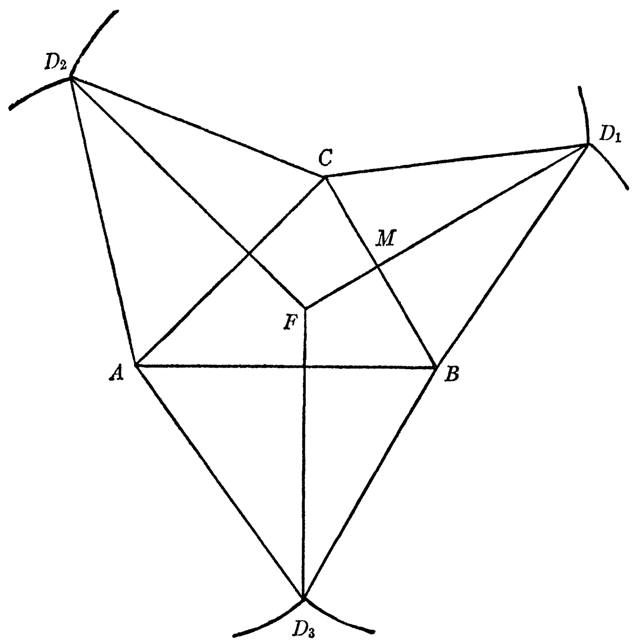

FIG. 3. The common destination of three travelers.

(6) We may visualize the transition from Fig. 1 to Fig. 2 with or without a cardboard model. Let us focus our attention upon one of the three lateral faces, upon $\triangle BCD_1$, which was originally located in the same plane as $\triangle ABC$, in the plane of Fig. 1 which we imagine as horizontal. Let us watch the triangle BCD_1 rotating about its fixed side BC and let our eyes follow its only moving vertex D_1. This vertex D_1 describes an arc of a circle. The center of this circle is a point of BC; the plane of this circle is perpendicular to the horizontal axis of revolution BC; thus D_1 moves in a vertical plane. Therefore, the projection of the path of the moving vertex D_1 onto the horizontal plane of Fig. 1 is a straight line, perpendicular to BC, passing through the original position of D_1.

Yet there are two more rotating triangles, three altogether. There are three moving vertices, each following a circular path in a vertical plane—to which destination?

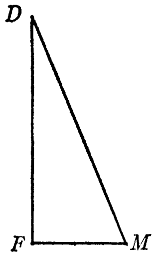

FIG. 4. The rest is easy.

(7) I think that by now the reader has guessed the result (perhaps even before reading the end of the foregoing subsection): the three straight lines drawn from the original positions (see Fig. 1) of D_1, D_2, and D_3 perpendicularly to BC, CA, and AB, respectively, meet in one point, the point F, our supplementary aim (b), see Fig. 3. (It is enough to draw two perpendiculars to determine F, but we may use the third to check the precision of our drawing.) And what remains to do is easy. Let M be the point of intersection of D_1F and BC (see Fig. 3). Construct the right triangle FMD (see Fig. 4), with hypotenuse $MD=MD_1$ and leg MF. Obviously, FD is the altitude (our supplementary aim (a)) and $\angle FMD$ measures the dihedral angle included by the base $\triangle ABC$ and the lateral face $\triangle DBC$ which was required by our problem.

(8) One of the virtues of a good problem is that it generates other good problems.

The foregoing solution may, and should, leave a doubt in our mind. We found the result represented by Fig. 3 (that the three perpendiculars described above are concurrent) by considering the motion of rotating bodies. Yet the result is a proposition of geometry and so it should be established independently of the idea of motion, by geometry alone.

Now, it is relatively easy to free the foregoing consideration (in subsections (6) and (7)) from ideas of motion and establish the result by ideas of solid

geometry (intersection of spheres, orthogonal projection). Yet the result is a proposition of plane geometry and so it should be established independently of the ideas of solid geometry, by plane geometry alone. (How?)

(9) Observe that this tenth grade problem also illustrates various points about teaching discussed in the foregoing. For instance, the students could and should participate in the final formulation of the problem, there is an exploratory phase, and a rich background.

Yet here is the point I wish to emphasize: the problem is designed to deserve the attention of the students. Although the problem is not so close to everyday experience as our seventh grade problem, it starts from a most familiar piece of knowledge (the construction of a triangle from three sides) it stresses from the start an idea of general interest (analogy) and it points to eventual practical applications (mechanical drawing). With a little skill and good will, the teacher should be able to secure for this problem the attention of all students who are not hopelessly dull.

7. Learning teaching. There remains one more topic to discuss and it is an important topic: teacher training. In discussing this topic, I am in a comfortable position: I can almost agree with the "official" standpoint. (I am referring here to the "Recommendations of the Mathematical Association of America for the training of mathematics teachers," this MONTHLY, 67 (1960) 982–991. Just for the sake of brevity, I take the liberty to quote this document as the "official recommendations.") I shall concentrate upon just two points. To these two points I have devoted a good deal of work and thought in the past and practically all my teaching in the last ten years.

To state it roughly, one of the two points I have in mind is concerned with "subject matter" courses, the other with "methods" courses.

(1) *Subject matter.* It is a sad fact, but by now widely recognized, that our high school mathematics teachers' knowledge of their subject matter is, on the average, insufficient. There are, certainly, some well-prepared high school teachers, but there are others (I met with several) whose good will I must admire but whose mathematical preparation is not admirable. The official recommendations of subject matter courses may not be perfect, but there is no doubt that their acceptance would result in substantial improvement. I wish to direct your attention to a point which, in my considered opinion, should be added to the official recommendations.

Our knowledge about any subject consists of information and know-how. Know-how is ability to use information; of course, there is no know-how without some independent thinking, originality, and creativity. Know-how in mathematics is the ability to do problems, to find proofs, to criticize arguments, to use mathematical language with some fluency, to recognize mathematical concepts in concrete situations.

Everybody agrees that, in mathematics, know-how is more important, or even much more important, than mere possession of information. Everybody demands that the high school should impart to the students not only informa-

tion in mathematics but know-how, independence, originality, creativity. Yet almost nobody asks these beautiful things for the mathematics teacher—is it not remarkable? The official recommendations are silent about the mathematical know-how of the teacher. The student of mathematics who works for a Ph.D. degree must do research, yet even before he reaches that stage he may find some opportunity for independent work in seminars, problem seminars, or in the preparation of a master's thesis. Yet no such opportunity is offered to the prospective mathematics teacher—there is no word about any sort of independent work or research work in the official recommendations. If, however, the teacher has had no experience of creative work of some sort, how will he be able to inspire, to lead, to help, or even to recognize the creative activity of his students? A teacher who acquired whatever he knows in mathematics purely receptively can hardly promote the active learning of his students. A teacher who never had a bright idea in his life will probably reprimand a student who has one instead of encouraging him.

Here, in my opinion, is the worst gap in the subject matter knowledge of the average high school teacher: he has no experience of active mathematical work and, therefore, he has no real mastery even of the high school material he is supposed to teach.

I have no panacea to offer, but I have tried one thing. I have introduced and repeatedly conducted a *problem solving seminar* for teachers. The problems offered in this seminar do not require much knowledge beyond the high school level, but they require some degree, and now and then a higher degree, of concentration and judgment—and, to that degree, their solution is "creative" work. I have tried to arrange my seminar so that the students should be able to use much of the material offered in their classes without much change; that they should acquire some mastery of high school mathematics; and so that they should have even some opportunity for practice teaching (in teaching each other in small groups). I can not enter here upon details; I gave a detailed description in a recently published book [2].

(2) *Methods.* From my contact with hundreds of mathematics teachers I gained the impression that "methods" courses are often received with something less than enthusiasm. Yet so also are received, by the teachers, the usual courses offered by the mathematics departments. A teacher with whom I had a heart to heart talk about these matters found a picturesque expression for a rather widespread feeling: "The mathematics department offers us tough steak which we can not chew and the school of education vapid soup with no meat in it."

In fact, we should once summon up some courage and discuss publicly the question: Are methods courses really necessary? Are they in any way useful? There is more chance to reach the right answer in open discussion than by widespread grumbling.

There are certainly enough pertinent questions. Is teaching teachable? (Teaching is an art, as many of us think—is an art teachable?) Is there such a

thing as the teaching method? (What the teacher teaches is never better than what the teacher is—teaching depends on the whole personality of the teacher—there are as many good methods as there are good teachers.) The time allotted to the training of teachers is divided between subject matter courses, methods courses, and practice teaching; should we spend less time on methods courses? (Many European countries spend much less time.)

I hope that people younger and more vigorous than myself will take up these questions some day and discuss them with an open mind and pertinent data.

I am speaking here only about my own experience and my own opinions. In fact, in this hour, I have already implicitly answered the main question raised: I believe that methods courses may be useful. In fact, what I have presented to you in this hour was a sample of a methods course, or rather an outline of some topics which, in my opinion, a methods course offered to mathematics teachers should cover.

In fact, all the classes I have given to mathematics teachers were intended to be methods courses to some extent. The name of the class mentioned some subject matter, and the time was actually divided between that subject matter and methods: perhaps nine tenths for subject matter and one tenth for methods. If possible, the class was conducted in dialogue form. Some methodical remarks were injected incidentally, by myself or by the audience. Yet the derivation of a fact or the solution of a problem was almost regularly followed by a short discussion of its pedagogical implications. "Could you use this in your classes?" I asked the audience. "At which stage of the curriculum could you use it? Which point needs particular care? How would you try to get it across?" And questions of this nature (appropriately specified) were regularly proposed also in examination papers. My main work was, however, to choose such problems (like the two problems I have here presented) as would illustrate strikingly some pattern of teaching.

(3) The official recommendations call "methods" courses "curriculum-study" courses and are not very eloquent about them. Yet you can find there one recommendation that is excellent, I think. It is somewhat concealed; you must put two and two together, combining the last sentence in "curriculum study courses" and the recommendations for Level IV. But it is clear enough: A college instructor who offers a methods course to mathematics teachers should know mathematics at least on the level of a Master's degree. I would like to add: he should also have had some experience, however modest, of mathematical research. If he had no such experience how could he convey what may be the most important thing for prospective teachers, the spirit of creative work?

You have now listened long enough to the reminiscences of an old man. Some concrete good could come out of this talk if you give some thought to the following proposal which results from the foregoing discussion. I propose that the following two points should be added to the official recommendations of the Association:

I. *The training of teachers of mathematics should offer experience in independent ("creative") work on the appropriate level in the form of a Problem Solving Seminar or in any other suitable form.*

II. *Methods courses should be offered only in close connection either with subject matter courses or with practice teaching and, if feasible, only by instructors experienced both in mathematical research and in teaching.*

This paper was presented as an address at the 46th Annual Meeting of the Mathematical Association of America at Berkeley, California on January 27, 1963, and will appear in Volume 2 of the author's *Mathematical Discovery*. It is printed here by kind permission of the publishers, John Wiley & Sons, Inc.

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ORDERED GROUPS

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1. Introduction. One of the more recent movements in pure mathematics (see [1], pp. 214–237) has been concerned with the "remarriage" of algebraic systems based upon the properties of certain operations to those based upon order relations. The union is achieved by means of an axiom or axioms relating the operation(s) to the order relation. It is interesting to note that, appearing as special cases, we find such systems as the integers and real numbers whose order relations and algebraic operations were at one time "divorced" in order that venturesome mathematicians might freely explore the delights of each.

An important example is the *partially ordered group*, a system P satisfying (i) P is a group under the operation $+$; (ii) P is a partially ordered set under the relation $\geqq$; and (iii) if a and b are elements in P such that $a \geqq b$, then $x+a+y \geqq x+b+y$ for any pair of elements x and y in P. Thinking of ways in which this study might be extended, Frink proposed a definition of an *ideal* in a partially ordered set and suggested that one might consider systems in which the algebraic operations preserve ideals (see [2]). We are to explore the consequences of this and other generalizations of postulate (iii) while leaving the very general axioms, (i) and (ii), practically unchanged.

2. Preliminary definitions.

DEFINITION 1. *If F is a nonvoid subset of a partially ordered set P, F^* denotes the set of all elements x of P such that $x \geqq f$ for every element f of F, and F^+ denotes*

* Now with the General Electric Company, Huntsville, Alabama.

the set of all x in P such that $x \leqq f$ for every f of F. (If F consists of a single element f, we shall sometimes write f^* and f^+ to mean F^* and F^+, respectively.)

The following definition is a generalization of the concept of an ideal in a lattice ([1], p. 21). It differs from the definition suggested by Frink in the requirements that F^* be nonvoid.

DEFINITION 2. *A subset J of P is an ideal if for every finite nonvoid subset F of J, F^* is nonvoid and $(F^*)^+ \subset J$. A subset J is a dual ideal if for every finite nonvoid subset F of J, F^+ is nonvoid and $(F^+)^* \subset J$.*

Sets expressible as a^+, for some $a \in P$, are simple examples of ideals. These are known as *principal ideals.*

DEFINITION 3. *Elements a and b in P are connected if there exists a finite sequence*

$$a = x_1, x_2, \cdots, x_n = b$$

of elements of P such that x_i is comparable (i.e., related by $\geqq$) to x_{i+1} for $i=1, 2, \cdots, n-1$. A set C is said to be a connected set if for every pair of elements a and b in C, a is connected to b. A maximal connected set is called a component of P.

DEFINITION 4. *A partially ordered set P is said to be of length n if n is the largest integer for which there exist $n+1$ elements $x_1, x_2, \cdots, x_{n+1}$ in P such that*

$$x_1 > x_2 > \cdots > x_{n+1}.$$

If no such integer exists, P is said to be of infinite length.

3. Postulate systems. The following are several postulates with which the authors have replaced postulate (iii) in an effort to arrive at more general classes of ordered groups than the partially ordered group:

(iii)′ for any ideal J of P, $x+J+y$ is an ideal when x and y are any elements of P;

(iv) for any ideal J of P, $x+J+y$ is either an ideal or a dual ideal;

(v) for any ideal or dual ideal J of P, $x+J+y$ is either an ideal or a dual ideal;

(v)′ for each component C of P and each pair of elements x and y in P, either it is true that $x+a+y \geqq x+b+y$ for all elements a and b in C such that $a \geqq b$ or it is true that $x+a+y \leqq x+b+y$ for all pairs of elements a and b in C such that $a \geqq b$.

Assuming (i) and (ii), we shall discuss briefly the relationships among the other postulates. It is asserted that (iii) and (iii)′ are equivalent, that (v) and (v)′ are also equivalent, and that (iv) is equivalent to (v) provided a rather trivial class of systems defined by (iv) is excluded. These trivial cases are the systems of length one whose components are cosets of the component containing 0 and principal ideals having at least three distinct elements.

That postulate (v)′ implies (v), and consequently (iv), may be demonstrated by means of the two equalities

$$x + F^{*+} = (x + F)^{*+} \text{ (for } x \text{ order preserving)}$$

$$x + F^{*+} = (x + F)^{+*} \text{ (for } x \text{ order reversing)}$$

concerning any finite subset F of P. These equalities can be easily derived from (v)′. Employing only the first of the two equalities, we can show that (iii) implies (iii)′.

On the other hand, assume postulate (iii)′ and let a and b be any elements of P such that $a \geqq b$. Now, since $(x+a)^+$ is an ideal, $-x+(x+a)^+$ must also be one and, knowing that a is in $-x+(x+a)^+$, it follows from Definition 2 that $a^{*+} \subset -x+(x+a)^+$. Using the transitive property of the order relation, it may be shown that $b \in a^{*+}$. Consequently, $b \in -x+(x+a)^+$ and $x+b \in (x+a)^+$. The latter implies that postulates (iii) and (iii)′ are actually equivalent.

In order to justify all of the assertions concerning equivalence of postulates, it remains to be shown that postulate (iv) implies (v) (excluding the trivial systems referred to) and that (v) implies (v)′. The proofs for these latter implications are considerably involved and will be left to other papers.

Now we turn to the problem of representing the mathematical systems, introduced herein, in terms of more familiar systems. In order that a complete theory may be developed in this paper, we restrict ourselves to those systems P satisfying (i), (ii), (v)′, and (vi) P is a connected set (i.e., there is only one component of P). Omission of postulate (vi) leads to considerably complicated statements of the theorems and proofs. Some effects of its omission will be mentioned. The systems satisfying (i), (ii), and (v)′ are referred to as *partially ordered ideal-preserving groups*. The principal results show that any such system is order isomorphic to the ordinal product of a partially ordered ideal-preserving group of length zero or one and a partially ordered group.

Example 1. The multiplicative group of real numbers ordered according to magnitude is a connected partially ordered ideal-preserving group. It is the union of the set N of order preserving numbers (i.e., those elements x such that $a>b$ implies $x+a>x+b$ and $a+x>b+x$) and the set R of order reversing elements. The subset N is a normal subgroup of P with cosets N and R. It is a partially ordered group and is order isomorphic to R. Furthermore, every element of N is greater than every element of R.

All partially ordered ideal-preserving groups have order properties similar to those of the real numbers.

Example 2. The additive group of rational integers I or the rational integers modulo $2K$ (denoted by I_{2K}) are partially ordered ideal-preserving groups of length one when ordered in such a manner that $m>n$ if and only if m is even and $n=m\pm 1$.

Example 3. The direct sum $I_{2K} \dotplus P$ of I_{2K} ordered as in the previous example and any partially ordered group P forms a partially ordered ideal-preserving group when ordered as follows: (i) If $m \neq n$, $(m, x)>(n, y)$ if and only if $m>n$;

(ii) if $m=n$ and m is even, $(m, x)>(n, y)$ if and only if $x>y$; (iii) if $m=n$ and m is odd, $(m, x)>(n, y)$ if and only if $x<y$.

4. Strong constituents.

LEMMA 1. *If x is order preserving on the left (i.e., if $a>b$ implies that $x+a>x+b$), then $-x$ is order preserving on the left.*

LEMMA 2. *If x is order preserving on the left (right), then it is order preserving on the right (left).*

Proof. Let y be any element of P which is order preserving on the left and on the right (e.g., the 0 element) or order reversing on both sides. Using Lemma 1, it may be shown with little difficulty that any element comparable to y must be either order preserving on both sides or order reversing on both sides. The remainder of the proof of Lemma 2 must employ finite induction and the fact that P is connected. The lemma is not generally true when P is not connected.

DEFINITION 5. *The element a is strongly connected to b if and only if there exists a sequence $a=x_1, x_2, \cdots, x_n=b$ in which x_i is comparable to x_{i+1} and such that x_i is order preserving for $i=1, 2, \cdots, n$ or such that x_i is order reversing for $i=1, 2, \cdots, n$. A set S is strongly connected if and only if every two elements of S are strongly connected. A maximal strongly connected set is called a strong constituent of P, and S_a will denote the strong constituent containing a.*

Strong connectivity is an equivalence relation on P but does not necessarily have meaning for a partially ordered set per se.

LEMMA 3. *If a and b are strongly connected, then $x+a$ and $x+b$ are strongly connected.*

Proof. If a and b are strongly connected, there exists by Definition 5 a sequence

$$a = y_1, y_2, \cdots, y_n = b$$

such that y_i is comparable to y_{i+1} and such that the elements of the sequence are all order preserving or all order reversing. Either $x+y_i$ is order preserving for all i or order reversing for all i $(i=1, 2, \cdots, n)$; for example, if x and y_i are both order reversing, then $a>b$ implies $y_i+a<y_i+b$, which in turn clearly implies that $(x+y_i)+a>(x+y_i)+b$. Furthermore, according to axiom (v)′, $x+y_i$ is comparable to $x+y_{i+1}$ for $i=1, 2, \cdots, n-1$.

THEOREM 1. *The subset S_0 is a normal subgroup of P, and the strong constituents of P are the cosets of S_0.*

Proof. By Lemma 3 if $a\in P$ and $b\in P$, then $a+b\in P$ and $-a\in P$, proving that S_0 is a subgroup. The set $a+S_0$ is contained in S_a because $a\in S_a$ and all elements of $a+S_0$ are strongly connected to a according to Lemma 3. Likewise,

$-a+S_a$ is contained in S_0 and, therefore, $a+S_0=S_a$. Similarly $S_0+a=S_a$ so that S_0 is a normal subgroup of P.

The quotient group P/S_0 is a partially ordered set when ordered according to the rule: $S_a \geqq S_b$ if and only if $S_a=S_b$ or every element in S_a is greater than every element in S_b. The symbol P/S_0 will henceforth represent the group P/S_0 ordered in this manner.

LEMMA 4. *If $a>b$ and $S_a \neq S_b$, then $a'>b'$ for every element a' in S_a and every element b' in S_b.*

Proof. It cannot be the case that a and b are both order preserving or both order reversing elements. Assume that a is order preserving and b is order reversing. Let a_2 be a member of S_a which is comparable to $a_1=a$. Then $a_2>(b-a_1)+a_2$. If $a_2 \geqq a_1$, then $a_2>b$ which is the desired result. So assume that $a_2 \leqq a_1$. Hence $-a_1+a_2 \leqq 0$ and, therefore, $(b-a_1)+a_2 \geqq b$. Thus $a_2>b$. There is a similar proof of the latter inequality for the case in which a is order reversing and b is order preserving. Using mathematical induction it can be shown that $a'>b$.

A proof very similar to that of the previous paragraph shows that $a'>b'$. The lemma follows.

THEOREM 2. *The mapping h given by $h(x)=S_x$ is a group homomorphism of P onto P/S_0 such that $x \geqq y$ implies $h(x) \geqq h(y)$ and $h(x)>h(y)$ implies $x>y$. Furthermore, $h(x)$ is order preserving (reversing) in P/S_0 if and only if x is order preserving (reversing) in P.*

Proof. Clearly, h is a group homomorphism. Consider elements x and y such that $x \geqq y$. If x and y are in the same strong component, then $h(x)=h(y)$. However, if they are not, according to Lemma 4 every element of S_x is over every element of S_y. Hence $x \geqq y$ implies $h(x) \geqq h(y)$. That the inequality $h(x)>h(y)$ implies $x>y$ follows from the definition of the order relation on P/S_0.

Suppose $h(a)>h(b)$. When $h(x)+h(a)>h(x)+h(b)$, it is true that $h(x+a) >h(x+b)$ and $x+a>x+b$. On the other hand, the inequality $h(x)+h(a)<h(x) +h(b)$ implies $h(x+a)<h(x+b)$ and $x+a<x+b$. Thus, when $h(x)$ is order preserving (reversing), so is x. The converse can be easily proven.

COROLLARY 2.1. *The system P/S_0 is a connected partially ordered ideal-preserving group. Moreover, either P/S_0 is totally unordered or every maximal chain of P/S_0 is of length one.*

Proof. The first statement follows directly from the theorem and the fact that P is a connected partially ordered ideal-preserving group.

Assume now that $h(a)>h(b)>h(c)$. Then $a>b>c$ and—since some pair of a, b, and c must be either order preserving or order reversing—at least two of the three must be in the same strong constituent; that is, at least two of $h(a)$, $h(b)$, and $h(c)$ are equal, a contradiction. Therefore, P/S_0 can have no chain of length greater than one. It is easily shown that, if P/S_0 has at least one chain of length one, then all maximal chains of P/S_0 are of length one.

In fact, any partially ordered ideal-preserving group of finite length must either be totally unordered or such that every maximal chain is of length one, because finite length implies the existence of a maximal element m and the translation $x+b=m$, of the central element b of a chain $a>b>c$ of length three into m, creates the impossible situation in which

$$x + b > x + a, \qquad x + b > x + c.$$

Observe that, if P/S_0 is totally unordered, then P is a partially ordered group. (This is not generally true when P is not connected.)

THEOREM 3. *The strong constituent S_0 is a partially ordered group. Furthermore, every coset of S_0 is order isomorphic to S_0, and if a is any element of the coset S_a of S_0, such an isomorphism may be described as follows*: (i) *If a is order preserving, then the element b in S_a corresponds to $-a+b$ in S_0*; (ii) *if a is order reversing, then the element b in S_a corresponds to $-b+a$ in S_0.*

Proof. According to Theorem 1, S_0 is a strong constituent of P. All of its elements are order preserving because 0 is such an element. From axiom (ii) and Theorem 1, it follows that S_0 is both a partially ordered set and a group. Therefore, S_0 is a partially ordered group.

The correspondence indicated is clearly one-to-one and, from the properties of the cosets of a normal subgroup, it is known that $-a+S_a=S_0$ and $-S_a+a=S_0$ since $-a+a=0$ and $0\in S_0$. Hence, the correspondence maps all of S_a onto S_0. If a is order reversing, then $b>c$ implies $-a+b<-a+c$, which in turn implies $-(-a+b)>-(-a+c)$. The latter implication follows from the observation that, if x, $y\in S_0$ and $x>y$, then $0>y-x$ and $-y>-x$. Now $-(-a+b)=-b+a$ and $-(-a+c)=-c+a$. Therefore, $b>c$ implies $-b+a>-c+a$. The converse is obtained by reversing the proof. The case in which a is order preserving can be proved in a similar but simpler manner.

5. Representation theorem.

THEOREM 4. *Any partially ordered ideal-preserving group P (connected or otherwise) is order isomorphic to the ordinal product $(P/S_0)\circ S_0$ of the partially ordered ideal-preserving group P/S_0 of length zero or one and the partially ordered group S_0. If A is a set of coset representatives, the isomorphism may be described as follows*: (i) *If the element b of P is order preserving, then b corresponds to $(S_b,-a+b)$ where $a\in S_b\cap A$*; (ii) *if the element b of P is order reversing, then b corresponds to $(S_b, -b+a)$ where $a\in S_b\cap A$.*

Proof. Suppose b and c correspond to (S_b, y) and (S_b, z), respectively. Then $b\in S_b$ and $c\in S_b$. It follows from Theorem 3 that $b\geqq c$ if and only if $y\geqq z$. Hence $b\geqq c$ if and only if $(S_b, y)\geqq(S_b, z)$ since, according to the definition of the ordinal product of two partially ordered sets, when $u=v$, $(u, y)\geqq(v, z)$ if and only if $y\geqq z$.

Suppose now that b and c correspond to (S_b, y) and (S_c, z) respectively,

where $S_b \neq S_c$. Then $b \neq c$. If $b > c$, by definition of the ordering on P/S_0, $S_b > S_c$. Hence, if $b > c$, $(S_b, y) > (S_c, z)$. When $(S_b, y) > (S_c, z)$, it follows that $S_b > S_c$. Thus the inequality $(S_b, y) > (S_c, z)$ implies that $b > c$. Therefore, $b > c$ if and only if $(S_b, y) > (S_c, z)$, completing the proof.

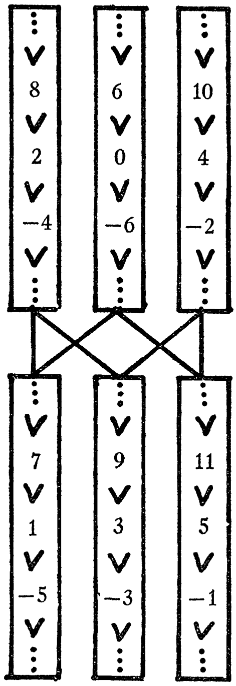

FIG. 1 (a)

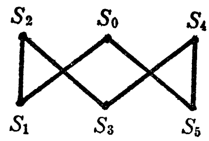

FIG. 1 (b)

Example 4. Let P be the partially ordered ideal-preserving group composed of the additive group of integers ordered as shown in Fig. 1(a). Each block of integers in that figure represents a class of integers modulo 6. A straight line connecting a block to one below it signifies that all elements contained in the first block are greater than those contained in the latter. The partially ordered ideal-preserving group P/S_0 is the group of integers modulo 6 ordered as indicated in Fig. 1(b).

6. Systems of finite length. By virtue of Theorem 4, we turn to a discussion of partially ordered ideal-preserving groups of finite length. Since any one of these systems which is connected consists of a single element or is such that every maximal chain is of length one, consideration will be given only to those of length one. Furthermore, we will consider, with no loss of generality, only those systems having the identity element 0 as a maximal element. For every element a of a partially ordered set P, T_a will be used to represent the set of all elements x of P, $x \neq a$, such that x is comparable to a.

THEOREM 5. *A connected partially ordered ideal-preserving group P having length one and the identity element as a maximal element satisfies the following conditions*: (i) *If* $t\in T_0$, *then* $-t\in T_0$; (ii) *any element a of P can be expressed as a linear combination* $t_1+t_2+\cdots+t_n$, *where* $t_i\in T_0$; (iii) *if* $t_1+t_2+\cdots+t_n=0$, *where* $t_i\in T_0$, *then n is even*; (iv) $a+T_0=T_0+a$ *for all elements a in P*; (v) $a+T_0=T_0$ *for all elements a in P*; (vi) *the element* $t_1+t_2+\cdots+t_n$, *where* $t_i\in T_0$, *is maximal* (*minimal*) *if and only if n is even* (*odd*). *Conversely, let G be a group containing a nonvoid subset* T_0 *satisfying conditions* (i)–(iv). *Define an ordering on G using conditions* (v) *and* (vi). *Under this ordering G is a connected partially ordered ideal-preserving group P having length one and the identity element as a maximal element, and* T_0 *is the set of all elements of G which are less than* 0.

Proof. Conditions (i), (iv), and (v) can be easily proven. The argument for condition (ii) employs finite induction and the fact that, for any two comparable elements, x_{k-1} and x_k, there exists, in accordance with condition (v), an element t in T_0 such that $x_k=x_{k-1}+t$.

Condition (vi) can be proven with the use of an inductive argument, whereas (iii) follows immediately from (vi).

Conversely, let G be a group satisfying (i) to (iv). Suppose that

$$t_1 + t_2 + \cdots + t_m = t_1' + t_2' + \cdots + t_n',$$

where all of the elements t_i and t_i' are in T_0. Then

$$t_1 + t_2 + \cdots + t_m - t_n' - t_{n-1}' - \cdots - t_1' = 0.$$

The latter equation, along with conditions (i) and (iii), implies that $m+n$ is even; i.e., that m and n are both even or both odd. Thus, (v) and (vi) establish a well-defined partial ordering on G. It can be shown that under this ordering G is a connected partially ordered ideal-preserving group.

In summary, the structure of P as a partially ordered set can be determined up to order isomorphism when the partially ordered group S_0 and the partially ordered ideal-preserving group P/S_0 of length zero or one are known. Partially ordered groups have been the subject of previous investigations [1], and partially ordered ideal-preserving groups of length zero or one have been analyzed in detail (Theorem 5).

7. Conditions that a group may be ordered to form a partially ordered ideal-preserving group.

THEOREM 6. *Necessary and sufficient conditions that a group G may be ordered so that it forms a connected partially ordered ideal-preserving group P, in which the identity element of* P/S_0 *is maximal and such that P is not a partially ordered group, are* (i) *G contains a normal subgroup which can be ordered so that it forms a partially ordered group R*; (ii) *the maximal connected subgroup S of R is normal and the quotient group G/S can be ordered so that it forms a connected partially ordered ideal-preserving group P of length one with a maximal identity element*;

(iii) $a+0^*=0^*+a$ *for all elements* a *in* G. *Furthermore, the only ordering which can be defined on* G *such that* G *would be a connected partially ordered ideal-preserving group* P *with* $S_0=S$ *and* $P/S_0=P$, *where the equalities imply that the orderings on* S *and* G/S *are unchanged, may be described as follows*: (1) *If* $a+S\neq b+S$, *then* $a>b$ *if and only if* $a+S>b+S$ *in* G/S; (2) *if* $a+S=b+S$ *is maximal* (*minimal*) *in* G/S, *then* $a\geqq b$ *if and only if* $a-b\in 0^*(b-z\in 0^*)$.

Condition (i) can be expressed in terms of necessary and sufficient algebraic conditions (see [1], p. 214). According to Theorem 5 condition (ii) can also be written in terms of such conditions.

Proof. We have already shown that conditions (i) and (iii) of Theorem 6 are necessary. If $b>0$, then $a+b=c+a$ where $c=a+b-a>0$, because $-a$ is order preserving (reversing) if and only if a is order preserving (reversing). Thus condition (iii) is necessary.

Assume conditions (i)–(iii) hold true of G and order G in the manner of (1) and (2). It can be easily shown that G is a partially ordered set and that the order preserving or reversing properties of any element x in G are identical to those of $x+S$ in G/S. Therefore, if G/S is ordered so as to form a partially ordered ideal-preserving group, then G is such a group.

Let P be a partially ordered ideal-preserving group like that described in Theorem 6. That (1) is true of P follows from Theorem 2. Suppose $a+S_0=b+S_0$ is maximal in P/S_0. It follows from Theorem 5 that the maximal elements of P/S_0 are its order preserving elements. Thus b is order preserving, and the inequality $a\geqq b$ implies $a-b\in 0^*$. Similarly, if $a+S_0=b+S_0$ is minimal, then the inequality $a\geqq b$ implies $b-a\in 0^*$. This establishes the uniqueness of (1) and (2).

It must not be concluded that the ordered systems S_0 and P/S_0 determine P as a group. That this is not the case is demonstrated by the two following examples.

Example 5. Let I_2 be the additive group of integers modulo 2 ordered so that $0>1$, and let S be the additive group of even integers in their natural order. The direct sum $I_2\dotplus S$ becomes a partially ordered ideal-preserving group P when ordered as follows:

$$\cdots>(0,2)>(0,0)>(0,-2)>\cdots>(1,-2)>(1,0)>(1,2)>\cdots.$$

In this example S_0 consists of the number pairs $(0, a)$, where $a\in S$, ordered as indicated, and $P/S_0=I_2$. Clearly S and S_0 are isomorphic in both the order sense and the group sense.

Example 6. Order the additive group of integers as follows:

$$\cdots>2>0>-2>\cdots>-3>-1>1>3>\cdots.$$

The result is a partially ordered ideal-preserving group P. Again S_0 is isomorphic to the additive group of even integers S ordered in the usual manner, and P/S_0 is isomorphic to I_2 ordered as in Example 5. However, P is not isomorphic in

the group sense to the system P of Example 5, for in that case there is an element $(1, 0)$, not equal to the identity element $(0, 0)$, such that $(1, 0)+(1, 0)=(0, 0)$.

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SAMPLING PROPERTIES OF THE MEDIAN OF A LAPLACE DISTRIBUTION

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Introduction. In sampling situations in which the median is a more appropriate statistic than the arithmetic mean, it would be useful to have a symmetric distribution function having the sample median as the maximum likelihood estimator of the central point of the distribution, and for which a statistic dependent on the median is the maximum likelihood estimator of the dispersion parameter of the distribution function. A distribution function having these properties is the Laplace distribution function, [1, p. 247],

$$f(x; \mu, \lambda) = \frac{1}{2\lambda} e^{-|x-\mu|/\lambda}, \qquad \text{where } \lambda > 0. \tag{1}$$

This distribution is symmetric about μ and decreases exponentially to right and left, with λ the dispersion parameter. It will be seen that the sample median $\tilde{x}$ is the maximum likelihood estimator of μ. The maximum likelihood estimator of λ will be shown, and the distribution functions of each of these statistics will be found.

Known properties of $f(x; \mu, \lambda)$. It is known that the sample median $\tilde{x}$ is the maximum likelihood estimator of the parameter μ, and that the average absolute deviation from the sample median,

$$\lambda = \sum_{i=1}^{n} \frac{|x_i - \tilde{x}|}{n} \tag{2}$$

is the maximum likelihood estimator of λ, the dispersion parameter.

Proofs of these properties follow from the usual analysis of the likelihood function

$$\phi = \frac{1}{(2\lambda)^n} e^{-\Sigma|x_i-\mu|/\lambda}. \tag{3}$$

Maximizing ϕ with respect to μ is equivalent to minimizing $\sum_{i=1}^{n}|x_i-\mu|$. It is well known that this sum is a minimum for a set of $\{x_i\}$ when μ is the median of the set. Furthermore, setting $\partial\phi/\partial\lambda=0$ readily yields the estimator for λ, shown in (2).

The distribution of the sample median. Cramér [1, p. 370] has shown that for any distribution $f(x)$, with cumulative distribution $F(x)$, the distribution of the rth value, x_r, from the top of a sample of n, is given by,

$$g(x_r) = n\binom{n-1}{r-1}[F(x_r)]^{n-r}[1-F(x_r)]^{r-1}f(x_r). \tag{4}$$

We now apply this general result to the problem of finding the distribution of the median of a sample from a Laplace distribution. First, a mild restriction will be placed on the sample, namely, that its size be an odd integer. Hence n in (4) is replaced by $2m+1$, and $r=m+1$ is then associated with the sample median. Furthermore, we transform $f(x)$ as shown in (1) to a "standard" Laplace distribution as follows:

Restate (1) in the equivalent form

$$f(x;\mu,\lambda) = \begin{cases} \dfrac{1}{2\lambda}\, e^{(x-\mu)/\lambda} & \text{for } x \leqq \mu \\[2ex] \dfrac{1}{2\lambda}\, e^{-(x-\mu)/\lambda} & \text{for } x \geqq \mu. \end{cases} \tag{1a}$$

Then using the transformation $z=(x-\mu)/\lambda$ we have the standard Laplace distribution, (note that the $1/\lambda$ factor is absorbed in transforming the implied differential $dx=\lambda dz$),

$$f^*(z) = \begin{cases} \tfrac{1}{2}e^{z} & \text{for } z \leqq 0 \\ \tfrac{1}{2}e^{-z} & \text{for } z \geqq 0. \end{cases} \tag{5}$$

The cumulative standard distribution is then easily found to be

$$F^*(z) = \begin{cases} \tfrac{1}{2}e^{z} & \text{for } z \leqq 0 \\ 1-\tfrac{1}{2}e^{-z} & \text{for } z \geqq 0. \end{cases} \tag{6}$$

Applying the general theorem (4) we find the distribution of the median $\tilde{z}$ from a sample of size $2m+1$ taken from the standard Laplace distribution to be

$$g^*(\tilde{z}) = \frac{(2m+1)!}{(m!)^2}[F^*(\tilde{z})]^m[1-F^*(\tilde{z})]^m f^*(\tilde{z}), \tag{7}$$

where $f^*(\tilde{z})$ and $F^*(\tilde{z})$ are obtained from (5) and (6) respectively replacing z by $\tilde{z}$. Thus, in detail,

$$(8)\qquad g^*(\tilde{z}) = \begin{cases} \dfrac{(2m+1)!}{(m!)^2} (\tfrac{1}{2}e^{\tilde{z}})^m(1-\tfrac{1}{2}e^{\tilde{z}})^m\tfrac{1}{2}e^{\tilde{z}} & \text{for } \tilde{z} \leqq 0 \\ \dfrac{(2m+1)!}{(m!)^2} (1-\tfrac{1}{2}e^{-\tilde{z}})^m(\tfrac{1}{2}e^{-\tilde{z}})^m\tfrac{1}{2}e^{-\tilde{z}} & \text{for } \tilde{z} \geqq 0. \end{cases}$$

By means of the binomial theorem this reduces to

$$(9)\qquad g^*(\tilde{z}) = \begin{cases} \sum_{r=0}^{m} K(m,r)\dfrac{m+r+1}{2} e^{\tilde{z}(m+r+1)} & \text{for } \tilde{z} \leqq 0 \\ \sum_{r=0}^{m} K(m,r)\dfrac{m+r+1}{2} e^{-\tilde{z}(m+r+1)} & \text{for } \tilde{z} \geqq 0, \end{cases}$$

where

$$(10)\qquad K(m,r) = (-1)^r\frac{(2m+1)!}{m!}\,\frac{1}{2^{m+r}}\,\frac{1}{r!(m-r)!(m+r+1)}\,.$$

If we transform to x variates by $\tilde{z}=(\tilde{x}-\mu)/\lambda$, where $\tilde{x}$ is the sample median, the distribution function for $\tilde{x}$ is

$$(11)\qquad g(\tilde{x}) = \begin{cases} \sum_{r=0}^{m} K(m,r)\dfrac{1}{2\lambda_{\tilde{x}}} e^{(\tilde{x}-\mu)/\lambda_{\tilde{x}}} & \text{for } \tilde{x} \leqq \mu \\ \sum_{r=0}^{m} K(m,r)\dfrac{1}{2\lambda_{\tilde{x}}} e^{-(\tilde{x}-\mu)/\lambda_{\tilde{x}}} & \text{for } \tilde{x} \geqq \mu, \end{cases}$$

where $\lambda_{\tilde{x}}=\lambda/(m+r+1)$ is the dispersion parameter associated with $\tilde{x}$.

Finally, we have

$$(12)\qquad g(\tilde{x}) = \sum_{r=0}^{m} K(m,r)\frac{1}{2\lambda_{\tilde{x}}} e^{-|\tilde{x}-\mu|/\lambda_{\tilde{x}}}.$$

Thus the sample median, unlike the sample mean in the theory of the normal distribution, is not distributed by a single function of the type of the original population, but rather by a finite set of $m+1$ such functions, each member of the set with a different dispersion parameter $\lambda_{\tilde{x}}$, and each multiplied by a weighting factor $K(m, r)$ so that the superposition of all members of the set is a distribution function.

The distribution of $\tilde{\lambda}$ with λ and μ known. If we consider the likelihood function (3) as a function of λ only, $d\phi/d\lambda=0$ yields

$$(13)\qquad \lambda = \sum_{i=1}^{n}\frac{|x_i-\mu|}{n}\,.$$

Using the transformation $z_i = (x_i - \mu)/\lambda$, we obtain

$$\tilde{\lambda} = \frac{\lambda}{n} \sum_{i=1}^{n} | z_i | . \tag{14}$$

Define

$$s \equiv \sum_{i=1}^{n} | z_i | . \tag{15}$$

If we obtain the distribution of s, the distribution of $\tilde{\lambda}$ will easily follow. The moment generating function of s is,

$$m(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{st} \prod_{i=1}^{n} f^*(z_i) dz_i, \tag{16}$$

where $f^*(z)$ is given by (5). We see from (15) that

$$m(t) = \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{t|z_i|} f^*(z_i) dz_i.$$

Since the integrands are all even functions,

$$m(t) = 2^n \prod_{i=1}^{n} \int_{0}^{\infty} \frac{1}{2} e^{t|z_i| - z_i} dz_i$$

$$m(t) = \prod_{i=1}^{n} \int_{0}^{\infty} e^{-z_i(1-t)} dz_i$$

$$m(t) = \frac{1}{(1-t)^n}, \qquad \text{where } t < 1. \tag{17}$$

We note that $m(t)$ is in the form of a moment generating function of a Gamma distribution ([2], pp. 112–115), whence the distribution of s is

$$h^*(s) = \begin{cases} \dfrac{1}{(n-1)!} s^{n-1} e^{-s} & \text{for } s \geqq 0 \\ 0 & \text{for } s < 0. \end{cases} \tag{18}$$

Since $\tilde{\lambda} = (\lambda/n)s$ from (14) and (15), the distribution of $\tilde{\lambda}$ is

$$h(\tilde{\lambda}) = \begin{cases} \dfrac{1}{(n-1)!} \left(\dfrac{n\tilde{\lambda}}{\lambda}\right)^{n-1} e^{-(n\tilde{\lambda}/\lambda)} \left(\dfrac{n}{\lambda}\right), & \text{for } \dfrac{n\tilde{\lambda}}{\lambda} \geqq 0 \\ 0 & \text{for } \dfrac{n\tilde{\lambda}}{\lambda} < 0. \end{cases} \tag{19}$$

Thus $s = n\tilde{\lambda}/\lambda$ has a distribution which is a special case of the general Gamma distribution

$$\Gamma(x) = \begin{cases} \dfrac{1}{\alpha!\beta^{\alpha+1}} x^{\alpha}e^{-x/\beta} & \text{for } x \geqq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{20}$$

for $\alpha = n-1$ and $\beta = 1$.

The distribution of $\tilde{\lambda}$ with λ known and μ unknown. The preceding discussion assumed prior knowledge of μ. This requirement is now relaxed and the distribution of the more general maximum likelihood form of $\tilde{\lambda}$ given in (2) will be obtained as a finite sum of Gamma distributions.

As before, the problem is standardized by the transformation $z = (x-\mu)/\lambda$. The form of $\tilde{\lambda}$ in (2) is easily seen to become

$$\tilde{\lambda} = \frac{\lambda}{n} \sum_{i=1}^{n} | z_i - \tilde{z} | . \tag{21}$$

For simplicity we shall find the distribution of

$$s \equiv \sum_{i=1}^{n} | z_i - \tilde{z} | , \tag{22}$$

whence the distribution of $\tilde{\lambda}$ will easily follow.

The joint distribution function of a sample of $2m+1$ independent standard variates $\{z_i\}$ is from (5)

$$f^*(z_1, z_2, \cdots, z_{2m+1}) = \frac{2(2m+1)!}{2^{2m+1}} \exp\left[\sum_{i=1}^{m+r+1} z_i - \sum_{i=m+r+2}^{2m+1} z_i\right], \tag{23}$$

where r is the number of variates in a particular sample greater than the median and less than zero. Thus

$$z_1 < z_2 < \cdots < z_{m+r+1} < 0 < z_{m+r+2} < \cdots < z_{2m+1}.$$

Therefore, (23) is really $m+1$ different expressions as $r = 0, 1, 2, \cdots, m$. The formulation has symmetry about zero giving rise to the factor 2 in the numerator of the numerical coefficient. The factor $(2m+1)!$ takes into account all possible orderings of a particular type of sample. Note that $z_{m+1} \equiv \tilde{z}$.

For example, if $n=5$, $m=2$:

$$f^*(z_1, z_2, z_3, z_4, z_5) = \begin{cases} \dfrac{2(5!)}{2^5} e^{z_1+z_2+z_3-z_4-z_5} & \text{for } r = 0 \\ \dfrac{2(5!)}{2^5} e^{z_1+z_2+z_3+z_4-z_5} & \text{for } r = 1 \\ \dfrac{2(5!)}{2^5} e^{z_1+z_2+z_3+z_4+z_5} & \text{for } r = 2. \end{cases} \tag{24}$$

It may be verified that (24) is indeed a distribution function by integrating over all possible values of the $\{z_i\}$, remembering that they are ordered. Thus,

$$\frac{2(5!)}{2^5}\Bigg[\int_0^{\infty}\int_0^{z_5}\int_{-\infty}^{0}\int_{-\infty}^{z_3}\int_{-\infty}^{z_2} e^{z_1+z_2+z_3-z_4-z_5}dz_1dz_2dz_3dz_4dz_5$$

$$+\int_0^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{z_4}\int_{-\infty}^{z_3}\int_{-\infty}^{z_2} e^{z_1+z_2+z_3+z_4-z_5}dz_1dz_2dz_3dz_4dz_5$$

$$+\int_{-\infty}^{0}\int_{-\infty}^{z_5}\int_{-\infty}^{z_4}\int_{-\infty}^{z_3}\int_{-\infty}^{z_2} e^{z_1+z_2+z_3+z_4+z_5}dz_1dz_2dz_3dz_4dz_5\Bigg] = 1.$$

Similarly for the general case it may be verified that

$$\frac{2(2m+1)!}{2^{2m+1}}\sum_{r=0}^{m}\int_{2m+1}\int_{2m}\cdots\int_{m+r+3}\int_{m+r+2}\int_{m+r+1}\int_{m+r}\cdots \int_2\int_1 f(z_1, z_2 \cdots z_{2m+1})\prod_{i=1}^{2m+1} dz_i = 1, \tag{25}$$

where the symbol under each integral sign indicates the index of the z being integrated and the respective limits of integration are given by the following table.

TABLE I

Index	Variable	Limits
1	z_1	$-\infty$ to z_2
2	z_2	$-\infty$ to z_3
.	.	.
.	.	.
.	.	.
$m+r$	z_{m+r}	$-\infty$ to z_{m+r+1}
$m+r+1$	z_{m+r+1}	$-\infty$ to 0
$m+r+2$	z_{m+r+2}	0 to z_{m+r+3}
$m+r+3$	z_{m+r+3}	0 to z_{m+r+4}
.	.	.
.	.	.
.	.	.
$2m$	z_{2m}	0 to z_{2m+1}
$2m+1$	z_{2m+1}	0 to ∞

The integrand $f^*(z_1, z_2, \cdots, z_{2m+1})$ is, of course, the joint distribution of the $\{z_i\}$ of the sample and is given by (23).

Having the joint distribution of the $\{z_i\}$ we now proceed to derive $m(t)$, the moment generating function of s. Using (22) and (23) we have

$$m(t) = \frac{2(2m+1)!}{2^{2m+1}} \sum_{r=0}^{m} \int_{2m+1} \cdots \int_{1} \cdot \exp\left[\sum_{i=1}^{2m+1} |z_i - \bar{z}| t + \sum_{i=1}^{m+r+1} z_i - \sum_{i=1}^{2m+1} z_i \right] \prod_{i=1}^{2m+1} dz_i, \tag{26}$$

where the order of integration and the limits are those found in Table I. The absolute sign in the exponent may be disposed of if we realize that $\bar{z} = z_{m+1}$ and that for every $z_i < \bar{z}$ there is a corresponding $z_i > \bar{z}$. Thus, for example, when $i=2$ the term $|z_2 - \bar{z}| t = (\bar{z} - z_2)t$ is matched with $|z_{2m} - \bar{z}| t = (z_{2m} - \bar{z})t$ and the sum of the terms is $-z_2 t + z_{2m} t$. Thus the $\bar{z}t$ terms cancel out and (26) becomes, letting

$$K_1 \equiv \frac{2(2m+1)!}{2^{2m+1}}, \tag{27}$$

$$m(t) = K_1 \sum_{r=0}^{m} \int_{2m+1} \cdots \int_{1} \cdot \exp\left[t\left(- \sum_{i=1}^{m} z_i + \sum_{i=m+2}^{2m+1} z_i \right) + \sum_{i=1}^{m+r+1} z_i - \sum_{i=m+r+2}^{2m+1} z_i \right] \prod_{i=1}^{2m+1} dz_i,$$

or

$$m(t) = K_1 \sum_{r=0}^{m} \int_{2m+1} \cdots \int_{1} \cdot \exp\left[\sum_{i=1}^{m} (1-t) z_i + z_{m+1} + \sum_{m+2}^{m+r+1} (1+t) z_i - \sum_{m+r+2}^{2m+1} (1-t) z_i \right] \prod_{i=1}^{2m+1} dz_i. \tag{28}$$

The evaluation of this integral involves some complicated but not very profound algebra. It is seen that

$$m(t) = K_1 \sum_{r=0}^{m} \frac{1}{m!(m-r)!(1-t)^{2m-r} \prod_{k=0}^{r} [(m+1+k) - (m-k)t]}. \tag{29}$$

This is now in a form which may be separated, by the technique of partial fractions, into terms whose inverses are Gamma distributions. Thus

$$m(t) = K_1 \sum_{r=0}^{m} \frac{1}{m!(m-r)!(1-t)^{2m-r} \prod_{k=0}^{r} (m+1+k)[1 - (m-k)t/(m+k+1)]}$$

may be expressed as

$$m(t) = \sum_{i=1}^{2m} \frac{a_i}{(1-t)^i} + \sum_{i=1}^{m} \frac{b_i}{(h_i - t)}, \tag{30}$$

where

$$h_i = \frac{m+i}{m-i+1}$$

and the a_i and b_i are determinable from the theory of partial fractions.

From the theory of the Gamma distribution [2, p. 115] the distribution of s is seen to be,

$$h(s) = \sum_{i=1}^{2m} \frac{a_i}{(i-1)!} s^{i-1}e^{-s} + \sum_{i=1}^{m} b_i e^{-h_i s}. \tag{31}$$

Finally, since $\tilde{\lambda} = \lambda s/n$, the distribution of $\tilde{\lambda}$ is

$$f(\tilde{\lambda}) = \frac{n}{\lambda}\left[\sum_{i=1}^{2m} \frac{a_i}{(i-1)!}\left(\frac{n\tilde{\lambda}}{\lambda}\right)^{i-1} e^{-n\tilde{\lambda}/\lambda} + \sum_{i=1}^{m} b_i e^{-(nh_i\tilde{\lambda}/\lambda)}\right]. \tag{32}$$

Thus $\tilde{\lambda}$ from a sample of $n = 2m+1$ variates is seen to be distributed as a sum of $3m$ Gamma distributions.

To illustrate the above theory, we shall derive the distribution of $\tilde{\lambda}$ for a case of $n=5$. It will be seen that the arithmetical complexities in the determination of the coefficients of the partial fractions are considerable even for this simple case.

To determine the moment generating function for the case $n=5$, use equation (28), where $m=2$ since $n=2m+1$. This gives

$$\begin{aligned} m(t) = \frac{2\cdot 5!}{2^5}\Bigg[&\int_0^\infty\int_0^{z_5}\int_{-\infty}^{0}\int_{-\infty}^{z_3}\int_{-\infty}^{z_2} e^{t(z_5+z_4-z_2-z_1)+(z_1+z_2+z_3-z_4-z_5)}dz_1dz_2dz_3dz_4dz_5 \\ &+\int_0^\infty\int_{-\infty}^{0}\int_{-\infty}^{z_4}\int_{-\infty}^{z_3}\int_{-\infty}^{z_2} e^{t(z_5+z_4-z_2-z_1)+(z_1+z_2+z_3+z_4-z_5)}dz_1dz_2dz_3dz_4dz_5 \\ &+\int_{-\infty}^{0}\int_{-\infty}^{z_5}\int_{-\infty}^{z_4}\int_{-\infty}^{z_3}\int_{-\infty}^{z_2} e^{t(z_5+z_4-z_2-z_1)+(z_1+z_2+z_3+z_4+z_5)}dz_1dz_2dz_3dz_4dz_5\Bigg]. \end{aligned}$$

On integrating we find that

$$\begin{aligned} m(t) = \frac{2\cdot 5!}{2^5}&\ \frac{1}{2!2!(1-t)^4(3-2t)} + \frac{1}{2!(1-t)^3(3-2t)(4-t)} \\ &+ \frac{1}{2(1-t)^2(3-2t)(4-t)5}. \end{aligned}$$

Observe that this is equation (29), where

$$K_1 = \frac{2[(2m+1)!]}{2^{2m+1}} = \frac{2\cdot 5!}{2^5}.$$

Combining the three fractions we have

$$m(t) = \frac{2\cdot 5!}{2^5\cdot 20}\left[\frac{32 - 19t + 2t^2}{(1-t)^4(3-2t)(4-t)}\right] = \frac{2\cdot 5!}{2^5\cdot 20\cdot 2}\left[\frac{32 - 19t + 2t^2}{(1-t)^4(\frac{3}{2}-t)(4-t)}\right]$$

which in the form of (30) is

$$(33) \qquad m(t) = \frac{a_1}{1-t} + \frac{a_2}{(1-t)^2} + \frac{a_3}{(1-t)^3} + \frac{a_4}{(1-t)^4} + \frac{b_1}{\frac{3}{2}-t} + \frac{b_2}{4-t}\cdot$$

By the method of partial fractions one obtains the solution for the a's and b's as:

$$a_1 = -173/18 \qquad a_2 = 29/6 \qquad a_3 = -5/2$$

$$a_4 = 15/8 \qquad b_1 = 48/5 \qquad b_2 = 1/90.$$

Substituting in (31) we obtain

$$(34) \qquad h(s) = \left(a_1 + a_2 s + \frac{a_3}{2!}s^2 + \frac{a_4}{3!}s^3\right)e^{-s} + b_1 e^{-(3/2)s} + b_2 e^{-4s}.$$

It may be verified easily that $\int_0^\infty h(s)ds = 1$, and $h(s)$ is therefore a distribution function.

Applications to sampling. The utility of the distribution of the sample median is restricted by the fact that both μ and λ appear in the distribution. Thus, in order to obtain a confidence interval for μ, some estimate of λ would have to be used. The most appropriate estimate would be $\tilde{\lambda}$. The present state of this theory is therefore analogous to that of the normal theory before the development of the "student" t distribution.

If μ can be assumed to be known, the first distribution of $\tilde{\lambda}$, equation (19), may be used to estimate this parameter. If μ is not known, however, the second distribution of $\tilde{\lambda}$, equation (32), must be used.

Suggestions for further research. Any application of this theory to practical sampling would require that numerical tables be obtained for the distributions of $\tilde{x}$ and $\tilde{\lambda}$ for an ample range of sample sizes. This would be most easily done, of course, by computers. A distribution of $\tilde{x}$, independent of λ, would also be most useful.

Summary. The distribution functions of the maximum-likelihood estimators of the parameters μ and λ of the Laplace distribution have been obtained. The distribution of $\tilde{\mu}$ is seen to be a superposition of a set of Laplace distributions. The distribution of $\tilde{\lambda}$ in the case in which μ and λ are assumed to be known is a Gamma distribution and, in the case in which only λ is known, is a superposition of Gamma distributions.

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ON LINEAR SEQUENCES OVER A FINITE FIELD

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1. Introduction. Many modern electronic communication systems transmit information by means of coded signals. These signals, and the currents in the electrical appliances which generate them, are often likened to infinite sequences each of whose terms has a finite number of possible values. If the sequence is periodic then its (minimum) period gives a measure of the repetition time, cryptographic security, or reliability of the corresponding part of a communication system. These facts have motivated the study of the periods of linear sequences over a finite field. We discuss here the most general linear case, extending results in [3]. It is hoped that the work will have a mathematical interest independent of its practical value.

Let $\mathfrak{F}$ be a finite field, let A be a square matrix over $\mathfrak{F}$ of order α, and let (W_n) be an infinite sequence of $\alpha\times 1$ column vectors over $\mathfrak{F}$. Further, let (U_n) be the infinite sequence of $\alpha\times 1$ vectors over $\mathfrak{F}$ defined by U_1 and the recurrence relation

$$U_{n+1} = AU_n + W_n, \qquad \text{for } n \geqq 1. \tag{1}$$

Since $\mathfrak{F}$ has a finite number of elements, there is at most a finite number of distinct terms in (W_n), and in (U_n). Using these facts, it is not difficult to see that (U_n) will eventually be periodic if and only if (W_n) is eventually periodic, and we here assume that (W_n) has period w right from the start, so that

$$W_{n+w} = W_n, \qquad \text{for } n \geqq 1.$$

Given U_1 there will be a least integer u for which there is an integer m such that

$$U_{n+u} = U_n, \qquad \text{for } n > m. \tag{2}$$

Clearly u is the period of (U_n), but we will prefer the name cycle length for u. We may then call any sequence $U_n, U_{n+1}, \cdots, U_{n+u-1}$, with $n>m$, a cycle of (U_n), and we may call the least value of m for which (2) holds the distance of U_1 from a cycle of (U_n). When $m=0$ we say that U_1 is on a cycle of (U_n).

In this note we determine the possible cycle lengths of (U_n) and find under what conditions the initial vector U_1 must lie on a cycle of (U_n). Also, we show that the effect of (W_n) on the possible cycle lengths of (U_n) depends only on w and the matrix $E(w)$, where

$$E(r) = \sum_{i=1}^{r} A^{r-i}W_i, \qquad \text{for } r \geqq 1. \tag{3}$$

The most general linear recurrence relation for a sequence (U_n) of $\alpha\times\beta$

matrices has the form

$$U_{n+1} = \sum_{i=1}^{k} A_i U_{n+1-i} B_i + W_{n+1-k}, \qquad \text{for } n \geqq k, \tag{4}$$

where the A_i and B_i are given square matrices of orders α and β respectively, whilst $U_1, U_2, \cdots, U_k, W_1, W_2, \cdots$ are given $\alpha\times\beta$ matrices. In Section 4 we show how the case (4) can be reduced to the case (1).

The author would like to thank the referee for clarifying points in the text.

2. Cycle Lengths.

THEOREM 1. *For all choices of U_1 the cycle length of (U_n) is divisible by w.*

Proof. If u is the cycle length of (U_n) obtained from an arbitrary U_1, then for all sufficiently large n,

$$0 = U_{n+u+1} - U_{n+1} = AU_{n+u} + W_{n+u} - AU_n - W_n = W_{n+u} - W_n,$$

and therefore w divides u.

From now on p will denote the characteristic of $\mathfrak{F}$. By inspection of (1) and (3) we have

$$U_{n+1} = A^n U_1 + E(n) \qquad \text{for } n \geqq 1,$$

a relation which we use in

THEOREM 2. *The vector U_1 is on a cycle of (U_n), for all choices of U_1, if and only if A is nonsingular. If A is nonsingular with period a, and if c is the least common multiple of a and w, then the least common multiple λ of all the possible cycle lengths of (U_n) is c or pc according as $E(c)$ is or is not* 0.

The period of a nonsingular matrix A over $\mathfrak{F}$ is the least positive integer a such that $A^a = I$. The period of a polynomial $f(x)$ with $f(0) \neq 0$ and coefficients in $\mathfrak{F}$ is the least positive integer e such that $f(x)$ divides $x^e - 1$. The period of a nonsingular matrix is the period of its minimal polynomial.

Proof. Suppose first that U_1 lies on a cycle for all choices of U_1. Then $U_{\lambda+1} = A^\lambda U_1 + E(\lambda) = U_1$, for all U_1. Putting $U_1 = 0$ shows that $E(\lambda) = 0$, and therefore $A^\lambda = I$. Thus if U_1 always lies on a cycle then A is nonsingular and a divides λ.

Next assume that A is nonsingular. Then, because both w and a divide c, from (3) we have

$$E(tc) = \sum_{i=0}^{t-1} A^{ic} E(c) = tE(c),$$

for $t \geqq 1$, and therefore

$$U_{tc+1} = A^{tc} U_1 + E(tc) = U_1 + tE(c), \tag{5}$$

for $t \geqq 1$ and all U_1. In particular, because $pE(c)=0$, we have $U_{pc+1}=U_1$ for all U_1. Hence λ divides pc and every U_1 lies on a cycle. The first sentence of the theorem is now proved.

We continue to assume that A is nonsingular. Then every U_1 lies on a cycle and so, by the first part of the proof, a divides λ. This result, together with Theorem 1, establishes that c divides λ. If $E(c)=0$ then (5) shows that $U_{c+1} = U_1$ for all U_1. Hence, in this case, λ divides c and so must be equal to c. If, on the other hand, $E(c) \neq 0$, then (5) shows that $U_{c+1} \neq U_1$, and hence that λ does not divide c. We do know, however, that λ divides pc, and therefore, because p is a prime, λ is pc. This completes the proof of Theorem 2.

By examining the difference $U_{\mu w+1} - U_1$, we immediately obtain the following trivial result.

THEOREM 3. *If U_1 is on a cycle of (U_n), then the length of this cycle is the least positive multiple μw of w such that*

$$\sum_{i=0}^{\mu-1} A^{iw} V = 0, \tag{6}$$

where

$$V = (A^w - I)U_1 + E(w). \tag{7}$$

THEOREM 4. *If A is nonsingular with a period which divides w, then every cycle of (U_n) has length w or pw according as $E(w)$ is or is not* 0.

Proof. The V of Theorem 3 is independent of U_1, so that all cycles have the same length, and the result follows by Theorem 2.

If $f(x)$ is a polynomial

$$f(x) = x^\alpha - \sum_{i=1}^{\alpha} \eta_i x^{\alpha-i},$$

then we write $C\{f(x)\}$ for the companion matrix of $f(x)$, namely

$$C\{f(x)\} = \begin{bmatrix} 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 1 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \eta_\alpha & \eta_{\alpha-1} & \eta_{\alpha-2} & \cdots \eta_1 \end{bmatrix}$$

The following special cases are of value. In them q denotes the number of elements in $\mathfrak{F}$.

THEOREM 5. *If $A = C\{x^\alpha\}$ then (U_n) has only one cycle; the length of this cycle is w; no vector U_1 is at a distance greater than α from the cycle; and if $0 \leqq d \leqq \alpha$ there are at least $q^d(q^{\alpha-d}-1)$ vectors at distance at least d from the cycle.*

Proof. Now $A^\alpha = 0$, whence

$$U_{n+1} = \sum_{i=n-\alpha+1}^{n} A^{n-i} W_i$$

for $n \geqq \alpha$. Hence (U_n) has only one cycle, and the length of the cycle is w. Given U_1, if U_d is on the cycle, then the bottom $\alpha - d$ elements of U_1 must have specific values. The latter part of the theorem now follows.

Given the matrix A, and any $\alpha \times 1$ column vector V, a polynomial $g(x)$ such that $g(A)V = 0$, is said to annihilate V with respect to A. For each V there exists a unique, monic annihilating polynomial of lowest possible degree, and every such polynomial divides the characteristic polynomial of A.

Suppose now that $A = C\{f^\gamma(x)\}$, where $f(x)(\neq x)$ is a monic irreducible polynomial of positive degree and γ is a positive integer. We investigate the cycle lengths of (U_n) by determining which integers μ and vectors V satisfy the conditions of Theorem 3.

If V is any vector then the minimal annihilating polynomial of V with respect to A must be of the form $f^s(x)$, $0 \leqq s \leqq \gamma$. Moreover, if μw is the least multiple of w for which (6) holds, then μw is the lowest multiple of w for which

$$f^s(x) \mid x^{(\mu-1)w} + x^{(\mu-2)w} + \cdots + 1, \tag{8}$$

where we write $a \mid b$ to denote that a divides b. Hence we must now find the relationships between w, μ and s.

Let $w = p^\rho m$ where $(m, p) = 1$, and if e is the period of $f(x)$, put

$$\epsilon = \begin{cases} 1 & \text{if } e \mid m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x^m - 1$ has no repeated roots, and the highest power of $f(x)$ to divide $x^m - 1$ is $f^\epsilon(x)$. Since $(x^m - 1)^{p^\rho} = x^w - 1$, the highest power of $f(x)$ to divide $x^w - 1$ is therefore $f^{\epsilon p^\rho}(x)$. Hence if s is any integer, then (8) holds if and only if

$$f^{s+\epsilon p^\rho}(x) \mid x^{\mu w} - 1.$$

When $\mu = 1$ the only possible choice for s is clearly 0.

Now $f(x)$ divides $x^d - 1$ if and only if e divides d. Moreover, e is prime to p. Hence if k is the least common multiple of e and m, then the least multiple μw of w for which $f(x)$ divides $x^{\mu w} - 1$ is given by $\mu w = kp^\rho$. Since

$$x^{kp^\rho} - 1 = (x^k - 1)^{p^\rho},$$

and $x^k - 1$ has no repeated roots, the highest power of $f(x)$ to divide $x^{\mu w} - 1$ is $f^{p^\rho}(x)$. Note also that if ν and r are positive integers and

$$f^r(x) \mid x^{\nu w} - 1,$$

then νw is of the form kp^t with $\rho \leqq t$.

Next let τ be the integer for which

$$p^{\tau-1} < \gamma + \epsilon p^{\rho} \leqq p^{\tau}.$$

Then for $\rho \leqq t < \tau$ the highest value of s for which

$$f^{s+\epsilon p^{\rho}}(x) \mid x^{kp^t} - 1,$$

or, in other words, for which (8) holds, is given by

$$s = p^t - \epsilon p^{\rho},$$

while in the case $t=\tau$ we have

$$f^{\gamma+\epsilon p^{\rho}} \mid x^{kp^{\tau}} - 1.$$

For $1 \leqq r \leqq \gamma$ the number of vectors V over $\mathfrak{F}$ whose minimal annihilating polynomial is $f^r(x)$ is

$$q^{r\delta} - q^{(r-1)\delta},$$

where δ denotes the degree of $f(x)$ and, as before, q is the number of elements in $\mathfrak{F}$. The following theorem now follows easily.

THEOREM 6. *If $A = C\{f^{\gamma}(x)\}$ then the only possible cycle lengths of (U_n) are w and kp^t, $\rho \leqq t \leqq \tau$. For each of these cycle lengths the number of choices N for the V of Theorem 3 is given by*

$$N = \begin{cases} 1 & \text{if } \mu w = w, \\ q^{\delta(p^{\rho}-\epsilon p^{\rho})} - 1 & \text{if } \mu w = p^{\rho}k, \\ q^{\delta(p^{t}-\epsilon p^{\rho})} - q^{\delta(p^{t-1}-\epsilon p^{\rho})} & \text{if } \mu w = p^t k,\ \rho < t < \tau, \\ q^{\delta\gamma} - q^{\delta(p^{\tau-1}-\epsilon p^{\rho})} & \text{if } \mu w = p^{\tau}k. \end{cases}$$

The matrix $A^w - I$ is nonsingular when and only when $\epsilon = 0$, and then each choice of V leads to a choice of the U_1 of Theorem 3.

If $\epsilon=1$ then $A^w - I$ is singular, having a rank $\delta\gamma - \Delta$, where Δ is the degree of the highest common factor $(f^{\gamma}(x),\ x^w - 1)$, namely $\delta \min(\gamma,\ p^{\rho})$. (Cf. [2], Theorem 1, p. 37.) A given choice of V leads to q^{Δ} choices of U_1, provided only the equation (7) is solvable for U_1.

3. Similar sequences. We continue to study the sequences (U_n) defined by U_1 and (1). Suppose that D is a matrix similar to A. In other words, suppose that there is a nonsingular matrix M such that $M^{-1}DM = A$. Then the mapping $U \to MU$ is a one-to-one correspondence between the sequences defined by U_1 and (1) and those defined by $T_1 = MU_1$ and $T_{n+1} = DT_n + MW_n$ for $n \geqq 1$, because

$$M\{A^n U_1 + E(n)\} = M\left\{A^n U_1 + \sum_{i=1}^{n} A^{n-i} W_i\right\} = D^n(MU_1) + \sum_{i=1}^{n} D^{n-i}(MW_i).$$

We choose M so that D is the primary rational canonical form of A (cf. [1], p. 332). Then, by definition, D is a direct sum of companion matrices $C\{f^\gamma(x)\}$, where the polynomials $f(x)$ are monic and irreducible. Thus the sequences (T_n) are partitioned into independent subsequences of the kind discussed in Theorems 5 and 6, and the properties of the sequences (T_n) follow immediately.

4. Reduction of the general case. Next we observe that if F, G, H and K are matrices of types $\alpha\times\alpha$, $\alpha\times\beta$, $\beta\times\beta$ and $\alpha\times\beta$ respectively, then an equation $FGH=K$, can be written in the form

$$F^*H^*G^* = K^*,$$

where F^* and H^* are square matrices of order $\alpha\beta$, while G^* and K^* are $\alpha\beta\times 1$ column vectors. One obtains F^* from F by replacing each element θ of F by θI_β, where I_β is the unit matrix of order β. The matrix H^* is a direct sum of α matrices, each equal to the transpose of H. Finally, G^* and K^* are obtained by writing, one on top of the other, the transposes of the rows of G and K respectively. The product F^*H^* is nonsingular if and only if both F and H are nonsingular, and when it is nonsingular, the period of F^*H^* divides the least common multiple of the periods of F and H.

Using the above technique we can rewrite equation (4) in the form

$$U^*_{n+1} = \sum_{i=1}^{k} J_i U^*_{n+1-i} + W^*_{n+1-k}$$

for $n\geqq 1$, where $J_i=A_i^*B_i^*$. In turn this equation is equivalent to

$$\begin{bmatrix} U^*_{n-k+2} \\ U^*_{n-k+3} \\ \cdot \\ \cdot \\ \cdot \\ U^*_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ J_k & J_{k-1} & J_{k-2} & \cdots & J_1 \end{bmatrix} \begin{bmatrix} U^*_{n-k+1} \\ U^*_{n-k+2} \\ \cdot \\ \cdot \\ \cdot \\ U^*_{n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ W^*_{n-k+1} \end{bmatrix}$$

which is in the form (1). It now follows from Theorem 2 that every matrix U_n is on a cycle if and only if J_k, and hence both A_k and B_k, are nonsingular.

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MATHEMATICAL NOTES

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A FORMULA FOR THE DERIVATIVES OF LEGENDRE POLYNOMIALS

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In a recent paper [1] the following theorem was conjectured about derivatives of Legendre polynomials $P_n(x)$:

$$(1)\quad \frac{d^k}{dx^k} P_{n+k}(x) = 1\cdot 3\cdot 5 \cdots (2k-1) \sum_{i_1+i_2+\cdots i_{2k+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2k+1}}(x).$$

In calculating the sum on the right, it must be understood that the ordered set of indices $i_1, i_2, \cdots, i_{2k+1}$ must take on the values of all permutations of all sets of $2k+1$ nonnegative integers whose sum is n. For example, for $n=2$, $k=1$, equation (1) gives

$$\begin{aligned}\frac{d}{dx} P_3(x) &= P_0P_1P_1 + P_1P_0P_1 + P_1P_1P_0 + P_0P_0P_2 + P_0P_2P_0 + P_2P_0P_0 \\ &= 3P_0P_1^2 + 3P_0^2P_2.\end{aligned}$$

The author of [1] has given rather long and involved proofs for each of the separate cases $k=1$, $k=2$, $k=3$. The purpose of this note is to give a simple proof for general k. Let

$$(2)\qquad F(x, t) = (1 - 2xt + t^2)^{-1/2}$$

be the generating function for the Legendre polynomials. Then

$$(3)\qquad F(x, t) \equiv \sum_{n=0}^{\infty} t^n P_n(x)$$

is an identity in both x and t. Differentiating (2) repeatedly with respect to x, we get

$$(4)\qquad \begin{aligned}\frac{\partial}{\partial x} F(x, t) &= t(1 - 2xt + t^2)^{-3/2} = tF^3 \\ \frac{\partial^2}{\partial x^2} F(x, t) &= t\cdot 3F^2 \frac{\partial F}{\partial x} = 3t^2F^5 \\ &\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial^k}{\partial x^k} F(x, t) &= 3\cdot 5\cdot 7 \cdots (2k-1)t^kF^{2k+1}.\end{aligned}$$

Differentiating (3) successively with respect to x, we get

$$(5)\qquad \frac{\partial^k}{\partial x^k} F(x, t) = \sum_{n=0}^{\infty} t^n \frac{d^k}{dx^k} P_n(x) = \sum_{n=0}^{\infty} t^{n+k} \frac{d^k}{dx^k} P_{n+k},$$

since the kth derivative of any Legendre polynomial of order less than k is zero. Equating (4) and (5), cancelling t^k, and using (3), we have

$$(6)\qquad \begin{aligned} \sum t^n \frac{d^k}{dx^k} P_{n+k} &= 3\cdot 5\cdot 7 \cdots (2k-1)F^{2k+1} \\ &= 3\cdot 5\cdot 7 \cdots (2k-1)\left[\sum_{l=0}^{\infty} t^l P_l(x)\right]^{2k+1}. \end{aligned}$$

Since (6) is an identity in t, the coefficients of t^n on the two sides of the equation are equal. Let us equate the coefficients of t^n. On the left we get $(d^k/dx^k)P_{n+k}$. On the right, on multiplying out the product of $2k+1$ power series in t, we obtain a term in t^n for every (ordered) set of values of l (one from each series) whose sum is n. (Note that, say for $n=2$, t^0t^2 and t^2t^0 are separate terms.) Thus the coefficient of t^n on the right is the sum of all products of $2k+1$ Legendre polynomials whose indices are an ordered set of nonnegative integers whose sum is n. Thus we have (1).

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ON POINTS OF BOUNDED VARIATION

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Let $f(x)$ be a real function defined in the closed interval $[a, b]$. We subdivide the interval $[a, b]$ by means of the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and then form the sum

$$V = \sum_{k=0}^{n-1} \left| f(x_{k+1}) - f(x_k) \right|.$$

The least upper bound of the aggregate of the sums V for all possible modes of subdivision of the interval $[a, b]$ is called the total variation of $f(x)$ in $[a, b]$ and is denoted by $V_a^b(f)$. If $V_a^b(f) < +\infty$, $f(x)$ is said to be of bounded variation in $[a, b]$.

DEFINITION. *Let x be a point in $[a, b]$. If there exists a closed neighbourhood of x in which $f(x)$ is of bounded variation then we say that x is a point of bounded variation of $f(x)$. On the other hand, if there exists no closed neighbourhood of x in which $f(x)$ is of bounded variation, then x is said to be a point of nonbounded variation of $f(x)$.*

A point of bounded variation and a point of nonbounded variation will be called a B_V point and a NB_V point respectively.

Let x be an interior point of $[a, b]$. Choose a $\delta>0$ such that $[x-\delta, x+\delta] \subset [a, b]$. Let

$$V(x, \delta) = \frac{1}{1 + V_{x-\delta}^{x+\delta}(f)} \cdot$$

It is easy to see that as δ decreases, $V(x, \delta)$ increases. We denote by $v(x)$ the limit of $V(x, \delta)$ as $\delta \rightarrow 0$. If x is an end point of $[a, b]$, then $V(x, \delta)$ and accordingly $v(x)$ is defined by taking the closed neighbourhood of x in its usual restricted way. At each point x in $[a, b]$, $v(x)$ has a definite value and $0 \leqq v(x) \leqq 1$. We call $v(x)$ the *variation function* of $f(x)$. It is clear that if $v(x)=0$, then x is a NB_V point, while if $v(x) \neq 0$, x is a B_V point of $f(x)$.

It is easy to verify that if $f(x)$ is of bounded variation in $[a, b]$, then every point $x \in [a, b]$ is a B_V point, and conversely, if every point of $[a, b]$ is a B_V point, then $f(x)$ is of bounded variation in $[a, b]$.

R. L. Jeffery ([1], p. 137, Problem 5.2) has stated that the set of B_V points of $f(x)$, forms an open set. The purpose of the present note is to prove some properties of $v(x)$ and to show that Prof. Jeffery's statement comes as a corollary of our Theorem 1.

THEOREM 1. *$v(x)$ is lower semi-continuous in $[a, b]$.*

Proof. Let $c \in [a, b]$ and $\epsilon>0$ be arbitrary. There exists then a $\delta>0$ such that

$$V(c, \delta) > v(c) - \epsilon. \tag{1}$$

Let x be any point in $(c-\delta, c+\delta)$ and let $\delta'>0$ be chosen such that $[x-\delta', x+\delta'] \subset (c-\delta, c+\delta)$. Since

$$V_{x-\delta'}^{x+\delta'}(f) \leqq V_{c-\delta}^{c+\delta}(f),$$

it follows that

$$v(x) \geqq V(x, \delta') \geqq V(c, \delta). \tag{2}$$

Combining (1) and (2) we see that for every point $x \in (c-\delta, c+\delta)$

$$v(x) > v(c) - \epsilon.$$

Hence $v(x)$ is lower semi-continuous at $x=c$. Since c is any point in $[a, b]$, the theorem follows.

We have already remarked that if $v(\alpha)\neq 0$, then α is a B_V point of $f(x)$. Utilising a simple property of lower semi-continuous functions we see that any B_V point of $f(x)$ is an interior point of the set of B_V points of $f(x)$. So, we obtain

COROLLARY 1. *The set of B_V points of $f(x)$ in $[a, b]$ forms an open set.*

Consequently

COROLLARY 2. *The set of NB_V points of $f(x)$ in $[a, b]$ is a closed set.*

THEOREM 2. *If $f(x)$ is continuous at c and c is a B_V point then $v(c)=1$.*

Proof. Since c is a B_V point there exists a $\delta>0$ such that $f(x)$ is of bounded variation in $[c-\delta, c+\delta]$. Writing

$$\Pi(x) = V_{c-\delta}^{x}(f)$$

for $x\in[c-\delta, c+\delta]$, we see that

$$V(c,\delta') = \frac{1}{1+\Pi(c+\delta')-\Pi(c-\delta')}, \qquad 0<\delta'\leqq\delta.$$

Since $f(x)$ is continuous at c, the function $\Pi(x)$ is continuous at c ([2], p. 223).

Hence, letting $\delta'\to 0$ in the above, we see that $v(c)=1$.

Noting Theorem 1 and the fact $v(x)\leqq 1$, we get the following

COROLLARY. *If $f(x)$ is continuous at c and c is a B_V point, then $v(x)$ is continuous at c.*

It is clear that if $f(x)$ is continuous in $[a, b]$ and if the set of NB_V points is nondense in $[a, b]$, then every NB_V point is a point of discontinuity of $v(x)$.

THEOREM 3. *If $v(c)=1$, then $f(x)$ is continuous at $x=c$.*

Proof. Since $v(c)=1$, c is a B_V point of $f(x)$. There exists then $\delta_0>0$ such that $f(x)$ is of bounded variation in $[c-\delta_0, c+\delta_0]$. Since

$$\lim_{\delta\to 0} V_{c-\delta}^{c+\delta}(f) = 0,$$

if $\epsilon>0$ be chosen arbitrarily, there exists δ', with $0<\delta'<\delta_0$, such that

$$V_{c-\delta'}^{c+\delta'}(f) < \epsilon.$$

It follows, therefore, that if x be any point in $[c-\delta', c+\delta']$, we have $|f(x)-f(c)| <\epsilon$. Hence $f(x)$ is continuous at $x=c$.

It may be of interest to note that continuity at a point does not necessarily imply that the point is a B_V point. There are well-known examples that illustrate this. It is, however, not unnatural to expect that the continuity in an interval may be a guarantee for the existence of at least one B_V point in that interval.

But any continuous and everywhere nondifferentiable function shows that this is also not the case.

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SOME CARDINAL EQUIVALENCES WITH EXPLICIT FORMULAS

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Our objective is to define for each positive integer n a polynomial f_n which establishes a 1-1 correspondence from the set of all n-tuples of nonnegative integers onto the set of all nonnegative integers. Before presenting the details we offer some remarks. (1) The result in two dimensions has been known for some time (see [1] and [2]). (2) The formulas for $n=2$ and $n=3$ respectively were developed by counting along the sequence of lines with equations $x+y=k$, $k=1, 2, 3, \cdots$, in 2-space, and along the sequence of planes with equations $x+y+z=k$, $k=1, 2, 3, \cdots$, in 3-space. The lemmas proved below are geometrically obvious from the counting process.

For each positive integer n, let

$$f_n(x_1, \cdots, x_n) = \sum_{i=1}^{n} \frac{1}{(n-i+1)!} \prod_{\gamma=0}^{n-i} (x_i + \cdots + x_n + \gamma).$$

Note that

$$\begin{aligned} f_n(x_1, \cdots, x_n) &= \frac{1}{n!} \prod_{y=0}^{n-1} (x_1 + \cdots + x_n + y) + f_{n-1}(x_2, \cdots, x_n) \\ &= f_n\left(\sum_{1}^{n} x_i, 0, \cdots, 0\right) + f_{n-1}\left(\sum_{2}^{n} x_i, 0, \cdots, 0\right) + \cdots \\ &\quad + f_2(x_{n-1} + x_n, 0) + f_1(x_n). \end{aligned}$$

For each $(x_1, \cdots, x_n)$, $f_n(x_1, \cdots, x_n)$ is a sum of binomial coefficients, and is therefore an integer.

We now prove that for each positive integer n, f_n is a 1-1 correspondence from the set of all n-tuples of nonnegative integers onto the set of all nonnegative integers. In all the proofs below, small letters will denote nonnegative integers.

LEMMA 1. *For each* $n>0$ *and* $r\geqq 0$, $f_n(0, 0, \cdots, 0, r)+1=f_n(r+1, 0, 0, \cdots, 0)$.

Proof. The result is obvious for $n=1$. Assume that the result holds for $n-1$.

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Then

$$f_n(0, 0, \cdots, 0, r) + 1$$
$$= \frac{1}{n!} r(r+1) \cdots (r+n-1) + f_{n-1}(0, 0, \cdots, 0, r) + 1$$
$$= \frac{1}{n!} r(r+1) \cdots (r+n-1) + f_{n-1}(r+1, 0, 0, \cdots, 0)$$
$$= \frac{1}{n!} r(r+1) \cdots (r+n-1) + \frac{1}{(n-1)!} (r+1)(r+2) \cdots (r+n-1)$$
$$= \frac{1}{n!} (r+1) \cdots (r+n-1)(r+n) = f_n(r+1, 0, 0, \cdots 0).$$

The result follows by induction.

LEMMA 2. *If* $0 \leqq a \leqq b, f_n(a, 0, 0, \cdots, 0) \leqq f_n(b, 0, 0, \cdots, 0)$.

LEMMA 3. *Let* $x_1 + \cdots + x_n = r$. *Then* $f_n(r, 0, 0, \cdots, 0) \leqq f_n(x_1, \cdots, x_n) \leqq f_n(0, 0, \cdots, 0, r)$.

LEMMA 4. *If* $\sum_{i=1}^n x_i < \sum_{i=1}^n y_i$, *then* $f_n(x_1, \cdots, x_n) < f_n(y_1, \cdots, y_n)$.

Proof. Using lemmas 1, 2, 3, we have

$$f_n(x_1, \cdots, x_n) \leq f_n\left(0, 0, \cdots, \sum_{i=1}^n x_i\right) < f_n\left(0, 0, \cdots 0, \sum_{i=1}^n x_i\right) + 1$$
$$= f_n\left(\left(\sum_{i=1}^n x_i\right) + 1, 0, 0, \cdots, 0\right) \leq f_n\left(\sum_{i=1}^n y_i, 0, 0, \cdots, 0\right) \leq f_n(y_1, \cdots, y_n).$$

THEOREM 1. *For each* $n > 0$, f_n *is one-to-one.*

Proof. The result is obvious for $n=1$. Assume its truth for $n-1$. Suppose that $f_n(x_1, \cdots, x_n) = f_n(y_1, \cdots, y_n)$. Then $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ by Lemma 4. Hence $f_{n-1}(x_2, \cdots, x_n) = f_{n-1}(y_2, \cdots, y_n)$. By the inductive hypothesis $x_2 = y_2, \cdots, x_n = y_n$. Since $\sum_1^n x_i = \sum_1^n y_i$, $x_1 = y_1$, and it follows that f_n is one-to-one. The result follows by induction.

THEOREM 2. *For each* n, f_n *is onto.*

Proof. f_1 is onto. Assume that f_{n-1} is onto. Let $z \geqq 0$; then $\exists r \geqq 0$ such that $\alpha = f_n(r, 0, \cdots, 0) \leqq z < f_n(r+1, 0, \cdots, 0) = \beta$. By the inductive hypothesis, $\exists x_2, \cdots, x_n$ such that $x_i \geqq 0$, $i = 2, \cdots, n$, and $z - \alpha = f_{n-1}(x_2, \cdots, x_n)$. Hence $z = f_n(r, 0, \cdots, 0) + f_{n-1}(x_2, \cdots, x_n)$. Let $x_1 = r - \sum_2^n x_i$. We must show that $x_1 \geqq 0$, i.e. $\sum_2^n x_i \leqq r$. Now $z - \alpha < \beta - \alpha = f_{n-1}(r+1, 0, \cdots, 0)$. Therefore, $z - \alpha \leqq f_{n-1}(r+1, 0, \cdots, 0) - 1 = f_{n-1}(0, \cdots, 0, r)$ by Lemma 1. Also, $f_{n-1}(x_2, \cdots, x_n) = z - \alpha \leqq f_{n-1}(0, \cdots, 0, r)$. By Lemma 4, $\sum_2^n x_i \leqq r$. The result follows by induction.

The proof of Theorem 2 contains the germ of the algorithm to determine $x_1, \cdots, x_n$, once $z_n \geqq 0$ is given. Given $z_n \geqq 0$, there exist unique $x_1, \cdots, x_n$, all $\geqq 0$, such that $z_n = f_n(x_1, \cdots, x_n) = f_n(\sum_1^n x_i, 0, \cdots, 0) + f_{n-1}(\sum_2^n x_i, 0, \cdots, 0) + \cdots + f_2(x_{n-1}, +x_n, 0) + f_1(x_n)$. Let $X_i = \sum_{j=i}^n x_j$, $i = 1, \cdots, n$. We determine successively $X_1, X_2, \cdots, X_n$. Then

$$x_n = X_n,\ x_{n-1} = X_{n-1} - X_n, \cdots, x_1 = X_1 - X_2.$$

By Lemmas 1 and 3,

$$f_n(X_1, 0, \cdots, 0) \leqq f_n(x_1, \cdots, x_n) = z_n < f_n(X_1 + 1, 0, \cdots, 0).$$

Hence $X_1 = \max r_1[f_n(r_1, 0, \cdots, 0) \leqq z_n]$. Similarly $X_2 = \max r_2[f_{n-1}(r_2, 0, \cdots, 0) \leqq z_n - f_n(X_1, 0, \cdots, 0)]$. In general

$$X_i = \max r_i\left[f_{n-i+1}(r_i, 0, \cdots, 0) \leqq z_n - \sum_{y=0}^{i-2} f_{n-y}(X_{y+1}, 0, \cdots, 0)\right].$$

For those interested we note that the algorithm can be shown to be primitive recursive in both directions.

With regard to the problem of correspondences between integers, nonnegative integers, positive integers, and n-tuples thereof, we introduce two functions α and β which represent the following scheme:

$$\alpha \downarrow\!\uparrow \beta \quad \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \cdots \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 \cdots \end{matrix}$$

$$\alpha(n) = \frac{1 + (-1)^{n+1}}{2}\cdot\frac{n+1}{2} - \frac{1 + (-1)^n}{2}\cdot\frac{n}{2}$$

$$\beta(n) = \delta(n)(2n - 1) + (\delta(n) - 1)(2n), \quad \text{where } \delta(n) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n \leqq 0. \end{cases}$$

Many correspondences can now be obtained, of which three are listed below.

(1) n-tuples of positive integers$\leftrightarrow$positive integers:

$$P_n(x_1, \cdots, x_n) = f_n(x_1 - 1, \cdots, x_n - 1) + 1, \qquad x_i > 0 \text{ for each } i.$$

(2) n-tuples of integers$\leftrightarrow$nonnegative integers:

$$Q_n(x_1, \cdots, x_n) = f_n(\beta(x_1), \cdots, \beta(x_n)).$$

(3) n-tuples of integers$\leftrightarrow$integers:

$$R_n(x_1, \cdots, x_n) = \alpha(f_n(\beta(x_1), \cdots, \beta(x_n))).$$

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TWO INEQUALITIES INVOLVING ELLIPTIC FUNCTIONS

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Despite the existence of many reference books on elliptic functions, containing tables of identities, integrals, derivatives, series expansions, and numerical values, there does not seem to be an adequate source for *inequalities*. It is partly to draw attention to this need that we present here proofs of

$$\frac{\operatorname{sn} u}{(\operatorname{dn} u + \operatorname{cn} u)^2} \leqq \frac{u}{K(1 - k^2)}, \qquad 0 \leqq u \leqq K; \tag{1}$$

$$\frac{\operatorname{sn} u}{(\operatorname{dn} u + k \operatorname{cn} u)^2} \geqq \frac{2u}{\pi(1 + k)^2}, \qquad 0 \leqq u \leqq K. \tag{2}$$

In these expressions, k is the modulus of the given elliptic functions and

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta. \tag{3}$$

These inequalities arose in a study of conformal mapping of an annulus [2]. Inequality (1) becomes trivial if the constant $K(1-k^2)$ is replaced by $(1-k^2)$. For $k=0$ it reduces to $\sin u \leqq (2/\pi)u(1+\cos u)^2$, $0 \leqq u \leqq \pi/2$. Inequality (2) reduces, for $k=0$, to the familiar relation $\sin u \geqq (2/\pi)u$, $0 \leqq u \leqq \pi/2$. However, it is not a trivial consequence of the analogous inequality

$$\operatorname{sn} u \geqq u/K, \qquad 0 \leqq u \leqq K, \tag{4}$$

since $K \geqq \pi/2$.

The two sides of (1) are equal at the end-points $u=0$ and $u=K$, so it is enough to prove that $F(u) = \operatorname{sn} u[\operatorname{dn} u + \operatorname{cn} u]^{-2}$ is *convex* in $0 \leqq u \leqq K$. Differentiation, using [1, 731.01–.03; 121.00], gives

$$F'(u) = [1 + \operatorname{sn}^2 u\, f(u)](\operatorname{dn} u + \operatorname{cn} u)^{-2},$$

where

$$f(u) = (\operatorname{dn} u + k^2 \operatorname{cn} u)(\operatorname{dn} u + \operatorname{cn} u)^{-1}.$$

Since $\operatorname{sn} u$ increases and $(\operatorname{dn} u + \operatorname{cn} u)$ decreases, we will have proved $F'(u)$ increases in $0 \leqq u \leqq K$ if we show $f'(u) \geqq 0$ there. But

$$f'(u) = (1 - k^2)^2 \operatorname{sn} u(\operatorname{dn} u + \operatorname{cn} u)^{-2},$$

where [1, 121.00] has been used. This proves (1).

The proof of (2) will be divided into two parts. Suppose first $K/2 \leqq u \leqq K$. In view of (4), it suffices to prove

$$K(\operatorname{dn} u + k \operatorname{cn} u)^2 \leqq (\pi/2)(1 + k)^2, \qquad K/2 \leqq u \leqq K. \tag{5}$$

But cn and dn are decreasing in the given range, so the left-hand side of (5) is no larger than

$$K[\mathrm{dn}(K/2) + k\,\mathrm{cn}(K/2)]^2 = Kk'[1 + k(1 + k')^{-1/2}]^2 \leqq Kk'(1 + k^2),$$

where $k' = (1-k^2)^{1/2}$. (See [1, 122.10].) The proof of (5) is now completed by the estimate $K \leqq (\pi/2)(k')^{-1}$, which follows trivially from (3).

To prove (2) for $0 \leqq u \leqq K/2$, set $G(u) = \mathrm{sn}\, u(\mathrm{dn}\, u + k\,\mathrm{cn}\, u)^{-2}$, and compute

$$G'(u) = (\mathrm{cn}\, u\, \mathrm{dn}\, u + 2k\, \mathrm{sn}^2\, u)(\mathrm{dn}\, u + k\,\mathrm{cn}\, u)^{-2} \geqq H(u),$$

where $H(u) = (\mathrm{cn}\, u\, \mathrm{dn}\, u)(\mathrm{dn}\, u + k\,\mathrm{cn}\, u)^{-2}$. Since $G(0) = 0$, the inequality (2) will be established if we can show that

$$H(u) \geqq (2/\pi)(1 + k)^{-2}, \qquad 0 \leqq u \leqq K/2. \tag{6}$$

But $H(u)$ is *decreasing* in the given range, since

$$H'(u) = -\,\mathrm{sn}\, u(\mathrm{dn}\, u - k\,\mathrm{cn}\, u)^2(\mathrm{dn}\, u + k\,\mathrm{cn}\, u)^{-2} \leqq 0.$$

Therefore, in the interval $0 \leqq u \leqq K/2$, we have

$$\begin{aligned} H(u) \geqq H(K/2) &= (1 + k')^{-1/2}[1 + k(1 + k')^{-1/2}]^{-2} \\ &\geqq (1 + k')^{-1/2}(1 + k)^{-2} \geqq (1/\sqrt{2})(1 + k)^{-2} > (2/\pi)(1 + k)^{-2}. \end{aligned}$$

The relation (6) is therefore established, and the proof of inequality (2) is complete.

This work was supported in part by the National Science Foundation, Contract GP-281.

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2. P. L. Duren, Distortion in certain conformal mappings of an annulus, (to appear).

AN INTEGRAL EQUATION INVOLVING LEGENDRE'S POLYNOMIAL

A. ERDÉLYI, California Institute of Technology

The integral equation

$$\int_1^x P_n\left(\frac{x}{t}\right) g(t)dt = f(x) \qquad 1 < x < x_0 \tag{1}$$

was recently solved by R. G. Buschman [1] who used a different variable and employed certain integral formulas involving products of Legendre polynomials. It will be shown here that Rodrigues' formula for the Legendre polynomial,

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2 - 1)^n, \tag{2}$$

leads to a solution of (1) more directly and with fewer assumptions on f.

When $n = 0$, clearly $g = f'$. We shall assume that $n \geqq 1$, f is absolutely continuous on $[1, x_0]$ for some $x_0 > 1$, and $f(1) = 0$. These conditions on f are clearly

necessary, and they will turn out to be sufficient, for the existence of a unique solution g of (1).

On account of (2), the integral equation (1) may be re-written as

$$\frac{d^n}{dx^n}\int_1^x (x^2-t^2)^n t^{-n} g(t)\,dt = 2^n n!\,f(x) \tag{3}$$

and by n repeated integrations it follows that

$$\int_1^x (x^2-t^2)^n t^{-n} g(t)\,dt = 2^n n \int_1^x (x-t)^{n-1} f(t)\,dt$$

since the left hand side, together with its first $n-1$ derivatives, vanishes when $x=1$. Differentiating $n+1$ times with respect to x^2 we finally obtain

$$(n-1)!\,x^{-n-1} g(x) = \left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} \int_1^x (x-t)^{n-1} f(t)\,dt. \tag{4}$$

It is easy to confirm that g given by (4) does indeed satisfy (1) under the stated conditions on f.

An analogous integral equation involving a Chebyshev polynomial which has been studied previously by Ta Li [2] does not appear to have an equally elementary solution; Rodrigues' formula leads to differentiation and integration of fractional orders. There are other analogous integral equations with kernels involving Gegenbauer polynomials or associated Legendre functions, and these have also been solved by the present writer using fractional integrals and derivatives. The integral equation involving Gegenbauer polynomials has been investigated by T. P. Higgins [3] who used methods resembling those employed in [1] and [2].

References

1. R. G. Buschman, An inversion integral for a Legendre transformation, this MONTHLY 69 (1962) 288–289.

2. Ta Li, A new class of integral transforms, Proc. Amer. Math. Soc., 11 (1960) 290–298.

3. T. P. Higgins, An inversion integral for a Gegenbauer transformation, J. Soc. Indust. Appl. Math. (to appear).

Editorial Note. It has been pointed out by D. Ž. Đjoković that all theorems given by David Friedman in his article in this MONTHLY 69 (1962) 769–772 "are known and that the equation he considered was reduced to Cauchy's functional equation by Pexider in 1903. The full treatment . . . may be found in J. Aczél's 'Vorlesungen über Funktionalgleichungen und ihre Anwendungen', Basel, 1961 (pp. 116 and 44–47)."

CLASSROOM NOTES

EDITED BY JOHN M. H. OLMSTED, Southern Illinois University and
A. L. SHIELDS, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, Mathematics Department, University of Michigan, Ann Arbor, Michigan.

THE RANK OF AN R-MODULE

GEOFFREY A. KANDALL, Massachusetts Institute of Technology

Throughout the following discussion, R always denotes a commutative ring with an identity $1 \neq 0$, and M denotes a unitary R-module, i.e. an R-module such that $1x = x$ for every $x \in M$.

In studying linear algebra, one soon becomes acquainted with the following theorem: *Any vector space V possesses a basis and any two bases for V have the same cardinality.* Now the definition of a basis (an independent set of generators) makes sense in an R-module as well as in a vector space, since one still has the notions of *independence* and *generating set.* For example:

(i) The elements $x_1, \cdots, x_n$ of M are said to be *independent* if $\sum_{i=1}^{n} a_i x_i = 0$ $(a_i \in R)$ implies $a_i = 0$ $(i = 1, \cdots, n)$.

(ii) The elements $\{x_i\}$ (i ranging over an infinite indexing set) are said to be *independent* if every finite subset of them is independent as defined in (i).

It is therefore natural to ask whether the above theorem is also valid for R-modules. Unfortunately, it is not valid. An R-module M does not necessarily possess a basis. In fact, M need not contain a single independent element, even if R is a principal ideal domain. For example, if M is an (additive) abelian group of finite order k (considered in the natural way as a module over the ring of integers), then $ka = 0$ for every $a \in M$.

Then it is perhaps surprising to learn that the following theorem is true: *If an R-module M has one finite basis, then any two bases for M have the same cardinality.* Thus for such an R-module we may define the *rank* of M to be the cardinality of any basis. *Rank* is usually called *dimension* when R is a field, i.e. when M is a vector space.

This theorem can be found in several books (e.g. [1], [2]), but is deduced in each case from results on tensor products and exterior products of R-modules. It is the purpose of this note to show how the theorem may be proved in an elementary way, using only simple properties of determinants. While it is true that determinants are closely related to exterior products, it is also true that the results we need are very familiar ones which can be obtained by the usual elementary methods (as in [3], e.g.). The proof of the theorem is therefore easily accessible to students whose algebraic background is relatively limited.

Consider a matrix (a_{ij}), $a_{ij}\in R$ $(i, j=1, \cdots, m)$. A *p-rowed submatrix* of (a_{ij}) is the matrix consisting of those elements common to any p fixed rows and any p fixed columns of (a_{ij}), i.e. the matrix obtained when $m-p$ rows and $m-p$ columns of (a_{ij}) are deleted.

Denote the cofactor of a_{ij} by A_{ij}, i.e. A_{ij} is equal to $(-1)^{i+j}$ times the determinant of the $(m-1)$-rowed submatrix obtained from (a_{ij}) by deleting the ith row and jth column. We will use the well-known results:

$$\sum_{i=1}^{m} a_{ij}A_{ij} = \det (a_{ij}) \qquad (j = 1, \cdots, m) \tag{1}$$

$$\sum_{i=1}^{m} a_{ij}A_{ik} = 0 \qquad (j, k = 1, \cdots, m; j \neq k). \tag{2}$$

LEMMA. *Let M be any R-module. Suppose $x_1, \cdots, x_n \in M$, $a_{ij} \in R$ $(i,j=1, \cdots, n)$ and $y_1, \cdots, y_n\in M$ are related by $y_i= \sum_{j=1}^{n} a_{ij}x_j$ $(i=1, \cdots, n)$. Further, suppose that* $\det(a_{ij})=0$. *Then $y_1, \cdots, y_n$ are dependent.*

Proof. We assume that $y_1, \cdots, y_n$ are independent.

Consider the statement $P(m)$: The determinant of every m-rowed submatrix of (a_{ij}) is equal to 0.

$P(n)$ is certainly true, since $\det(a_{ij})=0$. We now show that $P(m)$ implies $P(m-1)$, $2\leqq m\leqq n$. So assume $P(m)$ to be true. Let A be the m-rowed submatrix of (a_{ij}) containing the elements common to the first m rows and m columns, i.e. A consists of those a_{ij} for which $i, j=1, \cdots, m$. Let A_{ij} denote the cofactor of a_{ij} $(i, j=1, \cdots, m)$ with respect to the matrix A. A simple computation shows that $\sum_{i=1}^{m} A_{i1}y_i= \sum_{j=1}^{n} c_jx_j$, where $c_j= \sum_{i=1}^{m} a_{ij}A_{i1}$. Now:

(i) $c_1=\det A=0$, by (1) and $P(m)$

(ii) For $2\leqq j\leqq m$, $c_j=0$ by (2)

(iii) For $m<j\leqq n$, $c_j=\pm$ the determinant of the m-rowed submatrix of (a_{ij}) containing the elements common to the first m rows and the columns $2, \cdots, m, j$. Hence $c_j=0$, again by $P(m)$.

It follows that $\sum_{i=1}^{m} A_{i1}y_i=0$. But $y_1, \cdots, y_m$ are independent (since $y_1, \cdots, y_n$ were assumed to be independent), and therefore $A_{i1}=0$ $(i=1, \cdots, m)$. Similarly $A_{i2}= \cdots =A_{im}=0$ $(i=1, \cdots, m)$. Thus the determinant of every $(m-1)$-rowed submatrix of A is equal to 0, since any such determinant differs from some A_{ij} $(i, j=1, \cdots, m)$ by at most a change of sign.

Now clearly this same result can be proved for any m-rowed submatrix of (a_{ij}) by a similar argument, A having been chosen above merely to simplify the notation. But this proves $P(m-1)$, since every $(m-1)$-rowed submatrix of (a_{ij}) is contained in at least one m-rowed submatrix of (a_{ij}).

It now follows that $P(m)$ is true for all m, $1\leqq m\leqq n$. In particular, taking $m=1$, we see that $a_{ij}=0$ for all $i, j=1, \cdots, n$. Hence $y_1= \cdots =y_n=0$, contradicting our assumption that they were independent.

Therefore $y_1, \cdots, y_n$ are dependent.

THEOREM. *Let M be an R-module. Suppose that M is generated by $x_1, \cdots, x_n$. Then M contains at most n independent elements.*

Proof. Let $y_1, \cdots, y_{n+1}$ be any $n+1$ elements of M. Write $y_i = \sum_{j=1}^{n} a_{ij}x_j$ $(i=1, \cdots, n+1)$. Choose any element x_{n+1} in M and set $a_{1,n+1} = \cdots = a_{n,n+1} = a_{n+1,n+1} = 0 \in R$. Then obviously we have $y_i = \sum_{j=1}^{n+1} a_{ij}x_j$ $(i=1, \cdots, n+1)$. The $(n+1)\times(n+1)$ matrix (a_{ij}) of coefficients is such that $\det(a_{ij})=0$, since the $(n+1)$st column contains only 0 entries. By the lemma, $y_1, \cdots, y_{n+1}$ are dependent, and the theorem is proved.

Our main result follows almost immediately from the above theorem.

THEOREM. *Suppose that M has a basis of n elements. Then any basis of M is also finite and contains exactly n elements.*

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THE GEOMETRIC REPRESENTATION OF $\sqrt{(z^2-k^2)}$ AND $z\pm\sqrt{(z^2-k^2)}$

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Suppose z is a complex number with nonzero imaginary part, while k is real and positive. It is frequently desirable to have a geometric representation in the complex plane of the quantities

$$\rho(z) = \sqrt{(z^2 - k^2)}, \qquad \zeta_{\pm}(z) = z \pm \rho(z). \tag{1}$$

For instance, the Helmholtz equation $\nabla^2\phi + k^2\phi = 0$ has solutions in rectangular and cylindrical coordinates which involve contour integrals in the z-plane whose integrands contain $\rho(z)$ and have branch points at $\pm k$. In solving boundary value problems involving the Helmholtz equation the contours frequently must be deformed in various ways. This process is greatly expedited if one has a geometric picture of the quantities in (1).

ζ_+ and ζ_- are roots of the equation

$$\tfrac{1}{2}(\zeta + k^2/\zeta) = z, \tag{2}$$

which occurs frequently. For instance, an application of the Method of Steepest Descent to Bessel functions of large order and index indicates the roots of (2) as saddle points.

We will take

$$\begin{aligned} |\rho(z)| &= |z-k|^{1/2}|z+k|^{1/2}, \\ \arg\rho(z) &= \tfrac{1}{2}\arg(z-k) + \tfrac{1}{2}\arg(z+k), \end{aligned} \tag{3}$$

where the absolute values of the arguments in the last line are all to be less than π.

Now plot the points $-k$, k, z, and draw lines connecting z to k and to $-k$. Bisect the angle $kz(-k)$ and erect a perpendicular to this bisector, Bz, at z. This perpendicular intersects the imaginary axis at O'. Now with O' as center and with radius $O'k$, draw a circle. We will prove that this circle intersects Bz extended, in the points $\zeta_+(z)$ and $\zeta_-(z)$, as shown in the diagram.

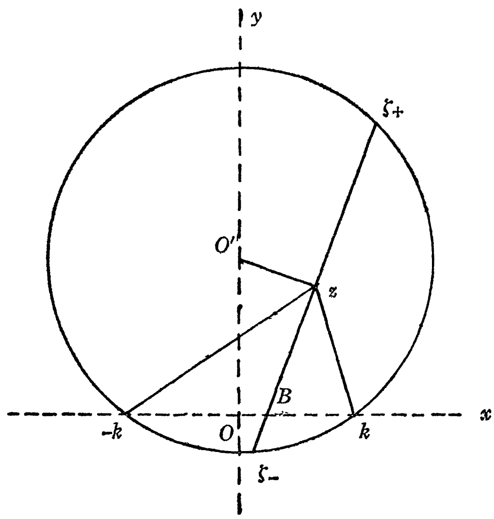

FIG. 1

To begin with, we see that the slope angle of Bz is the average of those of $z-k$ and $z+k$, and therefore, by (3), equals the argument of $\rho(z)$. Therefore ζ_+ and ζ_- must lie on Bz extended. Since they are at the same distance from z, they are either both inside, both on, or both outside the circle.

By (1), $\zeta_-=k^2/\zeta_+$. Therefore $\angle\zeta_+Ok=\angle kO\zeta_-$. Moreover $|\zeta_-|/k=k/|\zeta_+|$. Therefore $\triangle\zeta_-Ok$ is similar to $\triangle kO\zeta_+$. Therefore $\angle k\zeta_+O=\angle\zeta_-kO$. Similarly $\angle(-k)\zeta_+O=\angle\zeta_-(-k)O$. It follows that $\angle k\zeta_-(-k)$ is the supplement of $\angle k\zeta_+(-k)$. Therefore the points ζ_+ and ζ_- are both on the circle, and must be the points of intersection of the circle with Bz extended.

The author is indebted to the referee for an improvement in the method of the proof.

SOME PROBLEMS ON CONVEX FIGURES

H. GUGGENHEIMER, University of Minnesota

The following remarks grew out of an elementary course on Convex Figures. They might be of use and interest to others teaching the same material. Most of the theorems were given as problems in the final examination. They all can be

proved by continuity arguments such as those, e.g., in Chapter 3 of "Convex Figures" by Yaglom and Boltyanskii [2]. For a successful application of the method one has to note that an element of support of a convex curve is a couple consisting of a line (none of whose points is interior to the convex curve) and a point common to line and curve. One line of support of a curve which is convex, but not strictly convex, may belong to an infinity of elements of support. The elements of support of a strictly convex curve are continuous functions of the oriented angle of the line of support with a fixed axis. If the given curve is not strictly convex, one may proceed as follows: A closed convex curve contains at most countably many straight segments. Segment number i is given by the angle α_i of its line and has length s_i, and $\sum s_i < \infty$. If $\alpha \neq \alpha_i$ $(i = 1, 2, \cdots)$ we characterize the corresponding unique element of support by $\beta = \alpha + \sum_{\alpha_j < \alpha} s_j$. If $\alpha = \alpha_i$, the element of support consisting of the line of support and a point of contact at distance $t \leqq s_i$ from the starting point of the segment (counted in the positive sense on the convex curve) shall have coordinate $\beta = \alpha_i + \sum_{j<i} s_j + t$. This new variable β takes the place of the angle in all arguments about not strictly convex curves.

Our main interest is the study of parallelograms circumscribed to plane convex closed curves. For the purposes of this paper a parallelogram is a figure formed by four elements of support with pairwise parallel lines of support. Such a parallelogram exists for any pair of linearly independent directions in the plane; it can be given by its interior angles and the direction of one pair of parallel sides and eventually, in the not strictly convex case, a point of support on each one of the sides. If the curve is not strictly convex, one parallelogram in the sense of elementary geometry therefore may represent an infinity of circumscribed parallelograms. The four points of support define the *contact quadrilateral.* According to our conventions the contact quadrilaterals are in one-to-one correspondence with the circumscribed parallelograms.

1. *For any closed convex curve there exists a circumscribed parallelogram of given angles such that the diagonals of the contact quadrilateral meet in the midpoint of one of the diagonals.*

Sketch of proof. If the curve is strictly convex, the parallelogram (of fixed interior angles) may be given by the oriented angle θ which determines one of the sides. Let P, Q, R, S be the points of support, M the intersection of the diagonals. Then $f(\theta) = RM - MP$ is continuous and $f(\theta + \pi) = -f(\theta)$. The changes necessary for not strictly convex curves are routine.

2. *For any closed convex curve with piecewise monotone non-zero curvature there exists a circumscribed parallelogram such that the contact quadrilateral is a parallelogram.*

Proof (*sketch*). Monotonicity of the curvature implies continuity. Also, the curve is strictly convex. The circumscribed parallelograms are differentiable data on $0 \leqq \theta \leqq 2\pi$, $0 < \alpha < \pi$ in a (θ, α) plane, where θ is as in Problem 1 and α

measures one of the interior angles of the parallelogram. The support function is twice differentiable in θ (for all this see, e.g., [1], sec. 3-3). This is sufficient to show that the set of solutions $\theta_0=\theta_0(\alpha)$ for Problem 1 is connected in the (θ, α) plane. Therefore Bolzano's theorem may be used on the set $\{\theta_0(\alpha), \alpha\}$ to single out a parallelogram for which the diagonals of the contact quadrilateral bisect one another.

Remark. Strict convexity is necessary in Problem 2; the statement is false for triangles. I do not know whether it holds for C' (smooth) or C^0 (continuous) curves. An elementary proof of the proposition is certainly desirable.

The next propositions are proved in a way outlined for no. 1.

3. *For any closed convex curve there exists a parallelogram of given angles such that the diagonals of the contact quadrilateral meet on a diagonal of the parallelogram.*

4. *For any closed convex curve there exists a circumscribed rhombus of given angles.*

Let A, B, C, D be the vertices of the circumscribed parallelogram, P, Q, R, S those of the contact quadrilateral. P is on AB.

5. *For any closed convex curve there exists a circumscribed parallelogram of given angles such that* $\overline{PB}=\overline{DR}$, *one such that* $\overline{AP}:\overline{CR}=\overline{BQ}:\overline{DS}$, *and one such that* $\overline{PR}=\overline{QS}$.

6. *For any smooth strictly convex curve there exists a circumscribed parallelogram such that the midpoints of one pair of parallel sides are points of support.*

A simple proof of this statement was supplied by a student, M. Sayrafiezadeh, in the final examination of the course. Here is its outline. Given two parallel lines of support, it is possible to find a (unique) circumscribed quadrilateral having two sides on the given lines, and such that the other two sides touch the convex curve in their midpoints. Here the C' hypothesis is used. The difference of the segments cut off on the two parallel lines is a continuous function of their position (here strict convexity enters) and changes sign in a rotation of angle π. Simple examples show that both conditions are necessary.

Next we turn to parallelepipeds circumscribing closed convex surfaces in R^3. The projection of such a surface into any plane is a compact convex plane figure. No. 4 implies

7. *For any closed convex surface and any given direction there exists a circumscribed right prism whose lateral edges are in the given direction and whose base is a rhombus of prescribed angles.*

A closed convex surface has a circumscribed cube by Kakutani's theorem. The proof of Kakutani's theorem needs the Kronecker existence theorem in the plane and the fact that the fundamental group of the three dimensional rotation group is $\mathbf{Z}_2$. Using only Bolzano's theorem, one cannot get so far.

8. *Any closed convex surface has (uncountably many) circumscribed rectangular solids of edge lengths* $a \geqq b \geqq c$ *such that* $b = \sqrt{ac}$.

For any frame (system of three orthonormal vectors) there exists a circumscribed rectangular solid with edges parallel to the frame vectors **a**, **b**, **c**. Let $a(0) \geqq b(0) \geqq c(0)$ be the length of the respective edges. The data of the rectangular solid are continuous in a rotation of the frame since for a small angle of rotation the surface of the new rectangular solid is in a shell of small thickness about the old one. In a rotation of angle $\pi/2$ about **b** the function $f(\theta) = a(\theta)b(\theta)c(\theta) - a(\theta)^3$ is brought from $f(0) \leqq 0$ into $c(0)b(0)a(0) - c(0)^3 \geqq 0$, Q.E.D.

This research was supported by the Air Force Office of Scientific Research.

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UNCORRELATED GAUSSIAN DEPENDENT RANDOM VARIABLES

C. D. FIRESTONE AND J. E. HANSON, Applied Physics Laboratory, Silver Spring, Maryland

It is well known that two random variables can be uncorrelated but dependent. It is also well known (see, e.g., [1] p. 311) that if two *Gaussian* random variables are uncorrelated, then they are independent. It may, therefore, be of interest to the student to see an example of two random variables that are Gaussian and uncorrelated, but nevertheless dependent.

Let $a = (1/4\pi)e^{-4}$, so that for $-2 < x < 2$, $-2 < y < 2$, we have $0 < a < (1/2\pi)\exp(-\frac{1}{2}(x^2+y^2))$. Define $f(x, y)$ by the equations

$$f(x,y) = \begin{cases} a & \text{if } 1 < |x| < 2 \text{ and } 1 < |y| < 2, \text{ or if} \\ & \phantom{\text{if }} 0 < |x| < 1 \text{ and } 0 < |y| < 1 \\ -a & \text{if } 1 < |x| < 2 \text{ and } 0 < |y| < 1, \text{ or if} \\ & \phantom{\text{if }} 0 < |x| < 1 \text{ and } 1 < |y| < 2 \\ 0 & \text{otherwise,} \end{cases}$$

and let $p(x, y) = f(x, y) + (1/2\pi)\exp(-\frac{1}{2}(x^2+y^2))$. A diagram of $f(x, y)$ makes it obvious that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 0$, so that the nonnegative function $p(x, y)$ is a probability density function.

Now let X and Y be random variables with the joint probability density function $p(x, y)$. The probability density functions of X and Y are respectively $\int_{-\infty}^{\infty} p(x, y)dy = (1/\sqrt{2\pi})e^{-x^2/2}$ and $\int_{-\infty}^{\infty} p(x, y)dx = (1/\sqrt{2\pi})e^{-y^2/2}$, so that each of X and Y is Gaussian. Another glance at a diagram of $f(x, y)$, together with consideration of the symmetry around the origin of $p(x, y)$, makes it clear that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyp(x, y)dxdy = 0$, so that X and Y are uncorrelated. But if X and Y were independent, their joint probability density function would be the product $(1/2\pi)\exp(-\frac{1}{2}(x^2+y^2))$, which it is not. Hence, X and Y are dependent.

What is actually proved in [1], and what of course the author assumed the reader would understand, is that if *jointly* Gaussian random variables are uncorrelated then they are independent. This example may perhaps encourage the student to keep in mind that random variables can be Gaussian but not jointly Gaussian.

Reference

1. H. Cramér, Mathematical Methods of Statistics, Princeton, 1946.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N. W., Washington 5, D. C.

THE TRAINING OF A MATHEMATICS TEACHER IN THE NETHERLANDS

A Report to the International Commission on Mathematical Instruction

LUCAS N. H. BUNT, University of Utrecht

At the request of the Dutch Subcommittee of the International Commission on Mathematical Instruction a report has been composed under the above title. It is issued as No. VII of a series of publications of the Subcommittee and as No. XXII of the Series *Acta Paedagogica Ultrajectina*, published by the Institute of Education of the University of Utrecht. It gives an account of the way in which, in the Netherlands, a teacher of mathematics in an academic secondary school is being prepared for his task and it discusses certain general aspects of the training of a mathematics teacher.

There are six chapters, by different writers, dealing with general aspects of the training of a mathematics teacher, the subject-matter training of a mathematics teacher at a university or an institute of technology, the non-university training and the professional training of a mathematics teacher, the place of history and of philosophy in the training of a mathematics teacher. Ten appendices provide the reader with examination papers of several kinds, and with selections of problems to be solved in the laboratory courses in algebra and analysis.

The report concerns itself only with the training of teachers for a secondary school of the academic type, i.e. a school leading up to the university. The qualification to be a teacher of mathematics in such a school can be obtained in two ways: (a) by attending a university or an institute of technology and

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
Collaborating Editor: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1601. *Proposed by A. A. Mullin, University of Illinois*

If d is a nonnegative decimal integer, let d_n be the decimal integer obtained from d by keeping only the last n digits of d. Prove that $2^n | d$ if and only if $2^n | d_n$.

E 1602. *Proposed by M. C. Gemignani, University of Notre Dame*

Given a set of n points in Euclidean space such that no three of the points are collinear, prove that the set of all line segments determined by these n points can be disconnected by no set of less than $n-1$ points, and there is at least one set of $n-1$ points disconnecting the set of segments, where we require that no more than one point of the disconnecting set can come from any one segment.

E 1603. *Proposed by R. S. Luthar, University of Illinois*

If p is a prime not less than n, where n is a given positive integer, show that $(n-1)!(p-n)! \equiv (-1)^n$, mod p.

E 1604. *Proposed by A. P. Boblëtt, U. S. Naval Ordnance Laboratory, Corona, California*

Evaluate $\int_0^\infty [\sin^2 x/(\pi^2 - x^2)]dx$.

E 1605. *Proposed by R. E. Mikhel, Ball State Teachers College*

If A is a 3×3 involutoric matrix ($A^2=I$) with no zero elements, prove that the trace of A is $+1$ or -1.

E 1606. *Proposed by Leopold Flatto, Yeshiva University*

Let n denote the number of nonzero elements in a rectangular matrix of numbers each row and each column of which has a zero sum. What possible values can n assume?

E 1607. *Proposed by Ralph Greenberg, University of Pennsylvania*

Show that, for all integers $t \geqq 0$,

$$x^n/(n+1)^t + x^{n-1}/n^t + \cdots + x/2^t + 1 = 0$$

has no real root if n is even and exactly one real root if n is odd.

E 1608. *Proposed by G. A. Heuer and Dean Knudson, Concordia College*

Find the linear fraction $f(x)=(x+b)/(cx+d)$ which best approximates $\tan x$ in the interval $[0, \pi/2]$ in the sense that $\int_0^{\pi/2}|f(x)-\tan x|\,dx$ is minimized.

E 1609. *Proposed by Richard Sinkhorn, University of Houston*

Two numbers are chosen independently and at random from the closed interval $[-a, a]$. Show that the chance that their product will exceed their sum is the least when $a=[x_0/(x_0-1)]^{1/2}$, where x_0 is the real root of the equation $x=1+e^{-x}$.

E 1610. *Proposed by R. A. Olshen, University of California at Berkeley*

Prove that it is impossible to imbed a finite projective plane in the real affine plane.

SOLUTIONS

An Invalid Conjecture Concerning an Integral Domain

E 1537 [1962, 809]. *Proposed by Roy Dowling, University of Manitoba*

If the square root of an integer is not an integer, then it is not a rational number. This suggests a conjecture concerning an element a of an integral domain D: If $\sqrt{a}$ is not in D, then it is not in the field of quotients of D. Is this a valid conjecture?

I. *Solution by Joseph Beer, Radio Corporation of America, Moorestown, New Jersey.* Let D be the domain of integral multiples of 4. Then $\sqrt{4}\notin D$, but $\sqrt{4}=8/4$ is in the quotient field of D.

II. *Solution by R. E. Bowen, Fairfield, California.* Let D be the domain of polynomials in x with integral coefficients and with the coefficient of x zero. Then $x=\sqrt{x^2}\notin D$, but $x=x^3/x^2$ is in the quotient field of D.

Also solved by J. W. Baldwin, Walter Bluger, Leonard Carlitz, Alvin Hausner, C. V. Heuer, C. R. MacCluer, D. C. B. Marsh, R. H. C. Newton, J. L. Pietenpol, C. P. Seguin, E. L. Spitznagel, Jr., Arthur Steger, B. R. Toskey, W. C. Waterhouse, and the proposer.

Editorial Note. The conjecture is valid for a unique-factorization domain.

Simple Closed Curves with a Circle Property

E 1538 [1962, 809]. *Proposed by M. S. Klamkin, State University of New York at Buffalo, and Jerry Yos, Avco Corporation*

A simple closed curve has the property that there exist inscribed squares of the same dimension in every direction. Must the curve be a circle?

I. *Solution by Marlow Sholander, Western Reserve University.* No. Each unit square can be inscribed in an "eyepiece" of the spectacle-shaped boundary of the union of circles $(x\pm\sqrt{2})^2+y^2\leqq 1/2$ with the rectangle $-\sqrt{2}\leqq x\leqq\sqrt{2}$,

$1 \leqq 2y \leqq \sqrt{2}$. (A variation of the spectacles gives the same answer for regular n-gons.)

II. *Solution by Michael Goldberg, Washington, D. C.* The curve need not be a circle. Circumscribe a square about an oval of constant width. Holding the oval fixed, rotate the square about the oval. Then all four of the vertices of the square describe a new oval. This new oval is not a circle, yet it has the property that the inscribed square within it may be turned through all orientations. See Michael Goldberg, "Rotors tangent to n fixed circles," *J. Math. and Phys.*, 37 (1958) 70.

Also solved by J. W. Baldwin, Joseph Beer, D. G. Beverage, Walter Bluger, Norman Brenner, Brother Louis Zirkel, D. I. A. Cohen, C. M. Frye, Ralph Greenberg, R. A. Jacobson, E. L. Magnuson, D. A. Moran, Stanton Philipp, J. L. Pietenpol, Perry Scheinok, B. R. Toskey, Julius Vogel, and the proposers.

The proposers furnished a convex counter-example given by

$$x = a \cos \theta + \epsilon \cos 4\theta,$$

$$y = a \sin \theta + \epsilon \sin 4\theta,$$

where $16\epsilon < a$. Here there is only one square in each direction.

J. J. Schäffer, of the Instituto de Matematica y Estadistica, Montevideo, Uruguay, pointed out that this problem is considered by Günter Lumer in "Polígonos inscriptibles en curvas convexas," *Rev. Un. Mat. Argentina*, 17 (1955) 97–102. Considerable information, such as the fact that the curve may be convex and have area larger or smaller than the circle, is contained in the paper.

Minimum Number of Distinct Values Assumed by a Sum

E 1539 [1962, 809]. *Proposed by Leo Moser, University of Alberta*

Let $0 < a_1 < a_2 < \cdots < a_n$ and $\epsilon_i = \pm 1$. Prove that $\sum_{i=1}^{n} \epsilon_i a_i$ assumes at least $\binom{n+1}{2}$ distinct values as the ϵ_i range over the 2^n possible combinations of signs.

I. *Solution by B. R. Toskey, Seattle University.* We prove the result by setting $c = \sum_{i=1}^{n} (-a_i)$ and observing that $c < c+2a_1 < c+2a_2 < \cdots < c+2a_n < c+2a_n +2a_1 < \cdots < c+2a_n+2a_{n-1} < c+2a_n+2a_{n-1}+2a_1 < \cdots < c+2(\sum_{i=1}^{n} a_i) = \sum_{i=1}^{n} a_i$, so that there are at least $1+n+(n-1)+(n-2)+\cdots+2+1 = 1+n(n+1)/2$ distinct values in the list. Since each value is one of the given sums, we have shown that the expression assumes at least $\binom{n+1}{2}+1$ values.

II. *Solution by Ralph Greenberg, University of Pennsylvania.* We shall prove, by induction, that there are at least $\binom{n+1}{2}+1$ distinct values. Now $S \equiv \sum_{i=1}^{k-1} a_i \geqq \sum_{i=1}^{k-1} \epsilon_i a_i$. Suppose the last sum assumes at least $\binom{k}{2}+1$ distinct values. Take $a_k > a_{k-1}$. Then a_k+S, a_k+S-a_{k-1}, a_k+S-a_{k-2}, $\cdots$, a_k+S-a_1 are all distinct and greater than S. Therefore $\sum_{i=1}^{k} \epsilon_i a_i$ assumes at least $k+\binom{k}{2}+1 = \binom{k+1}{2}+1$ distinct values. Since $\epsilon_1 a_1$ assumes two values, and $2 = \binom{2}{1}+1$, the stated result follows.

Also solved by Joseph Beer, Walter Bluger, Bruce Blum, W. J. Blundon, J. L. Brown, Jr., D. I. A. Cohen, Dennis Couzin, Stephen Fisk, P. M. Gibson, Michael Goldberg, R. A. Jacobson, E. L. Magnuson, D. C. B. Marsh, P. N. Muller, J. L. Pietenpol, Perry Scheinok, Dennis Travis, Andy Vince, Charles Wexler, K. S. Williams, and the proposer.

A Polynomial Multiple of a Polynomial

E 1540 [1962, 809]. *Proposed by Azriel Rosenfeld, Yeshiva University*

Prove that every polynomial has a nonzero polynomial multiple whose exponents are all divisible by 1,000,000.

I. *Solution by M. S. Klamkin, State University of New York at Buffalo.* Let the given polynomial be

$$P(x) = \prod_i (x - r_i).$$

Let

$$Q(x) = x^a \prod_i [(x^3 - r_i^a)/(x - r_i)].$$

It follows immediately that $P(x)Q(x)$ is a polynomial whose exponents are all divisible by a. Now let $a = 10^6$.

II. *Solution by the proposer.* Suppose first that the polynomial f is irreducible over K, where K is a field containing the coefficients of f. Let θ be any root of f and n any natural number. Since $\theta^n \in K(\theta)$ must be algebraic over K, there exists a nonzero polynomial $g \in K[x]$ such that $g(\theta^n) = 0$. Let $h(x) = g(x^n)$; then every exponent of $h(x)$ is divisible by n. But $h(\theta) = g(\theta^n) = 0$; h must therefore be divisible by f, since f is a minimal polynomial of θ over K. For f reducible, let $f = \prod f_i$, where the f_i are irreducible; let g_i and h_i be as in the first part of the proof. Then f divides $\prod h_i$, which is clearly still nonzero and still has all its exponents divisible by n.

Also solved by Joseph Beer, D. G. Beverage, Norman Brenner, Leonard Carlitz, Stephen Fisk, Michael Goldberg, Ralph Greenberg, G. A. Heuer, H. W. Hickey, R. A. Jacobson, D. C. B. Marsh, J. B. Muskat, F. D. Parker, Stanton Philipp, J. L. Pietenpol, Perry Scheinok, B. R. Toskey, and Dennis Travis.

Marsh was the only solver, besides the proposer, who considered the problem in a more general setting than either the real or the complex number field.

ADVANCED PROBLEMS AND SOLUTIONS

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5111. *Proposed by W. A. Schneider, Milwaukee, Wisconsin*

If P_n/Q_n is the nth convergent of the continued fraction for $\sqrt{(x^2+1)}$, then arccot $P_{2n-1}=2$ arccot $Q_{2n}-$arccot P_{2n+1}.

5112. *Proposed by N. R. Riesenberg, University of Wisconsin*

In Dieudonné, *Foundations of Modern Analysis*, a real number system is defined as a field which (1) is Archimedean ordered, and (2) possesses the nested interval property. It is well known that neither (1) nor (2) alone suffices to give a real number system and many examples of Archimedean ordered fields which are not real number systems are in the literature. Give an example of a field which is non-Archimedean ordered but which possesses the nested interval property.

5113. *Proposed by J. S. Frame, Michigan State University*

Sum the series

$$S = \sum_{k=0}^{\infty}\binom{2k}{k}(-16)^{-k}(2k+1)^{-2}.$$

5114. *Proposed by W. E. Johnson and C. M. Petty, Lockheed Aircraft Corp., Sunnyvale, California*

Let the function $F(t)$ have a continuous derivative on $[0, 1]$ and set $S_1=\{t;\ F'(t)=0\}$, $S_2=\{F(t);\ t\in S_1\}$. Show, by example, that the set S_2 may be uncountable.

5115. *Proposed by Harley Flanders, Purdue University*

Let $k\leqq K$, $F\leqq\Omega$, all commutative fields. We may form the composite KF and it is known that $[KF:F]\leqq[K:k]$ if $[K:k]$ is finite. Prove that this inequality is true when $[K:k]$ is infinite, provided $[F:k]$, the linear dimension of F over k, is countable.

5116. *Proposed by David Greenstein, Northwestern University*

Let

$$S(A) = \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \frac{A^{j+k}}{j!k!}, \quad (A \text{ real}).$$

An engineer needs asymptotic information about $S(A)$ for $A \to \infty$. He conjectures that $e^{-2A}S(A) \to 1$ as $A \to \infty$. Prove or disprove his conjecture.

5117. *Proposed by L. Carlitz, Duke University*

Let η, ζ be roots of unity such that

$$a\eta + b\zeta + c = 0 \qquad (\eta^2 \neq 1, \zeta^2 \neq 1),$$

where a, b, c are nonzero integers. Show that the only possibilities are given by $a = b = c$, $\eta = \omega$, $\zeta = \omega^2$, $\omega^2 + \omega + 1 = 0$.

5118. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the integer n be given, $n \geqq 2$. Show that if

$$T(t) \equiv 1 + a_1 \cos t + b_1 \sin t + a_n \cos nt + b_n \sin nt \geqq 0 \tag{1}$$

for all real t, then

$$a_1 \leqq \sec(\pi/2n) \tag{2}$$

with equality if and only if

$$T(t) = 1 + \left(\sec\frac{\pi}{2n}\right) \cos t + \frac{(-1)^n}{n} \left(\tan\frac{\pi}{2n}\right) \cos nt. \tag{3}$$

5119. *Proposed by G. S. Glazer, Milwaukee, Wisconsin*

For every odd positive integer q there exists a set $\{(q^2+u^2)/2\}$ where u is odd and $q > u > 0$. It has been conjectured that at least one member of every such set is a prime.

Suppose the conjecture is false. Then for some q, every such quotient will have a least divisor c. Since $(q^2+u^2)/2 < q^2$, we can be sure $c < q$. In this case we can state the following theorem: *There exists a positive odd integer q such that, to every odd $u(q > u > 0)$ there corresponds c $(c < q)$ such that c divides $(q^2+u^2)/2$.*

Prove either the conjecture or the theorem.

5120. *Proposed by D. C. Olivier, Carleton College, Northfield, Minn.*

Define a sequence $\{v_n\} = \{v_n(x)\}$ recursively by $v_1 = x$, $v_{n+1} = (2 + 1/n)v_n - 1$, $n \geqq 1$. It is not hard to show that $\{v_n\}$ converges for at most one real value of x. Find x such that $\{v_n\}$ converges.

SOLUTIONS

A Real Function with Unusual Properties

5029 [1962, 438]. *Proposed by A. C. Zaanen, University of Leiden, and W. A. J. Luxemburg, California Institute of Technology*

Let $I=\{x: 0\leqq x\leqq 1\}$.

(a) Show that there exists a real function $f(x)$ on I having the following properties: (i) f is continuous and strictly increasing (i.e. $x_1<x_2$ implies $f(x_1)<f(x_2)$), (ii) the derivative $f'(x)=0$ for all $x\in I$ except on a subset of Lebesgue measure zero.

(b) Let $f(x)$ be continuous and nondecreasing on I and suppose that $f(0)=0$ and $f(1)=\alpha$. By L we denote the length of the graph of $f(x)$. Show that $L=1+\alpha$ if and only if $f'(x)=0$ for all $x\in I$ except on a subset of Lebesgue measure zero.

(c) Let $f(x)$ be continuous and strictly increasing on I, with $f(0)=0$ and $f(1)=\alpha$, and let $g(y)$ be the inverse function, defined on $I_1=\{y: 0\leqq y\leqq\alpha\}$. Show that $f'(x)=0$ almost everywhere on I if and only if $g'(y)=0$ almost everywhere on I_1.

Solution by the proposers. (a) Let $g(x)$ be the Cantor function corresponding to the closed interval $[0, 1]$, i.e. $g(x)=1/2$ on the open interval $(1/3, 2/3)$, $g(x)=1/4$ on $(1/9, 2/9)$, $g(x)=3/4$ on $(7/9, 8/9)$, and so on, and domain and range of $g(x)$ are extended to the whole of $[0, 1]$ by continuity. Furthermore, $g(x)=0$ for $x<0$ and $g(x)=1$ for $x>1$. Then, as is well known, $g'(x)=0$ almost everywhere on the real line. If we have the interval $[a, b]$, the corresponding Cantor function is defined similarly (for convenience let it assume the value zero at a and the value 1 at b). Now, let $I_1, I_2, \cdots$ be the intervals $[0, 1]$, $[0, 1/2]$, $[1/2, 1]$, $[0, 1/4]$, $\cdots$, $[0, 1/8]$, $\cdots$, and let $g_n(x)$ be the Cantor function corresponding to I_n. Then $f(x)=\sum_{n=1}^{\infty} g_n(x)/2^n$ is continuous and strictly increasing on $I=[0, 1]$, and $f'(x)=0$ almost everywhere by a theorem of Fubini (cf. Riesz-Nagy, *Analyse fonctionelle*, Sec. 5).

(b) We may assume $\alpha>0$ (for $\alpha=0$ the proof is trivial). Let $F(x)=x+if(x)$. By definition of arc length we have always $L=V(F)=V\{x+if(x)\}$, where V is the total variation. Now $V(F)\geqq V\{x-f(x)\}$ follows easily from the inequality $[(x_2-x_1)^2+\{f(x_2)-f(x_1)\}^2]^{1/2}\geqq|(x_2-x_1)-\{f(x_2)-f(x_1)\}|$ for $x_2>x_1$ by observing that f is nondecreasing. Now, if $f'(x)=0$ almost everywhere, then the Stieltjes-Lebesgue measure ν generated by $f(x)$ is orthogonal to (i.e. singular with respect to) Lebesgue measure μ, and $V\{x-f(x)\}$ is exactly the total variation of the signed measure $\mu-\nu$. For orthogonal measures μ and ν it is known that the total variations of $\mu+\nu$ and $\mu-\nu$ are equal, so $V\{x+f(x)\}=V\{x-f(x)\}$. But $V\{x+f(x)\}=1+\alpha$, since x and $f(x)$ are nondecreasing. Hence $1+\alpha=V\{x-f(x)\}\leqq L\leqq 1+\alpha$, and it follows that $L=1+\alpha$.

Assume now, conversely, that $L=1+\alpha$, and let $f=f_1+f_2$ be the Lebesgue decomposition of f, hence $f_1(x)=\int_0^x f'(t)dt$ and $f_2'(x)=0$ almost everywhere. Evi-

dently, since f_1 and f_2 are nondecreasing, $V(f)=V(f_1)+V(f_2)$. Then

$$1+\alpha = V(F) = V(x+if_1+if_2) \leqq V(x+if_1)+V(f_2)$$
$$\leqq 1+V(f_1)+V(f_2) = 1+V(f) = 1+\alpha,$$

so $V(x+if_1)=1+V(f_1)$. Since f_1 is absolutely continuous we may use the integral formula for $V(x+if_1)$, so

$$\int_0^1 \{1+(f_1')^2\}^{1/2}dx = 1+V(f_1) = \int_0^1 (1+f_1')dx.$$

But $\{1+(f_1')^2\}^{1/2}\leqq 1+f_1'$ for almost every x. Hence, these functions are equal almost everywhere, and so $f_1'(x)=0$ almost everywhere. In view of $f'=f_1'+f_2'$ and $f_2'=0$, we conclude that $f'(x)=0$ almost everywhere.

(c) $f'(x)=0$ almost everywhere on I is equivalent to $L=1+\alpha$, and this is equivalent to $g'(y)=0$ almost everywhere on I_1.

Also solved by J. M. Michael.

An Infinite Sequence of 3 Symbols with no Adjacent Repeats

5030 [1962, 439]. *Proposed by Hugh Noland, Chico State College, California*

I. Does there exist a nonterminating decimal, in which the only digits appearing are 1, 2, 3, such that no two adjacent ordered n-tuples of digits are identical for any integer n?

II. More generally, can three elements generate an infinite set which is closed under an associative multiplication, and has the property that every element is idempotent?

I. *Solution by C. H. Braunholtz, British Embassy, Washington, D. C.* We give a simple example of an infinite sequence on three symbols with no adjacent repeats. First define the sequence $S=a_0a_1a_2, \cdots$, where $a_n=0$ if there are an even number of 1's in the expression of n as a binary number, and $a_n=1$ if there are an odd number of 1's. Thus $S=0110100110010110 \cdots$. Then define the sequence $T=b_1b_2b_3, \cdots$, where b_i is the number of 1's between the ith and the $(i+1)$th occurrence of 0 in S. Thus $T=2102012 \cdots$. We show that T contains only the three symbols 0, 1, 2, and that T has no adjacent repeats.

From the definition of a_n we have at once

$$(1) \qquad a_{2r}=a_r, \qquad a_{2r+1}\equiv a_r+1 \pmod 2$$

$$(2) \qquad a_{2r}+a_{2r+1}\equiv 1 \pmod 2.$$

That $b_i\leqq 2$ for all i follows from the fact that a run of three consecutive 1's in S would imply $a_{2r}=a_{2r+1}=1$ for some r, contradicting (2).

A *proper adjacent repeat* in S is defined as an adjacent repeated sequence which starts to repeat a third time, i.e., the first symbol after the repeat is the first symbol of the repeat.

Suppose S has a proper adjacent repeat of length n. If n is odd it follows from (2) that the repeated sequence consists of alternating 0's and 1's, but then either at the beginning or end of the repeat there is a contradiction of (2). If, on the other hand, n is even, the existence of a proper adjacent repeat of length n starting at a_{2r} (or starting at a_{2r+1}) implies the existence of a proper adjacent repeat of length $\frac{1}{2}n$ starting at a_r (using equations (1)). Repeating this last argument as often as necessary we reach the first case and a contradiction.

We have proved that the sequence S has no proper adjacent repeats. It follows that T has no adjacent repeats, for it is clear that an adjacent repeat in T must arise from a proper adjacent repeat in S.

To obtain the proposed form, merely replace the symbol 0 by 3. Another example is given by John Leech, *A problem on strings of beads*, Math. Gazette, 41 (1957) 277–278. See also Morse and Hedlund, *Unending chess, symbolic dynamics, and a problem in semigroups*, Duke Math. J., 11 (1944) 1–7.

II. *Remark by T. Tamura, University of California, Davis.* In his paper, *Idempotent Semigroups*, this MONTHLY, 61 (1954) 110–113, D. McLean showed that a free idempotent semigroup generated by a finite number of elements is of finite order. Earlier, J. A. Green and D. Rees proved the same result in their paper, *On semigroups in which* $x^r = x$, Proc. Camb. Phil. Soc., 48 (1952) 35–40. They also showed that the order of a free idempotent semigroup generated by three elements is 159.

Also solved by J. W. Lindsay, and S. L. Segal.

A Property which Implies that a Matrix is the Identity

5031 [1962, 569]. *Proposed by Marvin Marcus, University of British Columbia*

Let A be an $n\times n$ matrix with n elements equal to 1 and the rest 0. Let S_r be the sum of all $\binom{n}{r}$ principal $r\times r$ subdeterminants of A. Show that $S_r = \binom{n}{r}$ for some r, $1 \leqq r < n$ if and only if $A = I$, the $n\times n$ identity matrix.

Solution by Sherwood Ebey, Mercer University, Macon, Ga. It is clear that $A = I$ implies $S_r = \binom{n}{r}$ for each r. On the other hand, let $S_r = \binom{n}{r}$ for some r. S_r is of the form

$$S_r = \sum \pm a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_r j_r}$$

where the sum is extended over the r-element subsets $\{i_1, i_2, \cdots, i_r\}$ of the set of indices $1 \leqq i \leqq n$ and over all permutations $\{j_1, j_2, \cdots, j_r\}$ of these subsets.

Since A has only n nonzero entries, there are at most $\binom{n}{r}$ nonzero terms in this sum. Since $S_r = \binom{n}{r}$ there must be exactly $\binom{n}{r}$ nonzero terms, each equal to 1. This implies each product of r nonzero entries of A must appear as a term in S_r.

It follows that no row or column of A can have more than one nonzero entry. Therefore A is a permutation matrix.

Any principal $r\times r$ subdeterminant of a permutation matrix will be 1, 0 or -1. Since $S_r=\binom{n}{r}$ it follows that each must equal 1. If $a_{ij}=1$, $j\neq i$, then any principal $r\times r$ subdeterminant containing row i and omitting column j equals 0. Therefore $A=I$.

Also solved by C. F. Evans, S. W. Williams, and the proposer.

Function with Positive Second Distribution Derivative

5032 [1962, 569]. *Proposed by Albert Wilansky, Lehigh University*

Let f be a real continuous function of one real variable such that $\int_{-\infty}^{\infty} fg''\geqq 0$ whenever g is a twice differentiable function such that the integral exists and such that $g(t)\geqq 0$ for all real t. Show that f is convex.

Solution by J. H. Michael, University of Adelaide, South Australia. For each positive integer n, let ψ_n be a nonnegative function on R^1, with a continuous second order derivative, with support contained in $(-1/n,\ 1/n)$ and with $\int_{-\infty}^{\infty}\psi_n(x)dx=1$. Put

$$f_n(x) = \int_{-\infty}^{\infty} f(\xi)\psi_n(\xi - x)d\xi.$$

Then

$$f_n''(x) = \int_{-\infty}^{\infty} f(\xi)\psi_n''(\xi - x)d\xi \geqq 0,$$

so that each f_n is convex. Since $f_n\to f$ pointwise on R^1, f is convex.

Also solved by P. T. Bateman, E. J. Burr, N. J. Fine, C. R. DePrima, Harley Flanders, A. G. Konheim, and L. A. Shepp.

Editorial Note. Flanders and DePrima prove the result with the weakened hypothesis, "whenever g is infinitely differentiable and of bounded support." The proposer notes that in the language of distributions the proposition says, if the second derivative of a continuous function is a positive distribution, then the function is convex.

Generalization of a Theorem on Field Extension

5033 [1962, 569]. *Proposed by C. C Faith, Rutgers-The State University*

E. Steinitz' well-known *Satz der Zwischenkörper* states: If K/L is a finite field extension, then K/L is simple if (and only if) the number of intermediate fields of K/L is finite. His proof (*Algebraische Theorie der Körper*, Berlin, 1930, p. 74) falls into two categories: (1) L is infinite, (2) L is finite. In case (2) K is a Galois field and the result follows from the theory of finite fields. If (1) holds, the result can be restated in greater generality. Prove the following theorem:

Let A be a ring with identity 1, and let D be a division subring, $1 \in D$, and suppose that A is algebraic of bounded degree over D in the sense that there exists a natural number N such that for each $a \in A$ the subring $D[a]$ generated by D and a has left dimension $\leqq N$ over D. Furthermore, assume that the collection $\{R\}$ of all intermediate rings of A/D, $A \supseteq R \supseteq D$ for each $R \in \{R\}$, has cardinality less than the cardinality of D. Then A/D is simply generated in the sense that there is an $a \in A$ such that $A = D[a]$.

Solution by Harley Flanders, Purdue University. Select an element a of A such that the left dimension over D of $D[a]$ is maximal. We shall prove that $A = D[a]$. If not, there is an element b in A such that $D[a, b] > D[a]$. Since there are fewer intermediate rings than elements of D, two of the rings, $D[a+\delta b]$, where $\delta \in D$, must coincide, say

$$E = D[a + \delta_1 b] = D[a + \delta_2 b].$$

Clearly

$$(\delta_1 - \delta_2)b = (a + \delta_1 b) - (a + \delta_2 b) \in E,$$

hence $b \in E$, $a \in E$, $E = D[a, b]$. This is impossible since $E = D[a+\delta_1 b]$ is simply generated and $E > D[a]$ forces larger dimension than the maximum.

Also solved by A. K. Charnow, M. M. Schacher, W. C. Waterhouse, and the proposer.

The Weak Law of Large Numbers

5034 [1962, 570]. *Proposed by Ronald Pyke, University of Washington*

We say that a random variable X satisfies the weak (strong) law of large numbers, denoted by W.L.L.N. (S.L.L.N.), if for a sequence of independent random variables, $\{X_n\}$, each with the same distribution function as X, $n^{-1}\sum_{i=1}^{n} X_i$ converges in probability (almost surely) to a finite constant. It is known that if one may write $X = Y + Z$, where Y and Z are independent, then X satisfies the S.L.L.N. if and only if both Y and Z do. Prove or disprove this proposition for the W.L.L.N.

Solution by the proposer. The proposition is false. Kolmogorov's necessary and sufficient condition for a random variable X with distribution function F to satisfy the W.L.L.N. is that (i) $\lim_{T\to\infty} \int_{-T}^{T} x\,dF(x)$ exists and is finite and (ii) $x\Pr[|X| > x] \to 0$ as $x \to \infty$. (Cf. *Foundations of the Theory of Probability*, A. N. Kolmogorov, Chelsea, New York, 1956, p. 65.) Suppose that $X = Y + Z$, where Y and Z are independent. Then condition (ii) above holds for X if and only if it holds for both Y and Z, as follows from the elementary inequalities

$$\begin{aligned} \Pr[|Y| < k]\Pr[|Z| > 2x] &\leqq \Pr[|Y + Z| > x] \\ &\leqq \Pr[|Y| > x/2] + \Pr[|Z| > x/2], \end{aligned}$$

which hold for all $x>k$, where k is any constant such that $\Pr[|Y|<k]>0$. (Notice that independence is used in obtaining the left hand inequality.) This remark is then sufficient for obtaining a wide class of counterexamples to the proposition. For if one lets Y be any random variable satisfying (ii) but not (i), and lets Z have the same probability distribution as $-Y$, then X becomes a symmetric random variable for which $\int_{-T}^{T} x\,dF(x)$ must be identically zero. Hence X satisfies the W.L.L.N. while Y does not.

Also solved by P. J. Bickel and S. W. Dharmadhikari, M. V. Menon, and L. A. Shepp.

Expansion of a Definite Integral

5035 [1962, 570]. *Proposed by Yoshio Matsuoka, Kagoshima-shi, Japan*

Let α be a fixed positive number. Prove that

$$\int_0^{\frac{1}{2}\pi} t^\alpha \cos^{2n} t\,dt = \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)\Big/ n^{(\alpha+1)/2} - \frac{1}{12}\Gamma\left(\frac{\alpha+5}{2}\right)\Big/ n^{(\alpha+3)/2} + O(1/n^{(\alpha+5)/2}) \qquad (n\to\infty).$$

Solution by M. S. Klamkin, State University of New York at Buffalo. First expand t^α into the series

$$t^\alpha = A_1 \sin^\alpha t + A_2 \sin^{\alpha+2} t + A_3 \sin^{\alpha+4} t + \cdots .$$

(This is a Lagrange reversion of a power series.) It follows immediately that $A_1=1$, $A_2=\alpha/6$. Since

$$\int_0^{\frac{1}{2}\pi} \sin^m t \cos^{2n} t\,dt = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma(n+\frac{1}{2})}{2\Gamma(\frac{1}{2}m+n+1)}$$

it follows that

$$\int_0^{\frac{1}{2}\pi} t^\alpha \cos^{2n} t\,dt = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma(n+\frac{1}{2})}{2\Gamma(\frac{1}{2}\alpha+n+1)} + \frac{\alpha}{12}\,\frac{\Gamma\left(\frac{\alpha+3}{2}\right)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+n+2)} + \cdots .$$

Expanding out the Gamma functions for large n by Stirling's approximation, i.e.,

$$\Gamma(n+1) = \left(\frac{n}{e}\right)^n (\sqrt{2n\pi})\left[1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right],$$

we obtain the proposed expansion.

Also solved by J. Boersma, J. Koekoek, S. Parameswaran, G. B. Parish, L. A. Shepp, and the proposer.

find this book interesting and, with some judicious selection, very teachable.

There are too many misprints, but these are mostly of a minor character that a careful reader will spot easily.

RICHARD S. PIETERS
Phillips Academy, Andover, Mass.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to the Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professors Jerzy Neyman, University of California, Berkeley, J. J. Stoker, New York University, and R. L. Wilder, University of Michigan, have been elected to membership in the National Academy of Sciences.

Professor Harriet F. Montague, SUNY at Buffalo (University of Buffalo), represented the Association at the inauguration of Dr. Vincent MacD. Barnett, Jr. as President of Colgate University on April 19, 1963.

Assistant Professor W. A. Al-Salam, Duke University, has been appointed Associate Professor at Texas Technological College.

Associate Professor Frances L. C. Amemiya, California Western University, has been appointed Associate Professor at Parsons College.

Dr. J. S. Bendat, Thompson Ramo Woolridge, Los Angeles, California, announces the formation of the Measurement Analysis Corporation, a research engineering and consulting group, in Los Angeles with Dr. Bendat as President.

Dr. J. W. Carlyle, Princeton University, has been appointed Assistant Professor of Engineering at the University of California, Los Angeles.

Assistant Professor W. R. Cooper, Grambling College, has been promoted to Associate Professor.

Colonel H. T. Darracott (U.S.A. Ret.) has accepted a position as Executive Engineer with Adler Electronics, New Rochelle, New York.

Dr. G. W. Day, Purdue University, has accepted a position as Project Engineer with the Western Development Laboratories of the Philco Corporation, Palo Alto, California.

Assistant Professor R. M. Freyre, North Carolina State College, has accepted a position as Research Physicist with National Company, Melrose, Massachusetts.

Professor W. T. Graybeal, Emory and Henry College, has been appointed Director of Admissions and Institutional Research.

Visiting Associate Professor Irwin Guttman, University of Wisconsin, has been appointed Professor, effective September 1, 1963.

Professor M. C. Hartley, University of Illinois, retired October 1962 with the title of Professor Emeritus. He has been appointed Professor of Mathematics and Chairman of the Department at the University of Tampa.

Associate Professor Howard Levi, Columbia University, has been appointed Professor at Hunter College.

Mr. Joel Levy, U. S. Navy Bureau of Supplies and Accounts, has accepted a position in the Applied Mathematics Division of the National Bureau of Standards.

Dr. P. E. Lewis, Convair, San Diego, California, has accepted a position as Chief of the Scientific Data Processing department of General Dynamics/Astronautics, San Diego, California.

Mr. T. N. McCreary, Milton Hershey High School, Hershey, Pennsylvania, has accepted a position as Mathematics Specialist with the Pennsylvania Department of Public Instruction.

Dr. Brockway McMillan has been appointed Under-Secretary of the Air Force.

Dr. Morris Ostrofsky, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has been appointed Director of Mathematical Sciences at Westinghouse Electric Corporation, Friendship International Airport, Baltimore, Maryland.

Mr. D. J. Smith, General Electric Company, Philadelphia, Pennsylvania, has accepted a position as Engineering Specialist with the Martin Company, Baltimore, Maryland.

Mr. H. G. Ternow, General Analysis, Los Angeles, California, has accepted a position as Senior Systems Specialist with the International Electric Corporation, Paramus, New Jersey.

Professor A. D. Wallace, Tulane University, has been appointed Professor at the University of Florida, effective September 1, 1963.

Assistant Professor B. H. Westfall, Berea College, has accepted a position as Systems Analyst with Douglas Aircraft, Santa Monica, California.

Professor Jessie W. Boyce, Augustana College, died on December 2, 1962. She was a member of the Association for 31 years.

Associate Professor M. R. Demers, University of Nevada, died on January 10, 1963. He was a member of the Association for 8 years.

Professor Emeritus Mabel M. Young, Wellesley College, died on March 4, 1963 at the age of 90. She was a Charter Member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fortieth annual meeting of the Louisiana-Mississippi Section of the MAA was held at Biloxi, Mississippi on February 15 and 16, 1963 with the University of Southern Mississippi as the host institution.

The general session on Friday afternoon featured an invited honor address by Professor A. D. Wallace of Tulane University entitled, "An Improbable Program for Education." The section offered appropriate recognition of Professor Wallace's leadership and example.

The technical papers were presented in two concurrent sessions. Professor Virginia Felder, Mississippi Vice Chairman and Professor T. K. Maddox, Louisiana Vice Chairman presided. Professor V. B. Temple, Section Chairman, presided at the three general sessions. There were 244 persons registered including 89 members of the Association.

The following officers were elected for the coming year: Chairman, Professor Roy Sheffield, University of Mississippi; Vice Chairman for Louisiana, Professor W. E.

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THE AMERICAN MATHEMATICAL MONTHLY

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THE KAKEYA PROBLEM

A. S. BESICOVITCH, Dartmouth College

Editorial Note. In 1958 a grant from the National Science Foundation enabled the Mathematical Association of America to establish a Committee on Production of Films for the purpose of exploring by experiment the possibilities of mathematical motion pictures. In all, the committee produced four films at different educational levels, employing a variety of production techniques. For information about rental or purchase of these films address Modern Learning Aids, 3 East 54th Street, New York 22, New York.

For the last of these films, the Committee invited Professor A. S. Besicovitch to lecture on his brilliant solution to the Kakeya Problem, first published in 1928. The technique of animation employed in this film is particularly appropriate for the geometric constructions involved. The following article approximates the script of the film.

In my paper "Sur deux questions d'intégrabilité," published in a Russian periodical in 1920, I considered the problem:

Given a function of two variables, Riemann-integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that the repeated simple integration along the two directions exists and gives the value of the integral over the domain?

The problem reduces to that of the existence of a set of Jordan plane measure zero which is the union of segments of all directions each of length $\geqq 1$.

The Russian periodical hardly reached other countries because of the isolation of Russia caused by the civil war and the blockade.

In 1917 a twin problem had been proposed by the Japanese mathematician S. Kakeya:

In the class of figures in which a segment of length 1 *can be turned around through* 360°, *remaining always within the figure, which one has the smallest area?*

This problem of course did not reach Russia. I call the two problems "twin problems" because each of them is concerned with sets containing segments of all directions, with the additional condition in Kakeya's case that there should be a continuous transition, within the set, from one position of the segment to any other one.

Let us look at a few figures of the above class.

Obviously a circle of diameter 1 (Fig. 1) is a figure of the class, for if we place the mid-point of the segment in the center of the circle and rotate the segment about the center through 360°, the segment will always remain within the circle. Another obvious figure of the class is an equilateral triangle ABC of height 1 (Fig. 2). For, placing the segment on the side AC so that one end is at A, we can rotate it about A through 60° bringing it onto AB; then we let it slide along AB until the other end of the segment reaches B, then rotate it about B, and so on.

The areas of the circle and of the triangle are $\pi/4 = .78$ and $.58$ respectively.

A three-cornered hypocycloid inscribed in a circle of diameter 3/2 also belongs to the class (Fig. 3). For it is well known that the tangent line at any point M of the hypocycloid meets the hypocycloid at two other points A and B distant 1 from each other. Thus if we let one end of the segment describe the hypocycloid while keeping the segment touching the hypocycloid, we have the other end of the segment also moving on the hypocycloid and so the whole of the segment remains all the time within the hypocycloid.

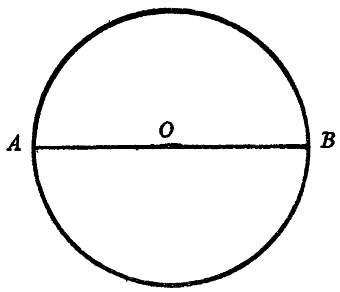

FIG. 1

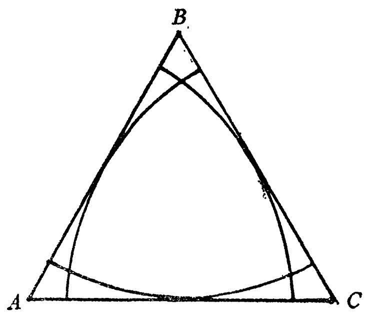

FIG. 2

The area of the hypocycloid is $\pi/8 = .39$, that is exactly half of the area of a circle of diameter 1. It was conjectured that the hypocycloid was the figure of minimum area. The problem aroused great interest. In 1925 G. D. Birkhoff, one of the greatest mathematicians of his time, writing about unsolved problems in his book *The Origin, Nature and Influence of Relativity*, first mentions the four-color problem, then adds: "Of like intriguing simplicity is the question raised a few years ago by the Japanese mathematician Kakeya."

My solution of the Kakeya problem was published in the *Mathematische Zeitschrift* in 1928 and was reported to the American Mathematical Society by J. D. Tamarkin.

The figure arrived at in solving my problem, together with the "joins" suggested by a Hungarian mathematician J. Pal, represent my solution of the Kakeya Problem.

My solution shows that the hypocycloid conjecture is false, and that in fact, there are figures of *arbitrarily small area* which permit a unit segment to change its direction by 360° while moving continuously within them.

The plan of the solution is this. We take a square of side 2 (see Fig. 4) and divide it into four congruent right triangles by joining the center to the vertices. The hypotenuse of each triangle is divided into a large number n of equal parts. Joining each point of division to the center of the square, we have $4n$ "elementary" triangles, each of height 1.

We enumerate the elementary triangles in the order in which they come as we move around the boundary of the square in the counterclockwise direction, starting at the vertex A. The directions of the various segments which

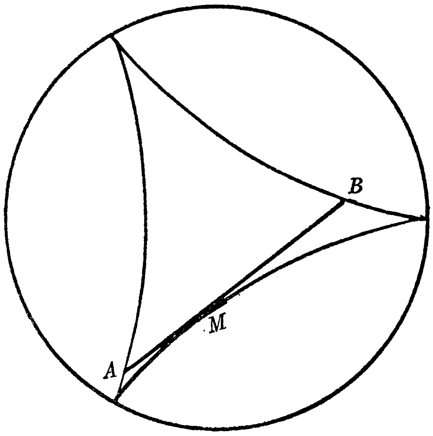

FIG. 3

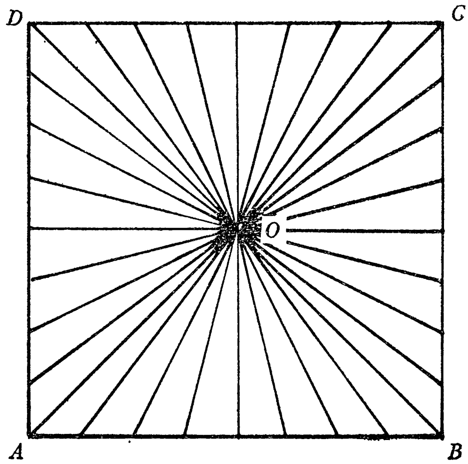

FIG. 4

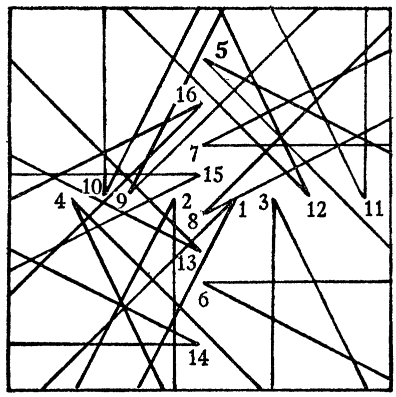

FIG. 4 (a)

join the vertex of each elementary triangle to every point of its base have a range of 360°. The same will remain true if we give arbitrary parallel translations to the elementary triangles (see Fig. 4a). As we shall show, parallel translations can be given to these elementary triangles which achieve such a degree of overlapping that the *total area covered by the triangles in their new position is as small as we please.*

Now if we place an end-point of the unit segment successively at the vertices $O_1, O_2, \cdots$ of the first triangle, the second one, and so on, in their position after translations and in each case rotate it in the positive directions from one side of the triangle to the other, the segment would turn through 360°. But this movement would not be continuous, for in moving from one triangle to the next

one the segment would not remain within the area of the figure. We eliminate this difficulty by means of Pal's joins, as follows:

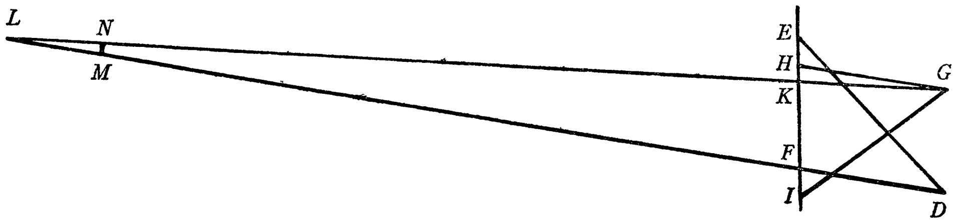

FIG. 5

Let DEF and GHI (Fig. 5) be a pair of consecutive elementary triangles after a parallel translation, and ϵ an arbitrarily small positive number. The sides DF and GH are parallel. Take a point K on HI so that $HK/HI<\epsilon/8$.

Suppose that the lines DF and GK meet in the point L and the triangle LMN is congruent to GHK. We have (denoting area by the sign $|\ \ |$)

$$|LMN| = |GHK| < \frac{\epsilon}{8}|GHI|.$$

The figure consisting of the lines GL, DL and of the triangle LMN will be called the join. We see that the area of the join is less than $\epsilon/8$ times the area of an elementary triangle. Connecting every pair of consecutive elementary triangles we shall get $4n$ *joins of total area less than* $\epsilon/8$ *times the area of the whole square, i.e.* $<\epsilon/2$. The join added to the triangles DEF, GHI permits the unit segment to come from the triangle DEF to GHI remaining always on the area of the triangles or of the join. For, from the position of the segment on the side DF we let the segment slide down along the line DL until its lower end-point reaches L, then rotate about L until it reaches the side LN and then slide up until its top end reaches G, that is, gets in the second triangle. Thus the problem is reduced *to finding parallel translations of elementary triangles such that the area covered by them be small.*

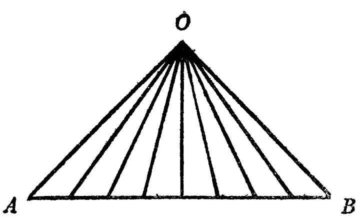

FIG. 6

Perron in 1929 published a new proof of my theorem in which the system of translations while essentially similar to mine is somewhat simpler. Recently Professor I. J. Schoenberg has carried out the Perron construction in the reverse

order, which has the advantage of being still simpler in details and easier to visualize. It is essentially this version that we shall adopt here.

We consider the coordinate plane and an integer $p \geqq 2$. We construct the isosceles right triangle $\Delta = OAB$ of our original square with its hypotenuse of length 2 on the x-axis. The base AB of Δ is divided into $n = 2^{p-2}$ equal parts and n elementary triangles with vertex O are constructed. Figure 6 shows the case $p = 5$, $n = 2^{5-2} = 8$.

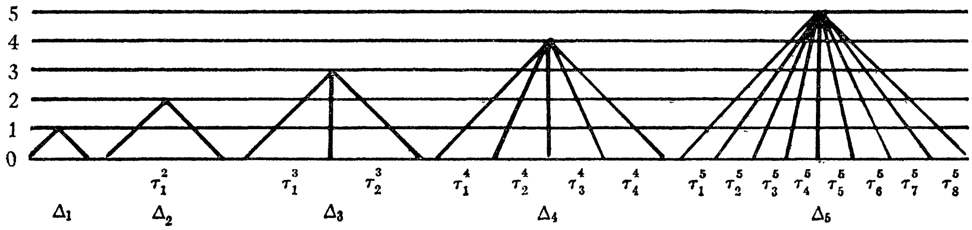

FIG. 7

We next draw the lines $y = k/p$, for $k = 1, 2, \cdots, p$ (Fig. 7) and call the line $y = k/p$ the line of level k, or simply *level k*. We then construct p isosceles right triangles $\Delta_1, \Delta_2, \cdots, \Delta_p$ each with hypotenuse on the x-axis, and the opposite vertices on the $1, 2, \cdots, p$ levels respectively. Note that $\Delta_p = \Delta$. For each k, $k = 3, \cdots, p$, the base of Δ_k is divided into 2^{k-2} equal parts and the elementary triangles are constructed on the subintervals of each base. Notice that Δ_{k+1} is divided into twice as many elementary triangles as Δ_k; Δ_2 is not divided into elementary triangles, Δ_3 is divided into two, Δ_4 into four, Δ_5 into eight, and so on (see Fig. 7).

We shall now assign labels to the elementary triangles in each Δ_k. These will be labeled from left to right as $\tau_1^k, \tau_2^k, \cdots, \tau_j^k, \cdots, \tau_{2^{k-2}}^k$. The superscript k shows that τ_j^k is part of Δ_k and the subscript j says that τ_j^k is the j-th elementary triangle in Δ_k counting from left to right. Δ_2 is not divided into elementary triangles. We shall say it coincides with the elementary triangle τ_1^2; Δ_3 has τ_1^3 and τ_2^3 as elementary triangles; Δ_4 has $\tau_1^4, \tau_2^4, \tau_3^4$, and τ_4^4; and so on.

Note a simple relationship between the elementary triangles of Δ_k and of Δ_{k+1}.

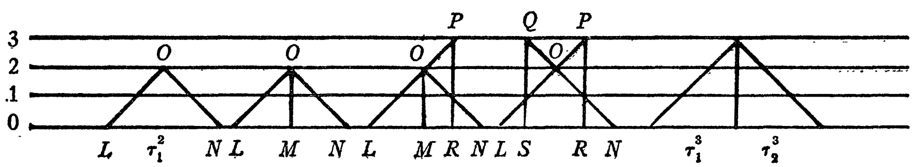

FIG. 8

Let us start with $\tau_1^2 = LON$ (Fig. 8). Bisect it by the median OM into two triangles OLM and ONM and expand them to similar triangles PLR and QNS to the level 3. We shall call this operation the bisection and expansion. The

result of this operation is a pair of triangles congruent to the pair τ_1^3 and τ_2^3 of Δ_3. Similarly is defined the operation of bisection and expansion of the triangles τ_j^k for any $k>2$: τ_j^k is bisected into two triangles by the median from its vertex, and each of the triangles is expanded to the next level. The operation transforms τ_j^k into parallel translates of τ_{2j-1}^{k+1} and τ_{2j}^{k+1} (Fig. 10), and applied to the set of all triangles τ_j^k, or to any set of their parallel translates, transforms the set into a set of parallel translates of elementary triangles of Δ_{k+1}. Figure 9 represents a particular case of $k=3$.

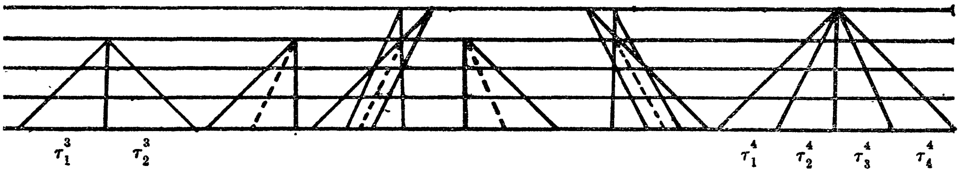

FIG. 9

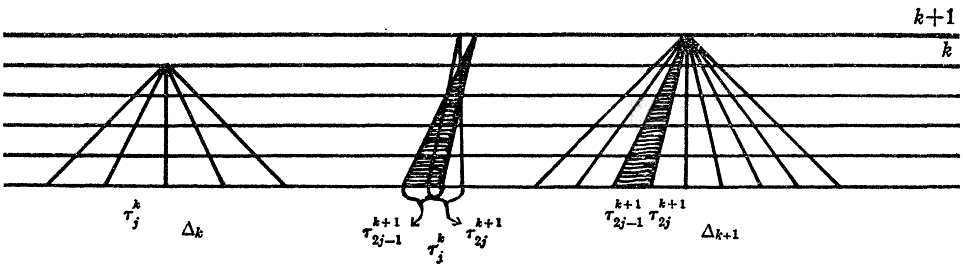

FIG. 10

The part of Δ_k (or of any elementary triangle τ_j^k of Δ_k) which lies between levels $k-1$ and k will be called the "top end" of Δ_k (or of τ_j^k). Notice that the top end of Δ_k is congruent to Δ_1 and that the sum of the areas of the top ends of all elementary triangles of Δ_k is equal to the area of the top of Δ_k, that is to $|\Delta_1|$ (see Fig. 11).

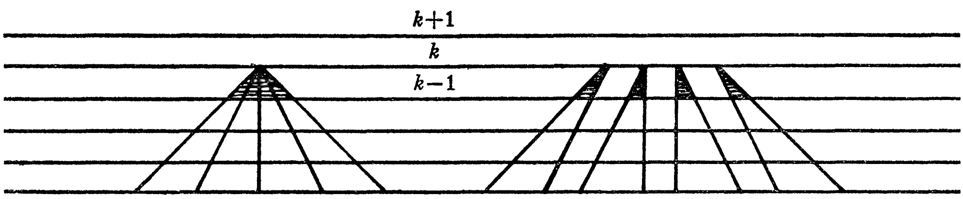

FIG. 11

Now let us look at the change in area when bisection and expansion are applied to a triangle. Consider an elementary triangle $\tau_j^k=LMN$ with vertex N at level k (see Fig. 12). Let NP be the median of LMN, bisecting it into the two subtriangles LPN and MPN. If we expand LPN upwards and to the right to the level $k+1$, we get a similar triangle LRQ. If we expand MPN upward to the left to level $k+1$, we get a similar triangle MTS.

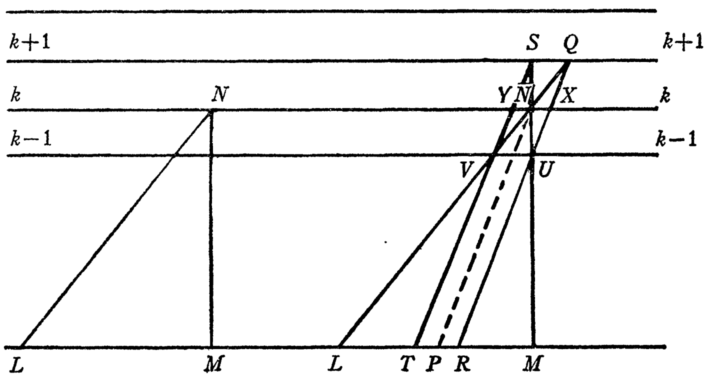

FIG. 12

The two triangles LRQ and MTS together cover the triangle LMN and the two "end pieces" SNV and QNU.

We have

$$|SNV| = |QNU| = |NUV|$$

so that the two overlapping triangles cover an area equal to the area of the original triangle LMN plus twice the area of the top end of LMN.

Observe that for constructing the end pieces to LMN, one merely has to know the top end of the triangle LMN. The rule is this: produce the sides LN and MN to the level $k+1$ and join the end-points to the points of the sides on the level $k-1$.

If we start with a complete set of elementary triangles of Δ_k or of their parallel translates and apply bisection and expansion to each of these triangles, we shall arrive at a set of parallel translates of all elementary triangles of Δ_{k+1} and introduce an increase in area at most equal to twice the sum of the areas of the top ends of these triangles. (The "at most" is necessary here because of possible overlapping.) The sum of the areas of the top ends of a complete set of elementary triangles is equal to the area of Δ_1, so that the total increase in area is at most $2|\Delta_1|$.

This conclusion leads immediately to the complete solution of our problem. We start with the triangle $\Delta_2 = \tau_1^2$ and apply to it the bisection and expansion which will transform it into parallel translates of the elementary triangles of Δ_3. Applying the same operation to each of the new triangles we shall get a set of parallel translates of elementary triangles of Δ_4, and so on.

After $p-2$ such operations we shall arrive at a set of parallel translates of elementary triangles of $\Delta_p = \Delta$. As the increment of the area at each operation is $\leqq 2|\Delta_1|$, the area of the final figure will be

$$\leqq |\Delta_2| + 2(p-2)|\Delta_1| = \frac{4}{p^2} + 2(p-2)|\Delta_1| = \frac{4}{p^2} + 2(p-2)\frac{1}{p^2} = \frac{2}{p}\cdot$$

Taking $p>16/\epsilon$ we shall get the area $<\epsilon/8$. With similar translations for the other $3n$ elementary triangles we shall get the total area covered by the translates $<\epsilon/2$. Adding $4n-1$ joins of total area $<\epsilon/2$ we shall get a figure of area $<\epsilon$ on which the unit segment can turn round through 360°, which represents a solution of the problem.

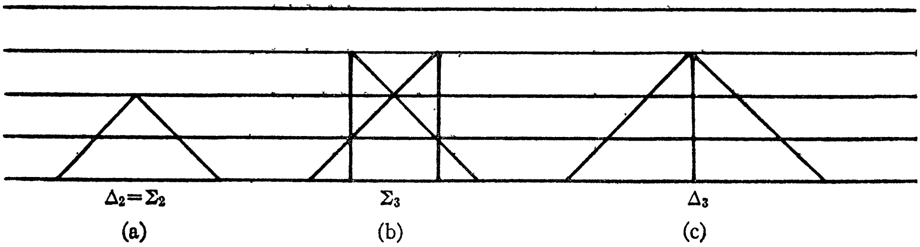

FIG. 13

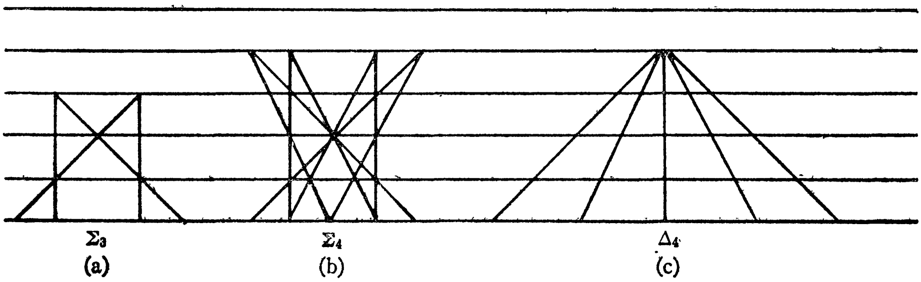

FIG. 14

In order to visualize the final picture of the set of translates of the elementary triangles of Δ we shall go through the process of its construction step by step. Figure 13 shows the beginning of the process of its construction: (a) is the triangle $\Delta_2=\tau_1^2$, we call it the set Σ_2; (b) shows the results of bisection and expansion, it is the set Σ_3 of parallel translates of elementary triangles of Δ_3; (c) represents Δ_3 divided into elementary triangles.

Figure 14 shows the second operation: (a) is Σ_3, (b) is the result of bisection and expansion of the two triangles of Σ_3, it is a set Σ_4 of translates of four elementary triangles of Δ_4, (c) is Δ_4 divided into elementary triangles.

Similarly Figure 15 shows the third operation: (a) is Σ_4, (b) is the result of bisection and expansion of each of the four triangles of Σ_4, it is a set Σ_5 of parallel translates of elementary triangles of Δ_5, (c) is the triangle Δ_5 divided into elementary triangles.

We continue in the same way arriving finally at Σ_p.

Pictures of consecutive Σ_k become more and more complicated. Observe however that for every $k\geqq 2$, Σ_{k+1} is obtained from Σ_k only by constructing the end-pieces of the triangles belonging to Σ_k. These end-pieces are constructed on the top triangles of Σ_k and lie between the levels $k-1$ and $k+1$. Thus each stage

of construction is confined only to the two top strips and there is no need for tracing fully all elementary triangles. Figure 16 represents the process.

The figure Σ_p of overlapping triangles has been called by Schoenberg the Perron tree; we shall call it the Perron-Schoenberg tree. The process of its construction can best be described as the "growth of the Perron-Schoenberg tree" (see Fig. 16).

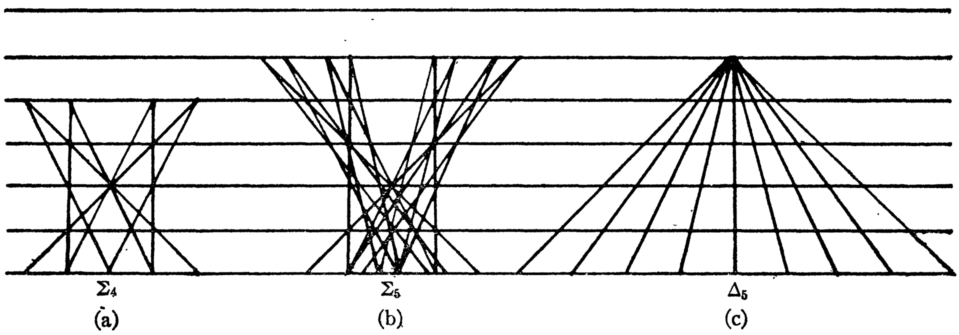

FIG. 15

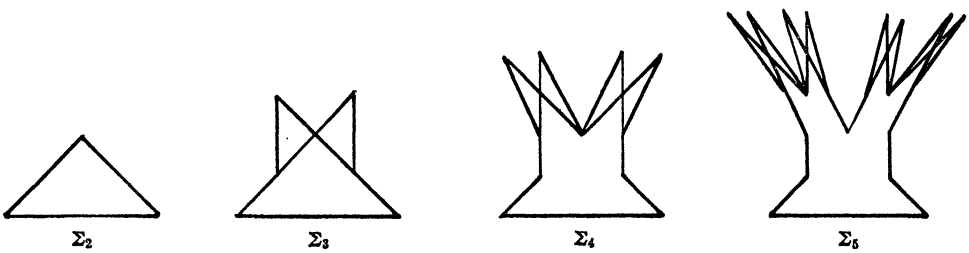

FIG. 16

REMARK. Observe that because of joins our domain is multiply connected, hence there arises a most interesting question: What will the result be if we confine ourselves only to simply connected domains?

I should like to say in conclusion a few words about the place of the Kakeya Problem in mathematics. It is one of the most intriguing problems in the class of extremal problems. On the other hand it is more important as a twin problem to the following problem of fundamental nature in the geometry of general sets of points.

Is the plane measure of a set of unit-segments of all directions in a plane, bounded from below by a positive number?

It is this twin problem that I was concerned with in the paper mentioned at the beginning of this lecture. The answer arrived at was negative.

In fact, I constructed such a set having plane measure zero. At that time one could not suspect that this result was only a first penetration into a much

greater problem. My later study of the geometry of sets of points has led me to the solution of the general problem on the plane measure of line-sets.

The study of point-sets has been carried out on the basis of measure. The most interesting class appears to be the class of sets of finite Hausdorff linear measure. A general set of that kind is the sum of a *regular set* and of an *irregular* one, a set being *regular or irregular according as at almost all of its points the density exists or does not exist.* The two classes are fundamentally different. The measure of a set of lines is defined by the measure of the set of the poles with respect to a fixed circle, and a line-set is regular or irregular according as its set of poles is regular or irregular. The problem on the plane measure of a line-set considered as a point-set is solved by the following result.

THEOREM. *The plane measure of any irregular set is* 0, *and of any regular one is* ∞.

The construction of irregular sets containing lines of all directions presents no difficulty.

The occasion of making a film may stimulate others to work on the Kakeya problem. I want only to express a hope that this new work will be crowned by success.

A REMARK ON THE KAKEYA PROBLEM

A. A. BLANK, New York University

Besicovitch [1] poses the question, what is the solution of the Kakeya problem if we restrict ourselves to simply connected domains. It is known, for example, that it is possible to turn a unit segment through 360° within the three-cornered hypocycloid of area $\pi/8$. Insofar as I am aware, no simply connected domain of lesser area is known to satisfy the Kakeya criterion. Here we demonstrate the existence of a class of domains, regular star polygons, which satisfy the Kakeya criterion and for which there exist areas exceeding $\pi/8$ by arbitrarily little. Is $\pi/8$ then the minimum area for simply connected domains, or at least, for star-like domains? If so, we have here a curious instance of a domain functional which is minimized in two entirely different ways.

We construct a sequence of star polygons in the following manner. First, a regular $(2n+1)$-gon is inscribed in a circle of radius $r<1/2$. Each side of the $(2n+1)$-gon is then used as the base of an exterior isosceles triangle with vertex at unit distance from the diametrically opposite vertex of the $(2n+1)$-gon. The domain of interest is the union of the inscribed polygon with the $2n+1$ triangles attached to the sides. In Fig. 1 we depict the case $n=3$.

Let us denote the center of the circle by O and label the vertices of the isosceles triangles by $V_0, V_1, \cdots, V_{2n}, V_{2n+1}, \cdots$, in counter-clockwise order, where if $i\equiv j \pmod{2n+1}$ then $V_i=V_j$. The diametrically opposite vertex of the

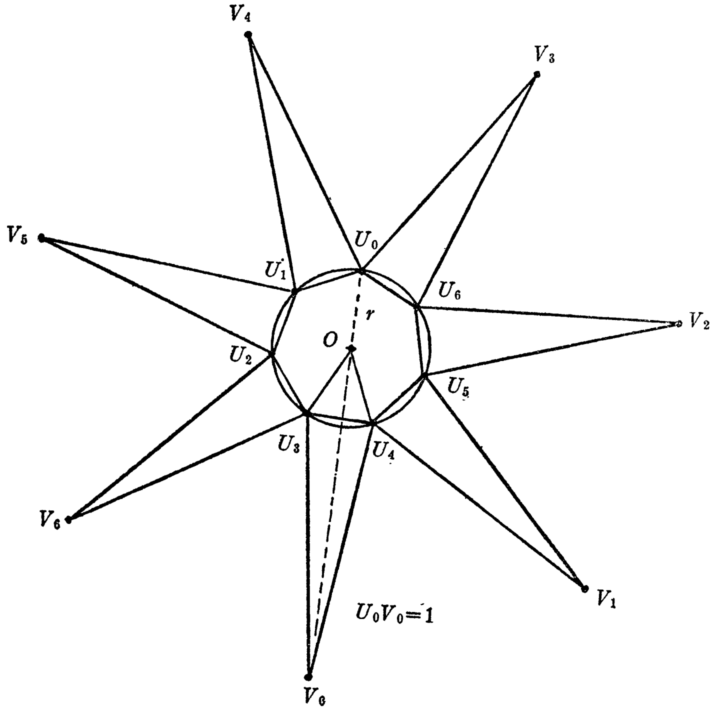

FIG. 1

inscribed polygon to V_i will be designated by U_i, $(i=0, 1, 2, \cdots,)$. The area A_n of the domain is easily calculated as the sum of the $2n+1$ congruent sectors each consisting of the union of the exterior isosceles triangle based on a given side of the inscribed polygon and the interior isosceles triangle with vertex at O based on the same side. For the common base of the triangles we have the length $2r \sin(\pi/(2n+1))$, and for the sum of the altitudes $(1-r)$. It follows that

$$A_n = (1-r)r(2n+1)\sin\frac{\pi}{2n+1}, \tag{1}$$

whence,

$$\lim_{n\to\infty} A_n = \pi(r-r^2). \tag{2}$$

We shall require that it be possible within the domain to move the unit directed segment U_0V_0 into U_1V_1 continuously with a small angle rotation. To do this it will be sufficient to move U_0V_0 into $V_{n+1}U_{n+1}$. (In Fig. 1, we move U_0V_0 into V_4U_4.) By symmetry, then, the following motions are possible,

$$U_kV_k \to V_{k+n+1}U_{k+n+1}, \qquad V_kU_k \to U_{k+n+1}V_{k+n+1}.$$

Applying these transformations successively, we get

$$U_kV_k \to V_{k+n+1}U_{k+n+1} \to U_{k+2n+2}V_{k+2n+2} = U_{k+1}V_{k+1}.$$

In this way we may step from one vertex to the next and so accomplish a complete rotation of the unit segment within the domain.

We shall now find the requisite condition that it be possible to move from U_0V_0 into $V_{n+1}U_{n+1}$.

First we extend the sides of the isosceles triangles with vertices V_0 and V_{n+1} to intersections on their line of bilateral symmetry. Let P denote the intersection of V_0U_n with $V_{n+1}U_1$ and Q that of V_0U_{n+1} with $V_{n+1}U_0$. In Fig. 2 the half of the symmetric figure on the side of U_0 is indicated schematically. We shall ensure that the motion $U_0V_0 \to V_{n+1}U_{n+1}$ is possible within the quadrilateral $V_0QV_{n+1}P$. For large enough n the quadrilateral is within the star polygon and the desired result is obtained.

The condition that the quadrilateral lie inside the star polygon is easily found. Referring to Fig. 2, we see that the result follows from the convexity of the inscribed polygon if we assume that P and Q lie within the inscribed polygon. For convenience, set $\alpha = \pi/(2n+1)$, $\phi = \measuredangle PV_0O = \measuredangle QV_0O$, $\beta = \measuredangle V_0QP$, $\gamma = \measuredangle QPV_0$. The angles subtended at O by the cited points are indicated in Fig. 2. Examining $\triangle OPV_0$ and $\triangle OQV_0$ we obtain

$$\beta = \pi - (n+1)\alpha - \phi, \qquad \gamma = \pi - n\alpha - \phi.$$

Observing that $n\alpha = \pi/2 - \alpha/2$, we have

$$(3) \qquad \begin{cases} \beta = \dfrac{\pi}{2} - \dfrac{\alpha}{2} - \phi \\[2ex] \gamma = \dfrac{\pi}{2} + \dfrac{\alpha}{2} - \phi. \end{cases}$$

Since $OV_0 = 1 - r$ we have on applying the law of sines to these triangles that

$$(4) \qquad |OP| = \frac{(1-r)\sin\phi}{\cos(\frac{1}{2}\alpha - \phi)}, \qquad |OQ| = \frac{(1-r)\sin\phi}{\cos(\frac{1}{2}\alpha + \phi)}.$$

We need only guarantee that these lengths are no greater than the radius of the inscribed circle to the inscribed polygon; i.e.,

$$(5) \qquad |OP|, \; |OQ| \leqq r\cos\alpha.$$

Clearly, for n large, α and ϕ are small and these conditions can be satisfied. We shall verify, in particular, that

$$(6) \qquad \alpha > 2\phi,$$

hence, that $\gamma > \pi/2$ and the quadrilateral is re-entrant at P. Since $OV_0 = 1 - r$,

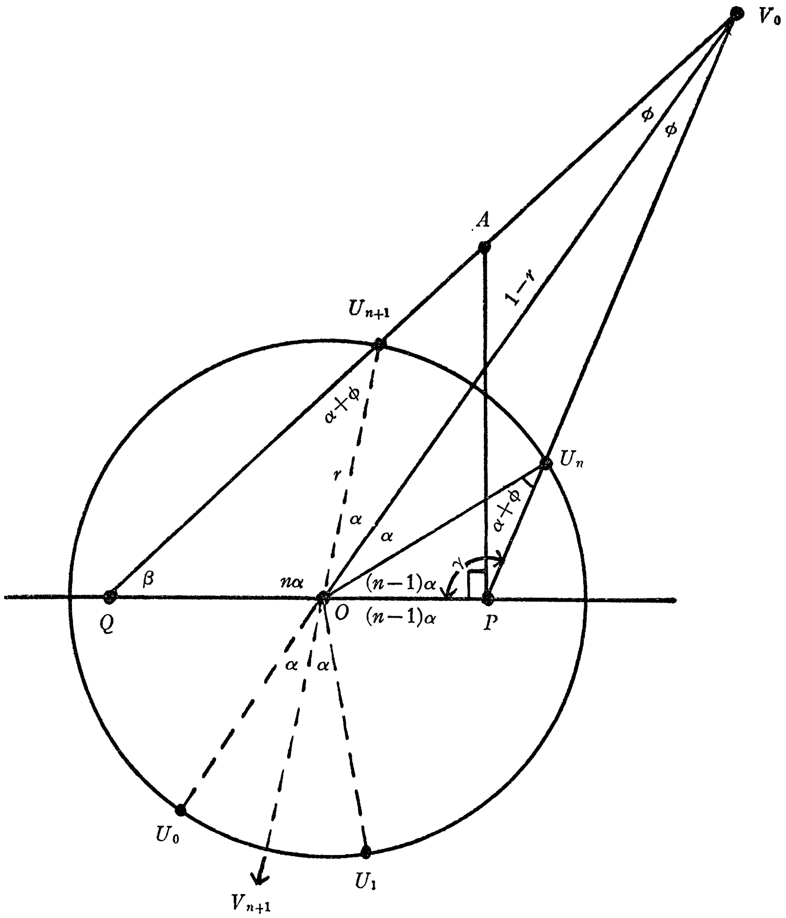

FIG. 2. Schematic diagram. (To be correct U_0, V_{n+1}, U_1 would have to be located symmetrically to U_{n+1}, V_0, U_n with respect to the axis QOP.)

the altitude from V_0 to U_nU_{n+1} has length $1-r-r\cos\alpha$. Consequently

$$\tan\phi = \frac{r\sin\alpha}{1-r(1+\cos\alpha)}\cdot \tag{7}$$

Using $\tan\phi>\phi$, $\sin\alpha<\alpha$, and $\cos\alpha<1$ in (7), we obtain

(8a) $\phi < \dfrac{r\alpha}{1-2r} < 2\alpha,$ provided that (8b) $r < 2/5.$

Under condition (8b) we shall satisfy (8a).

In order to be able to accomplish the desired motion within the quadrilateral it will be sufficient to require that the length of the perpendicular to QP at P intercepted by $\sphericalangle V_0QV_{n+1}$ is at least one. Denoting the intersection of this perpendicular with V_0Q by A (see Fig. 2), we require

$$|AP| \geqq \tfrac{1}{2}. \tag{9}$$

When this condition is satisfied, the motion can be executed by rotating the unit vector XY about V_0, starting from $X=U_0$, $Y=V_0$ until X lies on the line V_0P. We then let Y slide along V_0Q keeping P on the line of XY until Y reaches A. If in this motion the segment intercepted on the line of XY by $\sphericalangle V_0QV_{n+1}$ never becomes less than one, the motion of the unit segment XY remains entirely within the quadrilateral. We now prove under condition (9) that this is the case.

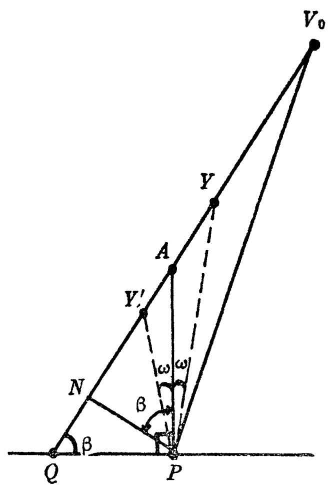

FIG. 3

In Fig. 3, PY' indicates the reflection in the axis QP of that part of the extension of YP below the axis intercepted by $\sphericalangle PQV_{n+1}$. We must show that $|PY|+|PY'|\geqq 1$. To this end let N denote the foot of the perpendicular from P on QV_0. Let $\omega=\sphericalangle YPA=\sphericalangle Y'PA$. From $\beta=\sphericalangle APN$, we have

$$|PY'| = |PN| \sec(\beta-\omega), \qquad |PY| = |PN| \sec(\beta+\omega),$$

and, from (7),

$$|PY'| = |PN| \sec\beta \geqq \tfrac{1}{2}.$$

Using the convexity of the secant we obtain

$$\tfrac{1}{2}[\sec(\beta+\omega)+\sec(\beta-\omega)] \geqq \sec\beta$$

and, therefore,

$$\tfrac{1}{2}\{ |PY'| + |PY| \} \geqq |PA| \geqq \tfrac{1}{2},$$

which proves the result.

To complete the proof we observe that if we are to satisfy (2) and $\lim_{n\to\infty} A_n = \pi/8$ then we must have $r(1-r)=1/8$. The only admissible solution of this equation, i.e., the solution satisfying (8b), is

$$r = \tfrac{1}{4}(2 - \sqrt{2}). \tag{10}$$

In order to verify that this is, in fact, attainable in the limit, we show for every $\epsilon > 0$ such that the radius $r' = r + \epsilon$ satisfies the criterion of (8b), $r' < 2/5$, that (9) follows for sufficiently large n. Referring to Fig. 2 we obtain from (3) and (4)

$$\begin{aligned} |AP| &= |QP| \tan\beta = (|OP| + |OQ|)\tan\beta \\ &= (1 - r')\sin\phi \left[\frac{1}{\cos(\frac{1}{2}\alpha - \phi)} + \frac{1}{\cos(\frac{1}{2}\alpha + \phi)}\right]\frac{1}{\tan(\frac{1}{2}\alpha + \phi)} \\ &= \frac{(1 - r')\sin\phi}{\sin(\frac{1}{2}\alpha + \phi)}\left[\frac{2\cos\frac{1}{2}\alpha\cos\phi}{\cos(\frac{1}{2}\alpha - \phi)}\right]. \end{aligned}$$

From (7) we obtain $(1-r')\sin\phi = r'\sin(\alpha+\phi)$, and using $\sin(\alpha+\phi)+\sin\phi = 2\sin(\frac{1}{2}\alpha+\phi)\cos\frac{1}{2}\alpha$, we find

$$\frac{\sin\phi}{\sin(\frac{1}{2}\alpha + \phi)} = 2r'\cos\frac{1}{2}\alpha\,\cdot$$

Entering this result in the equation for $|AP|$ we obtain

$$|AP| = 4(1 - r')r'\,\frac{\cos^2\frac{1}{2}\alpha\cos\phi}{\cos(\frac{1}{2}\alpha - \phi)}\,\cdot$$

From $r' = r + \epsilon$, we have $r'(1-r') = \frac{1}{8} + \epsilon(1-2r)$, where r is defined by (10). It follows that

$$|AP| = [\tfrac{1}{2} + 2\epsilon\sqrt{2}]\,\frac{\cos^2\frac{1}{2}\alpha\cos\phi}{\cos(\frac{1}{2}\alpha - \phi)}\,\cdot$$

Since (8b), and, hence, (8a) is satisfied, the cosine factors can be made arbitrarily close to one by taking n large enough, and, consequently, (9) is satisfied.

Reference

1. A. S. Besicovitch, The Kakeya Problem, this MONTHLY, the preceding article.

THE 1962 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twenty-third William Lowell Putnam Mathematical Competition held on December 1, 1962, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of California Institute of Technology, Pasadena, California. The members of the team were E. A. Bender, Kenneth Kunen and J. H. Lindsey; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of Dartmouth College, Hanover, New Hampshire. The members of the team were Sidney Marshall, R. W. Robinson and R. S. Strichartz; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Lawrence Corwin, J. N. Mather and W. C. Waterhouse; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Queen's University, Kingston, Ontario. The members of the team were W. A. E. Butler, D. A. E. Gregory and Michael Hindley-Smith; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of California, Los Angeles, California. The members of the team were Charles Goldberg, John Halpern and James Ulrich; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are E. A. Bender, California Institute of Technology; J. H. Lindsey, California Institute of Technology; R. S. Strichartz, Dartmouth College; W. C. Waterhouse, Harvard University; and John Wood, Harvard University. To each of these a prize of seventy-five dollars is awarded. The William Lowell Putnam Prize Scholarship to Harvard has been awarded to Mr. Waterhouse. The value of this scholarship has recently been increased to $2500.00 plus tuition ($1520.00), making a total monetary value of $4020.00.

The five persons ranking second highest in the examination, named in alphabetical order, are G. M. Bergman, University of California at Berkeley; R. C. Hill, California Institute of Technology; John Hirschfelder, University of Notre Dame; David Sachs, Washington Square College of New York University; and K. B. Stolarsky, California Institute of Technology. To each of these a prize of thirty-five dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being Joel Spencer, Alan Weinstein and John Wells; Michigan State University, East Lansing, Michigan, the members of the team being Robert Bartholomew, Stephen Crick, Jr., and R. E. Greene; Washington Square College of New York University, New York, N. Y., the members of the team being Leona Bassein, H. J. Bernstein and David Sachs; University of California, Berkeley, California, the members of the team being G. M. Bergman, Frank Henyey and R. J. Oberg; University of Manitoba, Winnipeg, Manitoba, the members of the team being Keith Powls, Larry Rice and Leslie Roberts; and the University of Toronto, Toronto, Ontario, the members of the team being R. J. Birgenau, J. M. Chambers and Thomas Wolter.

Honorable mention is given to the following twenty-one individuals, named in alphabetical order: Jeff Cheeger, Harvard University; P. R. Chernoff, Harvard University; R. K. Cole, Jr., Cornell University; G. E. Cooke, Dartmouth College; Steve Crocker, University of California, Los Angeles; James Dowling, Harvard University; John Halpern, University of California, Los Angeles; Frank Henyey, University of California, Berkeley; Michael Hindley-Smith, Queen's University; A. C. Hindmarsh, California Institute of Technology; Melvin Hochster, Harvard University; Kenneth Kunen, California Institute of Technology; Robert Latzer, Washington University; J. E. Marsden, University of Toronto; Sidney Marshall, Dartmouth College; J. N. Mather, Harvard University; Thomas Phinney, University of Notre Dame; R. W. Robinson, Dartmouth College; C. A. Ryavec, California Institute of Technology; George Shapiro, Harvard University; and Michael Weinless, Massachusetts Institute of Technology.

A total of one thousand five hundred eighty-five contestants from one hundred ninety-two colleges and universities entered the Competition. One thousand one hundred eighty-seven contestants from one hundred eighty-seven colleges and universities (one hundred fifty-seven having teams) participated in the examination on December 1, 1962.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

Those participating in the Competition were given the following problems to solve:

Part I

1. Given five points in a plane, no three of which lie on a straight line, show that some four of these points form the vertices of a convex quadrilateral.
2. Find every real-valued function f whose domain is an interval I (finite or infinite) having 0 as a left-hand end point, such that for every positive member x of I the average of f over the closed interval $[0, x]$ is equal to the geometric mean of the numbers $f(0)$ and $f(x)$.
3. In a triangle ABC in the Euclidean plane, let A' be a point on the segment from B to C, B' a point on the segment from C to A and C' a point on the segment from A to B such that

$$\frac{AB'}{B'C} = \frac{BC'}{C'A} = \frac{CA'}{A'B} = k,$$

where k is a positive constant. Let $\triangle$ be the triangle formed by parts of the segments obtained by joining A and A', B and B', and C and C'. Prove that the areas of the triangles $\triangle$ and ABC are in the ratio.

$$\frac{(k-1)^2}{k^2+k+1}.$$

4. Assume that $|f(x)| \leqq 1$ and $|f''(x)| \leqq 1$ for all x on an interval of length at least 2. Show that $|f'(x)| \leqq 2$ on the interval.
5. Evaluate in closed form

$$\sum_{k=1}^{n} \binom{n}{k} k^2.$$

Note:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}.$$

6. Let S be a set of rational numbers such that whenever a and b are members of S, so are $a+b$ and ab, and having the property that for every rational number r exactly one of the following three statements is true:

$$r \in S, \qquad -r \in S, \qquad r = 0.$$

Prove that S is the set of all positive rational numbers.

Part II

1. Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer and let $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

Note:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}.$$

2. Let R be the set of all real numbers and S the set of all subsets of the positive integers. Construct a function f whose domain is R and whose range is in S, such that $f(a)$ is a proper subset of $f(b)$ whenever $a<b$.
3. Let S be a convex region in the Euclidean plane containing the origin. Assume that every ray (that is, half-line) from the origin has at least one point outside S. Prove that S is bounded. (A region in the plane is defined to be convex if and only if the line segment joining every pair of its points lies entirely within the region.)
4. The Euclidean plane is divided into regions by drawing a finite number of circles. Show that it is possible to color each of these regions either red or blue in such a way that no two adjacent regions have the same color. (Two such regions are said to be adjacent if and only if their boundaries have an arc of a circle in common.)
5. Prove that for every integer n greater than 1:

$$\frac{3n+1}{2n+2} < \left(\frac{1}{n}\right)^n + \left(\frac{2}{n}\right)^n + \cdots + \left(\frac{n}{n}\right)^n < 2.$$

6. Let

$$f(x) = \sum_{k=0}^{n} (a_k \sin kx + b_k \cos kx),$$

where a_k and b_k are constants. Show that if $|f(x)| \leqq 1$ for $0 \leqq x \leqq 2\pi$ and $|f(x_i)| = 1$ for $0 \leqq x_1 < x_2 < \cdots < x_{2n} \leqq 2\pi$, then $f(x) = \cos(nx+a)$ for some constant a.

Solutions. Part I

1. Choose points A and B so that the other points lie in a half-plane. Then a third point, say C, may be chosen so that the remaining points, D and E, are

interior to the angle ABC. If at least one of them, say D, is exterior to ΔABC, $ABCD$ is convex. If both D and E are interior to ΔABC ,the line DE meets two sides of ΔABC. Then D, E and the ends of the third side of ΔABC form a convex quadrilateral.

2. $\int_0^x f(t)dt = x\sqrt{f(0)f(x)}$

(a) If $f(0)=0$, $f(x)\equiv 0$ is a trivial solution.

(b) If $f(0)\neq 0$, let $g(x)=\sqrt{[f(x)/f(0)]}$. Then

$$\int_0^x g^2(t)dt = xg(x).$$

Hence $g^2(x)=xg'(x)+g(x)$ whose solutions are

$$g(x) = \frac{1}{1-cx}, \qquad 0 \leqq x < \frac{1}{c}, \qquad c > 0$$

$$g(x) = \frac{1}{1+cx}, \qquad 0 \leqq x < \infty, \qquad c \geqq 0$$

including in the latter case, $g(x)\equiv 1$. Thus

$$f(x) = \frac{f(0)}{(1 \pm cx)^2} \quad \text{or} \quad f(x) \equiv f(0)$$

(which includes case (a)).

3. Let AA' and CC' meet at P. By vector algebra, it is easy to show that

$$AP = \frac{k+1}{k^2+k+1} AA'.$$

If T is the area of triangle ABC, the area of triangle ACA' is $kT/(k+1)$ and the area of triangle APC is

$$\frac{k+1}{k^2+k+1} \frac{k}{k+1} T = \frac{k}{k^2+k+1} T \cdot$$

Removing two other triangles of equal area we obtain

$$\Delta = \left(1 - \frac{3k}{k^2+k+1}\right) T = \frac{(k-1)^2 T}{k^2+k+1} \cdot$$

4. We can assume without loss of generality that the interval is $0 \leqq x \leqq 2$. Expanding $f(x)$ by Taylor's formula with the remainder around the points

0 and 2, and subtracting, we get $f(2)-f(0)=2f'(x)-\frac{1}{2}x^2f''(t_1)+\frac{1}{2}(2-x)^2f''(t_2)$, where t_1 and t_2 are points in the interval. This gives

$$\begin{aligned} 2|f'(x)| &\leqq |f(2)| + |f(0)| + \tfrac{1}{2}x^2|f''(t_1)| + \tfrac{1}{2}(2-x)^2|f''(t_2)| \\ &\leqq 2 + \tfrac{1}{2}[x^2 + (2-x)^2]. \end{aligned}$$

The maximum of the right side is attained for $x=0$, and is 4. Dividing by 2 we get the result.

5.
$$(1+x)^n = 1 + \sum_{k=1}^{n} \binom{n}{k} x^k$$

$$n(1+x)^{n-1} = \sum_{1}^{n} \binom{n}{k} k x^{k-1}$$

$$n(n-1)(1+x)^{n-2} = \sum_{1}^{n} \binom{n}{k} k(k-1)x^{k-2}.$$

For $x=1$, $n2^{n-1}=\sum_1^n \binom{n}{k}k$, and $n(n-1)2^{n-2}=\sum_1^n \binom{n}{k}(k^2-k)$.
By addition, $n(n+1)2^{n-2}=\sum_1^n \binom{n}{k}k^2$.

6. If n is a positive integer and $-1/n\in S$, then $(1/n)^2=1/n^2\in S$, and thus $\sum 1/n^2=1/n\in S$. (Contradiction.) Therefore, for every positive integer n, $-1/n\notin S$ and hence $1/n\in S$. Therefore, $\sum 1/n=m/n\in S$ for every positive rational number m/n. If r is a negative rational number, then $-r\in S$ and hence $r\notin S$. Finally, $0\notin S$ since if $0\in S$, $-0\in S$ also. Therefore S is the set of *all* positive rational numbers.

Solutions. Part II

1. By the binomial theorem, we have

$$\sum_{n=0}^{\infty} \frac{x^{(n)}}{n!} t^n = \sum_{n=0}^{\infty} \binom{x}{n} t^n = (1+t)^x.$$

The identity then follows at once from the identity $(1+t)^{x+y}=(1+t)^x(1+t)^y$, by multiplying the power series expansions. It is not necessary to check convergence of the series, because we are working with formal power series.

2. Let F be a one-to-one correspondence between the set of all rational numbers and the set of all positive integers, and for any nonempty set A of rational numbers let $F(A)$ be the corresponding (nonempty) set of positive integers. Let

$$f(a) = F(\{r \mid r < a,\ r \text{ rational}\}).$$

3. The word "region" is interpreted to mean an open connected set. Assume that S is not bounded, and for every integer n, let R_n be a ray through the origin O containing a point of S at a distance from O greater than n. Let R be a limit ray of the infinite set $\{R_n\}$, and let P be a point of R outside S. Let D be an open disk about O lying in S, let A and B be points of D on opposite sides of R,

and draw the lines AP and BP. By definition of R, the region formed by the rays from P that are the extensions of AP and BP (away from A and B) contains a point Q of S. For definiteness assume that A and Q are on the same side of R, and let the lines AQ and BP intersect in a point C. Then $C \in S$ and consequently $P \in S$. (Contradiction.)

4. Let $R, S \cdots$ be a list of the regions into which the plane is divided by the circles; each such region is clearly convex. For every such region R, let $n(R)$ denote the number of distinct circles which contain R in their interior. Now if two regions, R and S, are adjacent, the common part of their boundary will be a single arc of a circle, otherwise not both the regions could be convex. This is tantamount to saying that one of the regions is interior to exactly one more circle than the other, that is, say $n(R)=n(S)+1$. Then, the numbers assigned to two adjacent regions always have opposite parity. But then, color in red all even-numbered regions, and in blue all odd-numbered ones, q.e.d.

5. Consider the function $f(x)=x^n$ on $[0, 1]$, which is increasing and concave up. Hence by rectangles and trapezoids,

$$\left[f(0)+f\left(\frac{1}{n}\right)+\cdots+f\left(\frac{n-1}{n}\right)\right]\frac{1}{n} < \int_0^1 x^n dx < \left[f\left(\frac{1}{n}\right)+\cdots+f\left(\frac{n-1}{n}\right)+\frac{1}{2}f\left(\frac{n}{n}\right)\right]\frac{1}{n}$$

and hence:

$$\left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\cdots+\left(\frac{n-1}{n}\right)^n < \frac{n}{n+1} < \left(\frac{1}{n}\right)^n+\cdots+\left(\frac{n-1}{n}\right)^n+\frac{1}{2},$$

whence

$$\frac{n}{n+1}+\frac{1}{2}=\frac{3n+1}{2n+2} < \text{sum} < \frac{n}{n+1}+1=\frac{2n+1}{n+1} < 2.$$

6. Consider the trigonometric polynomials $f'(x)^2$ and $1-f(x)^2$. These are both of degree $2n$; furthermore, they both have *double* zeros at the points $x_0 < x_2 < \cdots < x_n$. Hence they must be constant multiples of each other. Hence $f'(x)^2=K[1-f^2(x)]$ for some constant $K \geqq 0$. Since at points other than the x_k we do not have $|f(x)|=1$, we choose $K>0$. Solving the differential equation we get the stated solution. The restriction on the degree of the polynomial gives $K=n$.

Caution. The statement of the problem implicitly assumes that $x_0 < \cdots < x_n$ is a complete list of the points where $|f(x)|=1$. If this assumption is dropped, then $K=0$ gives the constant $f(x)\equiv \pm 1$ as another possible solution.

QUASI-CONVERGENT SERIES OF INDEPENDENT RANDOM VARIABLES

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1. Introduction and summary. Since the Lévy concentration function of a distribution function is invariant under change of location parameter, it is reasonable to expect that quasi-convergence of a series of independent random variables should be characterizable in terms of the concentration function. This Paul Lévy did in terms of the concentration functions of the tails of such a series (see [2], pages 133–134). Another characterization was obtained by K. Ito ([1], page 46, Theorems 1 and 2) in terms of the concentration functions of the sequence of partial sums. Ito obtained his theorem, however, using characteristic functions. In probability theory, one obtains results by the use of characteristic functions if one has to, but it seems to be preferable to obtain results in a more direct manner. The purpose of this note is to show how Ito's theorem can be obtained directly from properties of concentration functions, and this is accomplished in Section 2. In Section 3, some well-known results are obtained as corollaries to Ito's theorem.

2. A characterization of quasi-convergence. The purpose of this section is to prove that a series of independent random variables is quasi-convergent if and only if the sequence of concentration functions of the sequence of partial sums converges everywhere to a function which is not identically zero.

DEFINITION. *Let F be a distribution function. For every $L \geqq 0$, the concentration function Q of F is defined by* $Q(L) = \sup_x\{F(x+L+0) - F(x-0)\}$.

The following lemma will be useful.

LEMMA 1. *If Q is the concentration function of a distribution function F, then for every $L \geqq 0$ there exists a real number x_L such that*

$$Q(L) = F(x_L + L + 0) - F(x_L - 0).$$

Proof. If $L=0$, the lemma is proved immediately by taking x_L as any number if F is continuous and, in case F is not continuous, by taking x_L as a point at which a largest discontinuity of F occurs. So we now assume $L>0$. In this case, $Q(L)>0$. Let us write

$$\mathfrak{X} = \{x \mid F(x + L + 0) - F(x - 0) > \tfrac{1}{2}Q(L)\}.$$

Clearly, $\mathfrak{X}$ is nonempty, and $\mathfrak{X} \subset [K-L, M]$, where $K<M$ are real numbers which satisfy $F(M)>1-\frac{1}{2}Q(L)$, $F(K)<\frac{1}{2}Q(L)$. By the Bolzano-Weierstrass theorem there exists a sequence of not necessarily distinct elements, $\{x_n\}$, of $\mathfrak{X}$ such that, for some x_0, $x_n \to x_0$ as $n \to \infty$ and $F(x_n+L+0)-F(x_n-0) \to Q(L)$. For $\epsilon>0$ there exists an N_ϵ such that for all $n>N$,

$$F(x_n + L + 0) < F(x_0 + L + 0) + \epsilon/2$$

and

$$F(x_n - 0) > F(x_0 - 0) - \epsilon/2.$$

Hence $F(x_0+L+0)-F(x_0-0)+\epsilon>Q(L)$, and from the arbitrariness of $\epsilon>0$ and the definition of Q we obtain $x_0=x_L$. Q.E.D.

LEMMA 2. *If X and Y are independent random variables, if Q_1 is the concentration function of $X+Y$, and if Q_2 is the concentration function of X, then $Q_1(L) \leqq Q_2(L)$ for all $L\geqq 0$.*

LEMMA 3. *If Q is the concentration function of a distribution function F, then F is continuous if and only if $Q(0)=0$.*

These lemmas are well known; see P. Lévy ([2], pages 90 and 92).

Lemma 4. *Let $\{X_n\}$ be a sequence of independent random variables such that $\sum_{n=1}^{\infty} X_n$ converges almost surely. Let Q_n be the concentration function of $X_1+ \cdots +X_n$. Then for any $L\geqq 0$, $Q_n(L)\to Q_0(L)$ as $n\to\infty$, where Q_0 is the concentration function of $\sum_{n=1}^{\infty} X_n$.*

Proof. We first prove the lemma in the case in which $L=0$ and $Q_0(0)=0$. If we denote the distribution function of $\sum_{n=1}^{\infty} X_n$ by F_0, then $Q_0(0)=0$ implies (by Lemma 3) that F_0 is continuous. Hence it is uniformly continuous, and so, for arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x' - x''| < \delta$, then $|F_0(x')-F_0(x'')| <\epsilon/3$. Since $F_n\to F_0$ uniformly, there exists an N_ϵ such that for $n>N_\epsilon$, $|F_0(x)-F_n(x)| <\epsilon/3$ for all x. Hence, for all $n>N_\epsilon$, $0<L<\delta$,

$$\sup_x \{F_n(x + L + 0) - F_n(x - 0)\} \leqq \sup_x \{ | F_n(x + L + 0) - F_0(x + L) | + | F_0(x + L) - F_0(x) | + | F_0(x) - F_n(x - 0) | \} \leqq \epsilon,$$

or $Q_n(0)\to 0$ as $n\to\infty$. We now consider the remaining case: $L\geqq 0$ and $Q_0(L)>0$. (Note: if $L>0$, then automatically $Q_0(L)>0$.) By Lemma 2 above, $Q_n(L) \geqq Q_0(L)$ for all n and all L. Also, by Lemma 2, $Q_n(L)\geqq Q_{n+1}(L)$ for all n and L, and therefore $\lim_{n\to\infty} Q_n(L)\geqq Q_0(L)$. For each n we know by Lemma 1 that there exists an x_n such that $Q_n(L)=F_n(x_n+L+0)-F_n(x_n-0)$, where F_n is the distribution function of $X_1+ \cdots +X_n$. It is easy to show that the sequence $\{x_n\}$ is bounded. Indeed, let $K<M$ be such that $F_0(K)<\frac{1}{2}Q_0(L)$, $F_0(M)>1-\frac{1}{2}Q_0(L)$. Second, there exists an N such that, for all $n>N$, we have $|F_0(K)-F_n(K)| <\frac{1}{4}Q_0(L)$, and $|F_0(M)-F_n(M)| <\frac{1}{4}Q_0(L)$. Let K, M be continuity points of F_0, and define $\alpha=\min\{K-L, x_1, \cdots, x_n\}$ and $\beta=\max\{M, x_1, \cdots, x_N\}$. Then it is easy to see that $\alpha\leqq x_n\leqq\beta$ for all n. Hence there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $x_{n'}\to$(some) x_0 as $n'\to\infty$. If $\{Q_{n'}\}$ is the corresponding sequence of concentration functions for the distribution functions $\{F_{n'}\}$, it is seen that $\lim_{n'\to\infty} Q_{n'}(L)=\lim_n Q_n(L)$. Without loss of generality we may assume that $x_n\to x_0$ as $n\to\infty$. We shall now prove that $Q_0(L)=F_0(x_0+L+0)-F_0(x_0-0)$

and $Q_n(L) \to Q_0(L)$. Note that for $\delta > 0$,

$$F_n(x_0 + L + \delta) \geqq F_n(x_0 + L + 0)$$

and

$$F_n(x_0 - \delta) \leqq F_n(x_0 - 0).$$

Let $\delta > 0$ be small and such that $x_0 + L + \delta$ and $x_0 - \delta$ are continuity points of F. Then for sufficiently large n, $x_n + L + \delta/2 < x_0 + L + \delta$ and $x_n - \delta/2 > x_0 - \delta$. Hence, for sufficiently large n,

$$F_n(x_0 + L + \delta) - F_n(x_0 - \delta) \geqq F_n(x_n + L + 0) - F_n(x_n - 0).$$

Taking limits as $n \to \infty$ we obtain

$$F_0(x_0 + L + \delta) - F_0(x_0 - \delta) \geqq \lim_{n \to \infty} Q_n(L).$$

Now take the limit as $\delta \to 0$ and we obtain

$$Q_0(L) \geqq F_0(x_0 + L + 0) - F_0(x_0 - \delta) \geqq \lim_{n \to \infty} Q_n(L).$$

Since, always, $Q_0(L) \leqq \lim_n Q_n(L)$, we obtain $Q_0(L) = \lim_n Q_n(L)$.

Let $\{X_n\}$ be a sequence of random variables, and let $q_{n,N}(L)$ denote the concentration function of $X_{n+1} + \cdots + X_N$, where $N > n$. Define

$$q_n(L) = \lim_{N \to \infty} q_{n,N}(L),$$

and

$$q(L) = \lim_{n \to \infty} q_n(L).$$

Both of these limits exist because of Lemma 2.

LEMMA 5. *The value of $q(L)$ does not depend on L and is identically* 0 *or* 1.

This lemma is stated and proved in Lévy [2], page 131.

DEFINITION. *A series of random variables $\sum_{n=1}^{\infty} X_n$ is said to be quasi-convergent if there exists a sequence of numbers $\{c_n\}$ such that $\sum_{n=1}^{\infty} (X_n - c_n)$ converges almost surely.*

LEMMA 6. *If $\{X_n\}$ are independent random variables, $q(L) = 1$ if and only if $\sum_{n=1}^{\infty} X_n$ is quasi-convergent.*

This lemma is stated and proved in Lévy [2], pages 133–134.

The following theorem is K. Ito's theorem which was referred to in Section 1.

THEOREM. *Let $Q_n(L)$ be the concentration function of $S_n = X_1 + \cdots + X_n$, where $\{X_n\}$ are independent random variables. For each L, $\{Q_n(L)\}$ converges to a not identically vanishing function $Q(L)$ if and only if $\sum_{n=1}^{\infty} X_n$ is quasi-convergent.*

In this case, $Q(L)$ is the concentration function of $\sum_{n=1}^{\infty}(X_n-c_n)$, where $\{c_n\}$ satisfy the above definition.

Proof. We first show that the condition is sufficient for quasi-convergence. Suppose there exists an $L\geqq 0$ such that $Q_n(L)\to$(some) $d>0$ as $n\to\infty$. Let $N>n>1$. Then, by Lemma 2,

$$Q_N(L) \leqq \min\{Q_n(L), q_{n,N}(L)\}.$$

Hence $q_{n,N}(L)\to q_n(L)\geqq d>0$ as $N\to\infty$, and $q(L)=\lim_{n\to\infty} q_n(L)\geqq d>0$. Hence $q(L)\equiv 1$ by Lemma 5, and Lemma 6 implies that $\sum_{n=1}^{\infty} X_n$ is quasi-convergent. Conversely, if $\sum_{n=1}^{\infty} X_n$ is quasi-convergent, then $\sum_{n=1}^{\infty}(X_n-c_n)$ converges for some sequence $\{c_n\}$. Since concentration functions remain invariant under change of location parameters, the conclusion follows from Lemma 4.

3. Some corollaries. Ito's theorem yields some interesting corollaries.

COROLLARY 1. *If $\{X_n\}$ are independent, and if $\sum_{n=1}^{\infty} X_n$ converges, then its distribution function is continuous if and only if $Q_n(0)\to 0$ as $n\to\infty$, where Q_n is as in Ito's theorem.*

Proof. This follows directly from Ito's theorem and Lemma 3.

COROLLARY 2. *If $\{X_n\}$ are independent and discrete random variables, and if $\sum_{n=1}^{\infty} X_n$ converges, then the distribution of this sum is discrete if and only if $Q_n(0)\to$(some) $d>0$ as $n\to\infty$.*

Proof. By Corollary 1, the distribution of $\sum_{n=1}^{\infty} X_n$ is not continuous if and only if $\lim_{n\to\infty} Q_n(0)>0$. By a theorem of Jessen and Wintner [4], this implies that the distribution of $\sum_{n=1}^{\infty} X_n$ is discrete.

The following corollary is given in Loève ([3], page 538) and is proved by him using characteristic functions.

COROLLARY 3. *If $X_1, \cdots, X_n, Y_n$ are independent random variables and if $X=X_1+\cdots+X_n+Y_n$ for each n, then $\sum_{n=1}^{\infty} X_n$ is quasi-convergent.*

Proof. Let Q_n denote the concentration function of $X_1+\cdots+X_n$, and let Q be the concentration function of X. Then by Lemma 2, $Q_n(L)\geqq Q(L)$ for all n and $L\geqq 0$, and therefore $\lim_{n\to\infty} Q_n(L)\geqq Q(L)>0$ for all $L>0$. The conclusion follows from Ito's theorem.

The following corollary is well known.

COROLLARY 4. *Let $0<a_{n+1}<a_n$ for all n, and let $\{X_n\}$ be a sequence of independent random variables such that $P[X_n=a_n]=P[X_n=-a_n]=\frac{1}{2}$. Then $\sum_{n=1}^{\infty} X_n$ converges and its distribution is continuous if and only if $\sum_{n=1}^{\infty} a_n^2<\infty$.*

Proof. Convergence is necessary and sufficient for $\sum_{n=1}^{\infty} a_n^2<\infty$ by Kolmogorov's 3-series theorem. If Q_n denotes the concentration function of $X_1+\cdots+X_n$, then it is easy to verify that $Q_n(0)\to 0$ as $n\to\infty$. Corollary 1 implies then that the distribution of $\sum_{n=1}^{\infty} X_n$ is continuous.

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A PROBLEM IN PERMUTATIONS

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1. Introduction. In his solution of problem E1485, C. N. Bhaskaranandha [1] determined the number of arrangements of m men, w women and d dogs in a row such that no two women are together and no two dogs are together, whereas men may be together; and remarked that he had been unable to extend his method of solution to deal with the case of more than three categories. In this note we give the solution to the general problem of any number of categories, such that no two members of a category may be together, plus one special category (corresponding to Bhaskaranandha's men) on whose members no restriction is placed. Obviously no further generality would be gained by having more than one such special category.

We shall disregard the identity of individual members within a category. If desired, this can of course be allowed for by multiplying our results by $m!a_1!a_2! \cdots a_n!$, where $m, a_1, a_2, \cdots, a_n$ are the numbers of members in the various categories.

Suppose, then, that we have m letters M, and a_i letters A_i $(i=1, 2, \cdots, n)$, and let $f(m; a_1, a_2, \cdots, a_n)$ be the number of permutations of these $m+\sum_{i=1}^{n} a_i$ letters in a row so that, apart from the M's, no two like letters are together. We shall derive an explicit formula for this number, and we shall obtain its generating function. Further, we shall show that in the "symmetrical" case when $m=0$ and there are n categories, the problem has an unexpected and remarkable connection with the well-known problem of derangements, i.e. that of enumerating the permutations of the $\sum_{i=1}^{n} a_i$ letters so that no letter occupies the position which was occupied by a similar letter in a standard arrangement.

2. Explicit formula and generating function. The a_i letters A_i can be divided into t_i groups (with at least one letter in a group) in

$$\binom{a_i - 1}{t_i - 1}$$

ways. Treating each group as a unit, we can then arrange the $\sum_{i=1}^{n} t_i$ groups and the m letters M in a row, with no two groups of like letters adjacent, in $f(m;\, t_1, t_2, \cdots, t_n)$ ways, by definition. Since the total number of ways of arranging the $(m+\sum_{i=1}^{n} a_i)$ letters without any restriction is

$$\frac{\left(m + \sum_{i=1}^{n} a_i\right)!}{m!a_1!a_2! \cdots a_n!}$$

it follows that

$$\frac{\left(m + \sum_{i=1}^{n} a_i\right)!}{m!a_1!a_2! \cdots a_n!} = \sum_{t_1=1}^{a_1} \sum_{t_2=1}^{a_2} \cdots \sum_{t_n=1}^{a_n} \binom{a_1 - 1}{t_1 - 1}\binom{a_2 - 1}{t_2 - 1} \cdots \binom{a_n - 1}{t_n - 1} f(m;\, t_1, t_2, \cdots, t_n)$$

because each side of this equation enumerates all the arrangements without any restriction. Now the writer has shown elsewhere [2] that if for all nonnegative values of $s_1, s_2, \cdots, s_n$,

$$U(s_1, s_2, \cdots, s_n) = \sum_{i_1=0}^{s_1} \sum_{i_2=0}^{s_2} \cdots \sum_{i_n=0}^{s_n} \binom{s_1}{i_1}\binom{s_2}{i_2} \cdots \binom{s_n}{i_n} F(i_1, i_2, \cdots, i_n)$$

then,

$$F(s_1, s_2, \cdots, s_n) = \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{s_2} \cdots \sum_{j_n=0}^{s_n} (-1)^{\Sigma s - \Sigma j} \binom{s_1}{j_1}\binom{s_2}{j_2} \cdots \binom{s_n}{j_n} U(j_1, j_2, \cdots, j_n).$$

It is easy to see that this powerful general inversion formula applies, and gives

$$f(m;\, a_1, a_2, \cdots, a_n) = \sum_{r_1=1}^{a_1} \sum_{r_2=1}^{a_2} \cdots \sum_{r_n=1}^{a_n} (-1)^{\Sigma a - \Sigma r} \binom{a_1 - 1}{r_1 - 1}\binom{a_2 - 1}{r_2 - 1} \cdots \binom{a_n - 1}{r_n - 1} \frac{\left(m + \sum_{i=1}^{n} r_i\right)!}{m!r_1!r_2! \cdots r_n!} \tag{1}$$

which is the required explicit formula. (An entirely independent derivation of the formula (1) is given in Section 4 below).

To obtain the generating function, we multiply (1) by $y^m x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and sum over all values of $m, a_1, a_2, \cdots, a_n$ from 0 to ∞. We find

$$\sum_{m,a_1,a_2,\cdots,a_n} f(m; a_1, a_2, \cdots, a_n) y^m x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

$$= \sum_{m,r_1,r_2,\cdots,r_n} \left\{ \prod_{i=1}^{n} \left[\sum_{a_i=0}^{\infty} (-1)^{a_i - r_i} \binom{a_i - 1}{r_i - 1} x_i^{a_i} \right] y^m \cdot \frac{\left(m + \sum_{i=1}^{n} r_i\right)!}{m! r_1! r_2! \cdots r_n!} \right\}$$

$$= \sum_{m,r_1,r_2,\cdots,r_n} \left\{ \prod_{i=1}^{n} \left(\frac{x_i}{1 + x_i} \right)^{r_i} y^m \frac{\left(m + \sum_{i=1}^{n} r_i\right)!}{m! r_1! r_2! \cdots r_n!} \right\}.$$

For any particular value of $(m + \sum_{i=1}^{n} r_i)$, this is the multinomial expansion of

$$\left(y + \frac{x_1}{1 + x_1} + \frac{x_2}{1 + x_2} + \cdots + \frac{x_n}{1 + x_n} \right)^{m + \Sigma r}.$$

Therefore the sum is

$$\sum_{m+\Sigma r=0}^{\infty} \left(y + \sum_{i=1}^{n} \frac{x_i}{1 + x_i} \right)^{m+\Sigma r} = \frac{1}{1 - y - \sum_{i=1}^{n} \frac{x_i}{1 + x_i}}$$

$$(2) \qquad = \frac{\prod_{i=1}^{n} (1 + x_i)}{(1 - y) \prod_{i=1}^{n} (1 + x_i) - \sum x_1(1 + x_2) \cdots (1 + x_n)}$$

$$= \frac{\prod_{i=1}^{n} (1 + x_i)}{1 - \sum x_1 x_2 - 2 \sum x_1 x_2 x_3 - \cdots - (n - 1) x_1 x_2 \cdots x_n - y \prod_{i=1}^{n} (1 + x_i)}.$$

Thus (2) is the generating function of $f(m; a_1, a_2, \cdots, a_n)$.

3. Bhaskaranandha's problem. In the special case dealt with by Bhaskaranandha, the required number of ways is given by (1) with $n = 2$, $a_1 = w$, $a_2 = d$. Alternatively, we can obtain more convenient formulas in this simple case by proceeding from the generating function (2). For the required number of ways is the coefficient of $y^m x_1^w x_2^d$ in the expansion of

$$(3) \qquad \frac{1}{1 - y - \frac{x_1}{1 + x_1} - \frac{x_2}{1 + x_2}} = \frac{(1 + x_1)(1 + x_2)}{1 - x_1 x_2 - y(1 + x_1)(1 + x_2)}.$$

To obtain the formula given by Bhaskaranandha, namely

$$\sum_r \binom{w-1}{r-1}\binom{m+1}{r}\binom{m+1+r}{r-w+d}, \tag{4}$$

we write (3) in the form

$$\frac{\left(\frac{1+x_1}{1-x_1x_2}\right)(1+x_2)}{1-y\left(\frac{1+x_1}{1-x_1x_2}\right)(1+x_2)} = \sum_{m=0}^{\infty} y^m \left(\frac{1+x_1}{1-x_1x_2}\right)^{m+1}(1+x_2)^{m+1}$$

$$= \sum_{m=0}^{\infty} y^m \left(1+\frac{x_1(1+x_2)}{1-x_1x_2}\right)^{m+1}(1+x_2)^{m+1}$$

$$= \sum_{m=0}^{\infty} y^m \sum_{r=0}^{m+1}\binom{m+1}{r} x_1^r(1+x_2)^{r+m+1}(1-x_1x_2)^{-r},$$

and the coefficient of $y^m x_1^w x_2^d$ is now easily seen to be as given by (4). An alternative formula, which has the advantage of appearing—as well as being—symmetrical in w and d, can be deduced more directly from (3). For if we expand (3) in the form

$$(1+x_1)(1+x_2)\sum_{r=0}^{\infty}\left\{y(1+x_1)(1+x_2)+x_1x_2\right\}^r,$$

the coefficient of y^m is $\sum_{r=m}^{\infty}\binom{r}{m}(1+x_1)^{m+1}(1+x_2)^{m+1}x_1^{r-m}x_2^{r-m}$, and hence the coefficient of $y^m x_1^w x_2^d$ is

$$\sum_r \binom{r}{m}\binom{m+1}{r-w+1}\binom{m+1}{r-d+1}. \tag{5}$$

It is evident from the denominator of (3), in the second form shown, that $f(m; w, d)$ satisfies the recurrence relation

$$\begin{aligned} f(m; w, d) &= f(m; w-1, d-1) + f(m-1; w, d) + f(m-1; w-1, d) \\ &\quad + f(m-1; w, d-1) + f(m-1; w-1, d-1) \end{aligned} \tag{6}$$

which is easy to obtain independently by general reasoning.

4. Alternative approach. Returning to the general problem, we now derive (2) and hence (1) by an entirely independent method which includes the "general reasoning" approach to (6) as a special case. Let H_i ($i=1$ to n) be an operator which reduces by unity the number of letters in the category of letters A_i; e.g.,

$$H_1 f(m; a_1, a_2, \cdots, a_n) = f(m; a_1-1, a_2, \cdots, a_n)$$

and let H_0 be an operator which reduces by unity the number in the category of letters M; i.e.,

$$H_0 f(m; a_1, a_2, \cdots, a_n) = f(m - 1; a_1, a_2, \cdots, a_n).$$

Also let K_i $(i=1$ to $n)$ be an operator which selects the arrangements beginning with A_i and let K_0 be an operator which selects those beginning with M. Then we obviously have

$$\begin{aligned} K_0 f(m; a_1, a_2, \cdots, a_n) &= f(m - 1; a_1, a_2, \cdots, a_n) \\ &= H_0 f(m; a_1, a_2, \cdots, a_n) \end{aligned}$$

whence

$$K_0 \equiv H_0.$$

Also, $K_1 f(m; a_1, a_2, \cdots, a_n)$ is clearly equal to the number of permissible arrangements of the set $(m; a_1-1, a_2, \cdots, a_n)$ which begin with letters other than A_1, because if the first letter of an arrangement is A_1 the second letter must not be A_1. Hence

$$\begin{aligned} K_1 f(m; a_1, a_2, \cdots, a_n) &= [K_0 + K_2 + K_3 + \cdots + K_n] f(m; a_1 - 1, a_2, \cdots, a_n) \\ &= [1 - K_1] f(m; a_1 - 1, a_2, \cdots, a_n) \\ &= [1 - K_1] H_1 f(m; a_1, a_2, \cdots, a_n) \end{aligned}$$

so that $K_1 \equiv (1-K_1)H_1$ and similarly $K_i \equiv (1-K_i)H_i$ $(i=1$ to $n)$. That is, $K_i \equiv H_i/(1+H_i)$ $(i=1$ to $n)$, while $K_0 \equiv H_0$, as shown above. But since every arrangement begins with some letter,

$$K_0 + \sum_{i=1}^{n} K_i \equiv 1, \quad \text{whence} \quad H_0 + \sum_{i=1}^{n} \frac{H_i}{1 + H_i} \equiv 1,$$

or,

$$(1 - H_0) \prod_{i=1}^{n} (1 + H_i) - \sum H_1(1 + H_2) \cdots (1 + H_n) \equiv 0.$$

Therefore $f(m; a_1, a_2, \cdots, a_n)$ satisfies the recurrence relation obtained by applying this operator-identity to $f(m; a_1, a_2, \cdots, a_n)$, and it follows immediately that the denominator of the generating function must be as given in (2). The numerator is easily found to be $\prod_{i=1}^{n} (1+x_i)$ by considering the "initial conditions," i.e. the values of $f(m; a_1, a_2, \cdots, a_n)$ when the number of letters in each category is 0 or 1. To complete this approach, (1) is of course easily deduced from (2) by selecting the coefficient of $y^m x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

5. The symmetrical case. A case of particular interest arises when $m=0$ and we have n categories all to be subject to similar restrictions. From formula

(1), the number of permissible arrangements is

$$f(0; a_1, a_2, \cdots, a_n) = \sum_{r_1=1}^{a_1} \sum_{r_2=1}^{a_2} \cdots \sum_{r_n=1}^{a_n} (-1)^{\Sigma a - \Sigma r} \binom{a_1 - 1}{r_1 - 1} \binom{a_2 - 1}{r_2 - 1} \cdots \tag{7}$$
$$\binom{a_n - 1}{r_n - 1} \frac{\left(\sum_{i=1}^{n} r_i\right)!}{r_1! r_2! \cdots r_n!}.$$

Now A. W. Joseph and the present writer have shown elsewhere [3] that the number of derangements of the same set of letters is

$$\sum_{r_1=0}^{a_1} \sum_{r_2=0}^{a_2} \cdots \sum_{r_n=0}^{a_n} (-1)^{\Sigma a - \Sigma r} \binom{a_1}{r_1} \binom{a_2}{r_2} \cdots \binom{a_n}{r_n} \frac{\left(\sum_{i=1}^{n} r_i\right)!}{r_1! r_2! \cdots r_n!}. \tag{8}$$

The similarity of (7) and (8) is more than apparent. For since

$$\binom{a - 1}{r - 1} = \binom{a}{r} - \binom{a - 1}{r}$$

substitution in (7) shows that (7) represents the sum of the 2^n expressions similar to (8), corresponding to deducting 0 or 1 in all possible ways from the numbers $a_1, a_2, \cdots, a_n$. (It will be seen that the factor $(-1)^{\Sigma a - \Sigma r}$ correctly accounts for all positive and negative signs, so that the 2^n expressions similar to (8) are in fact all to be *added*). We have therefore the following remarkable result: *The number of ways of arranging a_1 letters A_1, a_2 letters A_2, $\cdots$, and a_n letters A_n so that no two like letters are together is equal to the sum of the numbers of ways of deranging the 2^n sets obtained by deducting* 0 *or* 1 *in all possible ways from the numbers of the letters in the n categories.*

It follows that the table of derangement numbers given in [3] can be used to obtain the numbers of arrangements for the present problem. However, to the recurrence relations derived in [3] for derangement numbers, there correspond recurrence relations for the present problem from which the numbers of arrangements can be calculated directly. (Formula (1) is inconvenient for numerical work.) Such recurrence relations are easily derived from the generating function $V \equiv V(x_1, x_2, \cdots, x_n)$, say, given by formula (2) with $y = 0$, viz.

$$V = \frac{1}{1 - \sum_{i=1}^{n} \frac{x_i}{1 + x_i}}.$$

If we write, for brevity,

$$f(0; a_1, a_2, \cdots, a_n) \equiv [a_1, a_2, \cdots . a_n],$$

then $[1, a_1, a_2, \cdots, a_n]$ is the coefficient of $zx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ in

$$\frac{1}{1-\dfrac{z}{1+z}-\displaystyle\sum_{i=1}^{n}\frac{x_i}{1+x_i}}$$

which is seen to be the coefficient of $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ in V^2, i.e. in $(1+x_1)^2\,\partial V/\partial x_1$ since $\partial V/\partial x_i = V^2(1+x_i)^{-2}$, i.e. in

$$\frac{\partial V}{\partial x_1} + 2x_1\frac{\partial V}{\partial x_1} + x_1^2\frac{\partial V}{\partial x_1}\cdot$$

Hence we have the relation

$$\begin{aligned}[1, a_1, a_2, \cdots, a_n] = (a_1+1)[a_1+1, a_2, \cdots, a_n] &+ 2a_1[a_1, a_2, \cdots, a_n]\\ &+ (a_1-1)[a_1-1, a_2, \cdots, a_n].\end{aligned} \tag{9}$$

By similar methods it is easy to show that

$$\begin{aligned}&[1, a_1, a_2, \cdots, a_n]\\ &= (\sum_i a_i + 1)[a_1, a_2, \cdots, a_n] + (a_1-1)[a_1-1, a_2, \cdots, a_n]\\ &+ (a_2-1)[a_1, a_2-1, a_3, \cdots, a_n] + \cdots + (a_n-1)[a_1, a_2, \cdots, a_{n-1}, a_n-1]\end{aligned} \tag{10}$$

and that

$$\begin{aligned}&(a_1+1)[a_1+1, a_2, \cdots, a_n]\\ &= (1-a_1+a_2+\cdots+a_n)[a_1, a_2, \cdots, a_n] + (a_2-1)[a_1, a_2-1, a_3, \cdots, a_n]\\ &+ (a_3-1)[a_1, a_2, a_3-1, a_4, \cdots, a_n] + \cdots + (a_n-1)[a_1, a_2, \cdots, a_{n-1}, a_n-1]\end{aligned} \tag{11}$$

Equation (11) is the most convenient for numerical work.

6. Generalization of the problem. Finally, we use the method of Section 4 to solve a much more general problem. Let it be required to permute the a_i letters A_i ($i=1$ to n) in a row with the restriction that not more than b_i letters A_i may be together; thus at most b_1 letters A_1 may be together, at most b_2 letters A_2 may be together, and so on. (Clearly this reduces to the original problem of this note if one of the b_i, say b_j, is put equal to a_j, so that there is no restriction on the a_j letters A_j, and all the other b_i are put equal to 1.)

Let $g(a_1, a_2, \cdots, a_n)$ be the required number of permissible permutations, where for convenience we suppress $b_1, b_2, \cdots, b_n$ from the notation, although obviously g is a function of these also.

Let K_i ($i=1$ to n) be an operator which selects those permissible permutations whose first letter is an A_i. Then $K_1g(a_1, a_2, \cdots, a_n)$ equals the number of permissible permutations of the set $(a_1, a_2, \cdots, a_n)$ whose first letter is an A_1 (by definition), which is equal to the number of permissible permutations of the set $(a_1-1, a_2, \cdots, a_n)$, *less* those of them which begin with exactly b_1 A_1's,

which in turn equals $g(a_1-1, a_2, \cdots, a_n)$ minus the number of permissible permutations of the set $(a_1-1-b_1, a_2, \cdots, a_n)$ *not* beginning with an A_1, and hence:

$$K_1g(a_1, a_2, \cdots, a_n) = g(a_1 - 1, a_2, \cdots, a_n) \\ -\left[g(a_1 - 1 - b_1, a_2, \cdots, a_n) - K_1g(a_1 - 1 - b_1, a_2, \cdots, a_n)\right].$$

Therefore,

$$K_1 \equiv H_1 - H_1^{b_1+1} + K_1H_1^{b_1+1},$$

where H_1 is defined as in Section 4. Similar relations hold for $K_2, K_3, \cdots, K_n$. Thus,

$$K_i \equiv \frac{H_i(1 - H_i^{b_i})}{1 - H_i^{b_i+1}}.$$

Since every permutation begins with some letter, $\sum_{i=1}^n K_i \equiv 1$, and following the method of Section 4 we easily find that the required number $g(a_1, a_2, \cdots, a_n)$ is the coefficient of $x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ in

$$\frac{1}{1 - \sum_{i=1}^{n} \frac{x_i(1 - x_i^{b_i})}{1 - x_i^{b_i+1}}}. \tag{12}$$

If b_j, say, is put equal to a_j, the term in the sum in the denominator involving x_j can obviously be replaced by x_j since terms in $x_j^{a_j+1}$ cannot affect the coefficient sought. Also, if $b_i = 1$ for all other values of i, the term involving x_i reduces to

$$\frac{x_i(1 - x_i)}{1 - x_i^2} = \frac{x_i}{1 + x_i},$$

and it is easy to see that (2) results with suitable changes of notation (change n into $n+1$ and write $j=n+1$, $y=x_{n+1}$).

As a simple numerical example, the number of ways of arranging the letters *AAAABBBCC* so that at most 2 A's are together, at most 2 B's are together, and the C's are separated, is the coefficient of $x_1^4x_2^3x_3^2$ in

$$\left\{1 - \left(\frac{x_1(1 - x_1^2)}{1 - x_1^3} + \frac{x_2(1 - x_2^2)}{1 - x_2^3} + \frac{x_3}{1 + x_3}\right)\right\}^{-1},$$

and after discarding terms which cannot affect the coefficient of $x_1^4x_2^3x_3^2$, this may be written as

$$\sum_{t=0}^{\infty} \left\{x_1 - x_1^3 + x_1^4 + x_2 - x_2^3 + x_3 - x_3^2\right\}^t.$$

The coefficient is easily found to be 675.

It will be noticed that if in (12) b_j is put equal to a_j for every j, the coefficient of $x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ is the same as the coefficient of this term in $(1-\sum_{i=1}^n x_i)^{-1} = \sum_{t=0}^\infty (x_1+x_2+ \cdots +x_n)^t$, which is of course $(\sum_{i=1}^n a_i)!/a_1!a_2! \cdots a_n!$, corresponding to the case of unrestricted permutations.

From (12), it would be possible to write down an explicit formula for $g(a_1, a_2, \cdots, a_n)$, but it is too complicated to be worth exhibiting.

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MATHEMATICAL NOTES

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THE ALTERNATIVE ISOTOPES OF DIVISION ALGEBRAS

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Let A be an algebra over a field F. Then A is a vector space over F equipped with a bilinear mapping xy from $A \times A$ to A. The bilinear mapping xy is called the product in A. If $(xx)y = x(xy)$ and $y(xx) = (yx)x$ for any $x, y \in A$, then the algebra A is called *alternative*. If $(xy)z = x(yz)$ for any $x, y, z \in A$, then the algebra is called *associative*. Thus an associative algebra is necessarily alternative but not conversely.

Suppose the set of nonzero elements in an associative algebra A over F forms a multiplicative group. Then A is called an *associative division algebra* over F. Well-known examples of associative division algebras are the real number field R over R, the complex numbers over R and the quaternion algebra over R. The Cayley numbers over the reals are an alternative, nonassociative, division algebra. For a definition, although not the most general, of an arbitrary division algebra, consult [1] p. 253.

Now suppose that P and Q are nonsingular (i.e., invertible) linear transformations of the underlying vector space of an algebra A onto itself. We define a new bilinear product in A by writing $x \circ y = Px \cdot Qy$. It is easily verified that $x \circ y$ is actually bilinear. The algebra, with new product, resulting from A and the maps P and Q is called an *isotope* of A and may be denoted by $A(P, Q)$.

The algebra A is among its isotopes: $A=A(I, I)$ where I is the identity mapping in A. It is easy to see that an isotope of A is, in general, not alternative (and hence not associative) even if A is an associative algebra. The purpose of this note is to characterize the mappings P and Q which yield alternative isotopes of an associative division algebra A. It will turn out that such isotopes (different from A) exist and that they are all necessarily associative.

Before stating our first theorem, we define the right and left multiplication operators in an algebra A. If $x\in A$, then R_x and L_x denote the linear mappings in $A: R_xy=yx$, $L_xy=xy$ for any $y\in A$. If A is an associative division algebra and x a nonzero element in A, then it is easy to show that R_x and L_x are nonsingular linear mappings with inverses $R_x^{-1}=R_{x^{-1}}$, $L_x^{-1}=L_{x^{-1}}$.

THEOREM 1. *Let A be an associative division algebra over a field F and let $A(P, Q)$ be an isotope of A. Then $A(P, Q)$ is an alternative algebra if and only if $P=R_x$ and $Q=L_y$ for suitable nonzero $x,y\in A$. Further, an alternative isotope $A(P, Q)$ of A is necessarily associative.*

Proof. (i) We will denote multiplication in A by uv and in $A(P, Q)$ by $u\circ v$. Suppose $P=R_x$, $Q=L_y$ for certain nonzero $x, y\in A$. Then $(u\circ v)\circ w=u\circ(v\circ w)$ for any $u, v, w\in A(P, Q)$ because

$$(u\circ v)\circ w = R_x(R_xu\cdot L_yv)\cdot L_yw = [(ux)(yv)x]yw$$

and

$$u\circ(v\circ w) = R_xu\cdot L_y(R_xv\cdot L_yw) = ux[y(vx)(yw)].$$

Thus, if an isotope of A is given by right and left multiplications, then the isotope is not only alternative but associative.

(ii) Suppose now, that $A(P, Q)$ is an alternative isotope of A. Then, since $(u\circ u)\circ v=u\circ(u\circ v)$ and $v\circ(u\circ u)=(v\circ u)\circ u$, we have

$$P(Pu\cdot Qu)\cdot Qv = Pu\cdot Q(Pu\cdot Qv) \tag{1}$$

$$Pv\cdot Q(Pu\cdot Qu) = P(Pv\cdot Qu)\cdot Qu \qquad \text{for any } u, v\in A. \tag{2}$$

Since P is a (1-1) onto map, there exists a unique $r\in A$, $r\neq 0$, such that $Pr=1$, where 1 is the multiplicative identity in A. Setting $u=r$ in (1), we get

$$P(Qr)\cdot Qv = Q(Qv) \qquad \text{for any } v\in A. \tag{3}$$

Now, since Q is (1-1) onto, Qv ranges over all of A as v ranges over A. We see from this and (3) that $Qz=yz$ for any $z\in A$ with $y=P(Qr)\neq 0$. Thus $Q=L_y$.

Since Q is (1-1) onto, there exists a unique $s\in A$, $s\neq 0$, such that $Qs=1$ and if we put $u=s$ in (2), we get

$$Pv\cdot Q(Ps) = P(Pv) \qquad \text{for any } v\in A. \tag{4}$$

Equation (4) implies $Pz=zx$ for any $z\in A$ with $x=Q(Ps)\neq 0$. The reason is the same as for Q, above. Thus $P=R_x$.

Because P and Q are, respectively, right and left multiplications, we see from (i) that $A(P, Q)$ is not only alternative—it is an associative algebra. This proves Theorem 1.

Henceforth we will assume that A is an arbitrary associative division algebra over a field F. Suppose $A(P, Q)$ and $A(S, T)$ are identical alternative isotopes of A. Thus, assume $A(P, Q)=A(S, T)$. Then by Theorem 1,

$$A(P, Q) = A(R_x, L_y) = A(S, T) = A(R_{x'}, L_{y'})$$

for nonzero $x, y, x', y' \in A$. There is a relation between x, y, x', y' given in

THEOREM 2. *If $A(R_x, L_y)=A(R_{x'}, L_{y'})$, then $xy=x'y'$ and conversely. Further, both isotopes are identical with $A(R_z, I)$ or $A(I, L_z)$ where $z=xy$.*

Proof. If $R_xu\cdot L_yv=R_{x'}u\cdot L_{y'}v$ for any $u, v\in A$, then $(ux)(yv)=(ux')(y'v)$ and so $xy=x'y'$. Conversely, if $xy=x'y'$, then $uxyv=ux'y'v$ for any $u, v\in A$ and so $A(R_x, L_y)=A(R_{x'}, L_{y'})$.

If we set $z=xy$ then, obviously, $A(R_x, L_y)=A(R_z, I)=A(I, L_z)$, where I is the identity mapping in A. This establishes Theorem 2.

THEOREM 3. *Let $A(P, Q)$ be an alternative isotope of A. Then $A(P, Q)$ is an associative division algebra.*

Proof. We saw in Theorem 1 that $A(P, Q)$ is an associative algebra. By Theorem 2, we may take $A(P, Q)$ in the "canonical" form $A(R_z, I)$ for a certain unique $z\neq 0$ in A. We show that $A(R_z, I)$ has an identity and each nonzero element has a multiplicative inverse.

Consider the element $e=z^{-1}\in A$. We have $u\circ e=e\circ u=u$ for any $u\in A$, for

$$u\circ e = R_zu\cdot Ie = (uz)z^{-1} = u,$$

and similarly $e\circ u=u$.

Now let v be any nonzero element in A. If we define $v^-=ev^{-1}e$, then $v\circ v^- =v^-\circ v=e$, for

$$v\circ v^- = R_zv\cdot Iv^- = vz(ev^{-1}e) = vz(z^{-1}v^{-1}z^{-1}) = z^{-1} = e$$

and similarly $v^-\circ v=e$. This proves Theorem 3.

Theorem 3 leads naturally to the question: Are the alternative isotopes of A essentially different from A?

THEOREM 4. *Any two alternative isotopes of A are isomorphic. In particular, if $A(P, Q)$ is such an isotope, then A and $A(P, Q)$ are isomorphic since A is an alternative isotope of itself.*

Proof. It is sufficient to prove that any alternative isotope $A(P, Q)=A(R_z, I)$ of A is isomorphic to A because of the symmetry and transitivity of isomorphism. Toward this end, define the map $f: A(R_z, I)\to A$ by $f(u)=uz$, $u\in A(R_z, I)$.

Clearly f is (1-1) onto. We have, for any $u, v \in A, a \in F$,

$$f(au + v) = (au + v)z = auz + vz = af(u) + f(v)$$

and

$$f(u \circ v) = (u \circ v)z = (uzv)z = (uz)(vz) = f(u)f(v).$$

Thus f is an isomorphic mapping and this completes the proof.

In closing we remark that the division algebra over the reals, which is defined in [1] Example 1, page 252, is an associative isotope C' of the algebra C of complex numbers over the reals. In this example we have

$$C' = C(R_{(-1+2i)}, I) = C(I, L_{(-1+2i)}).$$

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A NOTE ON NILPOTENT GROUPS

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It is proved (see [1] p. 215) that if a group G is nilpotent, then any subgroup of G is accessible. For an arbitrary group G, the converse is an open problem. In this note the converse is proved for finite groups.

DEFINITION. *Suppose H is a subgroup of a group G. If there exists a normal series $E \subset G_1 \subset \cdots \subset G_i = H \subset \cdots \subset G$ of G, where E is the identity subgroup, then H is said to be accessible.*

THEOREM. *If every subgroup of a finite group G is accessible, then G is nilpotent.*

Proof. Let P be any Sylow subgroup of G, and let $N(P)$ be its normalizer. Since any subgroup of G is accessible there exist a normal series $E \subset G_1 \subset \cdots \subset G_i = N(P) \subset \cdots \subset G$ of G. If $N(P) \neq G$, this implies that the normalizer of $N(P)$ properly contains $N(P)$. But the normalizer of a Sylow subgroup of a group contains its normalizer. Hence $N(P) = G$, and P is normal in G. This shows that every Sylow subgroup of G is normal in G. Thus, the Sylow subgroups of G generate a direct product. Since the Sylow subgroups of a finite group generate the group, G is a direct product of its Sylow subgroups. Since G is a direct product of its Sylow subgroups by [1] p. 216, G is nilpotent.

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ASSOCIOMORPHIC MAPPINGS OF GROUPOIDS

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A homomorphic mapping of a groupoid G into a groupoid H is a special case of an associomorphic mapping, i.e. a mapping which preserves the associative structure of G. In this note, we examine some of the relations between homomorphisms and general associomorphisms.

DEFINITION. *An n-associomorphism of a groupoid G into a groupoid H is a mapping such that the image of a product of $n+1$ elements of G is the product of their $n+1$ images, associated in the same way.*

Thus, for example, if α is a 2-associomorphism and a, b, c, are in G, then $[a(bc)]\alpha = a\alpha\cdot(b\alpha\cdot c\alpha)$ and $[(ab)c]\alpha = (a\alpha\cdot b\alpha)\cdot c\alpha$. Clearly, a homomorphism is an n-associomorphism for every n, and an r-associomorphism is an s-associomorphism if r divides s. A 1-associomorphism is a homomorphism of G onto its image. The following theorem gives some basic facts about associomorphisms, in the case in which G has an identity element.

THEOREM. *If a groupoid G has an identity element e, the mapping α of G onto a groupoid H is an n-associomorphism if and only if:*

(1) *H has an identity element* 1,

(2) *H has a central element $z\alpha$, such that $(z\alpha)^n = 1$, and*

(3) *there is a homomorphism β of G into H, such that for x in G, $x\alpha = x\beta\cdot z\alpha$.*

Proof. Let α be an n-associomorphic mapping of G onto H.

(1) For any x in G, $x\alpha\cdot(e\alpha)^n = (xe^n)\alpha = x\alpha = (e^n x)\alpha = (e\alpha)^n\cdot x\alpha$. Thus H contains an identity element $1 = (e\alpha)^n$.

(2) Recall that the center of the groupoid G consists of those elements z that commute and associate with all elements of G (i.e., for a, b, in G, $az = za$, $a(bz) = (ab)z$, $a(zb) = (az)b$, and $z(ab) = (za)b$). Then n-associomorphisms preserve centrality, provided $H^n = H$. For, suppose z is central in G and $H^n = H$. Then for $n = 2$ and a, b, in G, we have

$$a\alpha\cdot z\alpha = (x\alpha\cdot y\alpha)\cdot z\alpha = [(xy)z]\alpha = [z(xy)]\alpha = z\alpha\cdot(x\alpha\cdot y\alpha) = z\alpha\cdot a\alpha,$$

where $x\alpha\cdot y\alpha = a\alpha$. Of course,

$$a\alpha\cdot(b\alpha\cdot z\alpha) = [a(bz)]\alpha = [(ab)z]\alpha = (a\alpha\cdot b\alpha)\cdot z\alpha, \text{ etc.}$$

The generalization to $n>2$ is clear if we note that $H^n = H$ implies

$$H = H^2 = \cdots = H^{n-1} = H^n.$$

Now, since e is central in G and $(e\alpha)^n = 1$ (and hence, $H^n = H$), it follows that $e\alpha$ is central in H.

(3) Define the mapping β of G into H to be such that for x in G, $x\beta = x\alpha\cdot(e\alpha)^{n-1}$. Let $f\alpha = 1 = (e\alpha)^n$. Then for a, b, in G, we have

$$\begin{aligned}(ab)\beta &= (ab)\alpha\cdot(e\alpha)^{n-1} = (ab)\alpha\cdot\left[(e\alpha)^{n-1}\cdot f\alpha\right]\\ &= \left[(ab)(e^{n-1}f)\right]\alpha = \left[(ab)(e^{n-2}f)\right]\alpha\\ &= (a\alpha\cdot b\alpha)\left[(e\alpha)^{n-2}\cdot f\alpha\right] = (a\alpha\cdot b\alpha)(e\alpha)^{2n-2}\\ &= \left[a\alpha\cdot(e\alpha)^{n-1}\right]\left[b\alpha\cdot(e\alpha)^{n-1}\right] = a\beta\cdot b\beta.\end{aligned}$$

Thus β is a homomorphism, and clearly $a\alpha = a\beta\cdot e\alpha$.

Conversely, suppose conditions (1), (2), and (3) are satisfied. Then for $n=2$, we have

$$\begin{aligned}\left[a(bc)\right]\alpha &= \left[a(bc)\right]\beta\cdot z\alpha = \left[a\beta\cdot(b\beta\cdot c\beta)\right](z\alpha)^3\\ &= (a\beta\cdot z\alpha)\left[(b\beta\cdot z\alpha)(c\beta\cdot z\alpha)\right] = a\alpha\cdot(b\alpha\cdot c\alpha), \text{ etc.}\end{aligned}$$

The generalization to $n>2$ is obvious. We may note that this converse is true even if G does not contain an identity element. If G does contain an identity element e, then $z\alpha = e\alpha$.

The restriction in the hypothesis of the above theorem, that α map G *onto* H, is perhaps rather severe in view of the fact that the n-associomorphic image of a groupoid is not necessarily a groupoid. For example, the mapping $x\rightarrow -|x|$ of nonzero numbers into negative numbers under multiplication is a 2-associomorphism whose image is not a groupoid. The homomorphic image of a groupoid is, of course, a groupoid. This raises the questions: Under what conditions is an n-associomorphism a homomorphism? Under what conditions is the n-associomorphic image of a groupoid a groupoid? The following discussion partially answers these questions.

We have noted that if α is an r-associomorphism and r divides s, then α is an s-associomorphism. Further number theoretical considerations lead to the result that if α is both r- and s-associomorphic, then α is t-associomorphic, where t is the greatest common divisor of r and s. It follows that a mapping α of a groupoid G into a groupoid H is homomorphic if and only if there exist integers r and s, which are relatively prime and such that α is both r- and s-associomorphic. If G contains an identity element e, this condition becomes: $(e\alpha)^r = (e\alpha)^s$ implies α is t-associomorphic, where t is the greatest common divisor of r and s. If H contains an identity element $1 = z\alpha$ and α is an n-associomorphism, then for a, b, in G, $(ab)\alpha = (a\alpha\cdot b\alpha)(z^2\alpha)$. Hence α is homomorphic if $z^2\alpha = 1$.

The above conditions for an n-associomorphic mapping of a groupoid G into a groupoid H to be a homomorphism are also, of course, conditions that $G\alpha$ be a groupoid. By a device used in the proof of the above theorem, we may also show that $G\alpha$ is a groupoid if it contains an identity element.

THE CHARACTERISTIC OF A RING

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In the existing literature on the theory of rings, the finite characteristic of a ring is defined in one of the following ways:

(A) A ring is said to have finite characteristic n if n is the least positive integer such that $nx=0$ for all $x \in R$, [2].

(B) A ring R is said to have finite characteristic n if n is the maximum of the orders of the elements of R, [1].

To the best of this author's knowledge, nowhere has it been shown that for a given ring R both of these definitions lead us to the same value of n. The purpose of this note is to show that the two definitions coincide.

Suppose that $n>0$ is the maximum of the orders of the elements of R. Then there exists an element u in R such that the order of u is n. Let v be an arbitrary element of R and let $m<n$ be the order of v. Assume that m is not a divisor of n. Then there must exist at least one prime p such that

$$m = m_1p^e, \quad n = n_1p^f, \quad (p, m_1) = 1, \quad (p, n_1) = 1 \text{ with } 0 < f < e.$$

Since elements u and v of R have orders n and m respectively, we have $nu=0$, and for no smaller positive integer n is this true; and $mv=0$, and for no smaller positive integer m is this true. That is, $n_1p^fu=0$, and $m_1p^ev=0$.

Set $x=m_1v$ and $y=p^fu$. These elements of R have orders p^e and n_1 respectively. Also $(p^e, n_1)=1$ because $(p, n_1)=1$. Then by a well-known property of abelian groups, the order of the element $x+y$ is $n_1p^e>n_1p^f=n$. This contradicts the assertion that n is the maximum of the orders of the elements of R. Hence m divides n. That is, the order of every element of R is a divisor of n. It follows that $nz=0$ for all z in R, and since there exists an element of order n, the two definitions coincide.

It may be remarked that now it is easy to show that in both definitions the finite characteristic n is the least common multiple of the orders of all elements of R.

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FUNCTIONS WHICH REPRESENT ALL INTEGERS

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Let α be any positive irrational number. Then every positive integer can be expressed in one, but not both, of the forms $m+[\alpha m]$ or $n+[n/\alpha]$ for a suitable positive integer m or n (see Vinogradov [3], Chapter II, Problem 3). Here [ ] denotes the integer part function. The theorem which follows generalizes this result by allowing other functions $\phi(m)$, $\psi(n)$ to replace $[\alpha m]$, $[n/\alpha]$.

THEOREM. *Let $F(1)$, $F(2)$, $F(3)$, $\cdots$ and $G(1)$, $G(2)$, $G(3)$, $\cdots$ be two strictly monotone sequences of real numbers. Suppose that $F(m)$ and $G(n)$ increase without limit as m and n increase and that the equation $F(m)=G(n)$ has no solution m, n. Let $f(u)$ be the largest integer m such that $F(m)\leqq u$ and $g(v)$ the largest n such that $G(n)\leqq v$. Define $\phi(m)=g(F(m))$ and $\psi(n)=f(G(n))$. Then every positive integer appears exactly once in the list $\{1+\phi(1),\ 1+\psi(1),\ 2+\phi(2),\ 2+\psi(2),\ 3+\phi(3),\ 3+\psi(3),\ \cdots\}$.*

For example take $F(n)=p(n)$, the nth prime, and take $G(m)=4m$. Then $f(u)=\pi(u)$, the number of primes $\leqq u$, and $g(v)=[v/4]$. Since no prime $p(n)$ can equal $4m$ the theorem shows that every positive integer has one of the forms $n+[\frac{1}{4}p(n)]$ or $m+\pi(4m)$.

To prove the theorem note that the set $\{F(1), F(2), \cdots\}$ contains no pair of equal numbers because $F(m)$ is strictly monotone. Then there are $f(L)$ different numbers $F(m)$ which are $\leqq L$. Similarly the set

$$S = \{F(1), F(2), \cdots, G(1), G(2), \cdots\}$$

contains $U(L)=f(L)+g(L)$ numbers $\leqq L$. These numbers are all different because $F(m)\neq G(n)$. $U(L)$ is a step function which increases by one at each value L belonging to S. To represent an integer N in the desired form find the value $L=L(N)$ at which the step-function jumps for the Nth time, i.e., $N=U(L(N))$. $L(N)$ is one of the numbers in S, either an $F(m)$ or a $G(n)$. If $L(N)=F(m)$ then

$$N = U(F(m)) = f(F(m)) + g(F(m)) = m + \phi(m).$$

Similarly if $L(N)=G(n)$ then $N=n+\psi(n)$.

The theorem assumes that $\{F(m)\}$ and $\{G(n)\}$ grow without bound; this ensures that $U(L)$, $f(u)$, and $g(v)$ are defined for all real L, u, and v. If the assumption is dropped the same proof still applies, but one must write $f(u)=\infty$ if all $F(m)$ are $\leqq u$. Then in addition to positive integers the list may contain some terms $n+\psi(n)$ which are ∞. In particular if $\lim_{m\to\infty} F(m)<\lim_{n\to\infty} G(n)$ then all but a finite number of positive integers have the form $n+\phi(n)$.

The theorem has a convenient special form if there exist real valued continuous functions $F(x)$, $G(x)$ defined for all real positive x and which become $F(m)$, $G(n)$ for integer x. Then F and G have inverse functions F^{-1} and G^{-1}. The theorem defines

$$f(u) = \begin{cases} 0 & \text{if } u \leqq F(0), \\ [F^{-1}(u)] & \text{otherwise} \end{cases}$$

and a similar formula gives $g(v)$. Then $\phi(m)=[G^{-1}(F(m))]$ if this number is defined (i.e., if $F(m)\geqq G(0)$) and $\phi(m)$ is zero otherwise.

Table I gives some continuous functions $F(x)$, $G(x)$ from which the theorem constructs $\phi(m)$, $\psi(n)$. In order to satisfy all the requirements of the theorem choose α a positive irrational, β a transcendental greater than one, and q a

quadratic nonresidue modulo p. Braces $\{\ \}^+$ appear in one of the functions $\phi(m)$ to indicate that $\phi(m)$ is to have the value zero if the expression enclosed in braces is negative. These braces are needed because certain values $F(m)$ are smaller than $G(0)$; the theorem defines the corresponding $\phi(m)$ to be zero.

TABLE I

$F(x)$	$G(x)$	$\phi(m)$	$\psi(n)$
αx	x	$[\alpha m]$	$[n/\alpha]$
β^x	x	$[\beta^m]$	$[\log_\beta n]$
2^{x+1}	$2x+1$	2^m-1	$[\log_2(n+\frac{1}{2})]$
x^2+x	$2x^2+1$	$[\{\frac{1}{2}(m^2+m-1)\}^{1/2}]$	$[\frac{1}{2}(8n^2+5)^{1/2}-\frac{1}{2}]$
x^2	$px+q$	$[\{(m^2-q)/p\}^+]$	$[(pn+q)^{1/2}]$

To generalize the theorem replace the two functions $F(x)$, $G(x)$ by several, say $F_i(x)$, $i=1, \cdots, k$. Permit none of the $k(k-1)/2$ equations $F_i(m)=F_j(n)$ to have positive integer solutions m, n. Then every positive integer will have exactly one of k possible representations. For $j=1, \cdots, k$ the jth representation is a sum

$$\sum_{i=1}^{k} f_i(F_j(m))$$

for some integer m. The functions f_i are the generalizations of $f(u)$, $g(v)$. If $u \leqq F_i(0)$ then $f_i(u)=0$; otherwise $f_i(u)$ is $[F_i^{-1}(u)]$.

For a simple example take $k=3$, $F_1(x)=x$, $F_2(x)=\alpha x$, $F_3(x)=\beta x$, where α, β are positive irrationals such that α/β is also irrational. Then every positive integer is expressible in just one way as either $m+[\alpha m]+[\beta m]$, $[m/\alpha]+m+[\beta m/\alpha]$, or $[m/\beta]+[\alpha m/\beta]+m$ with m a positive integer. By contrast, Uspensky [2] and Graham [1] proved that no triple of positive numbers (a, b, c) exists such that every positive integer has exactly one of the forms $[am,]$ $[bm]$, $[cm]$.

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CLASSROOM NOTES

EDITED BY A. L. SHIELDS, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, University of Michigan.

THE n-TH DERIVATIVE OF A PRODUCT

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The Leibniz rule for the n-th derivative of a product of two factors is well known [1]:

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}, \tag{1}$$

where u and v are functions of x and $u^{(k)}$ indicates the k-th derivative with respect to x. There is an obvious symbolic analogy between the right members of (1) and of the binomial theorem

$$(u + v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k};$$

the exponents in the latter correspond to orders of derivatives in (1).

Leibniz has shown also the relation between $d^n(xyz \cdots)$ and $(dx+dy+dz+ \cdots)^n$ [2]. J. A. Serret gives the formula but not the expansion [3].

In order to obtain the higher derivatives of a product of more than two factors we first give an expression for the expansion of the n-th power of a multinomial. The multinomial theorem, as usually treated, leaves to the user the problem of arranging a summation over partitions of the exponent. Also, the coefficient formula requires a division before the value is obtained. E. Netto [4] has shown that each coefficient in a multinomial expansion can be expressed as a product of binomial coefficients. For our purposes we give below a slightly different form of the coefficient. This form permits one to write down an expansion very easily. Moreover, especially by use of a table of binomial coefficients, it simplifies the numerical work.

I. We know [5] that if $k_1 + \cdots + k_m = n$ then the coefficient of the general term $u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}$ in $(u_1 + u_2 + \cdots + u_m)^n$ is

$$\frac{n!}{k_1!k_2! \cdots k_m!}.$$

Since

$$n! = n \cdots (n - k_m + 1)(n - k_m) \cdots (n - k_m - k_{m-1} + 1) \\ \cdots (n - k_m - \cdots - k_3 - k_2) \cdots 1,$$

we have

$$\frac{n!}{k_1!k_2! \cdots k_m!} = \binom{n}{k_m}\binom{n-k_m}{k_{m-1}}\binom{n-k_m-k_{m-1}}{k_{m-2}} \cdots \binom{n-k_m-\cdots-k_2}{k_1}$$
$$= \binom{n}{k_m}\binom{n-k_m}{k_{m-1}}\binom{n-k_m-k_{m-1}}{k_{m-2}} \cdots \binom{n-k_m-\cdots-k_3}{k_2},$$

because $n-k_m-\cdots-k_2=k_1$. For example: In $(u_1+u_2+u_3+u_4)^n$ the general term is

$$\frac{n!}{a!b!c!d!} u_1^a u_2^b u_3^c u_4^d = \binom{n}{d}\binom{n-d}{c}\binom{n-d-c}{b} u_1^a u_2^b u_3^c u_4^d.$$

Thus, the numerical factors of each term consist of $m-1$ binomial coefficients.

For the exponents of the literal factors we have the rule: The exponent of the first letter of the polynomial is the difference between the upper and lower indices of the last binomial coefficient of the term. The exponents of the following letters of the polynomial (read from left to right) are the lower indices of the binomial coefficients, read from right to left, starting with the last binomial coefficient. Then

$$(u_1 + u_2 + \cdots + u_m)^n$$
$$= \sum_k \binom{n}{k_m}\binom{n-k_m}{k_{m-1}} \cdots \binom{n-k_m-\cdots-k_3}{k_2} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}. \tag{2}$$

By use of (2) or by mathematical induction over the number of terms, one obtains a form of the expansion which is easy to write down. Here is an example.

$$(u_1 + u_2 + u_3 + u_4)^3$$
$$= \binom{3}{0}\binom{3}{0}\binom{3}{0}u_1^3 + \binom{3}{0}\binom{3}{0}\binom{3}{1}u_1^2u_2 + \binom{3}{0}\binom{3}{0}\binom{3}{2}u_1u_2^2 + \binom{3}{0}\binom{3}{0}\binom{3}{3}u_2^3$$
$$+ \binom{3}{0}\binom{3}{1}\binom{2}{0}u_1^2u_3 + \binom{3}{0}\binom{3}{1}\binom{2}{1}u_1u_2u_3 + \binom{3}{0}\binom{3}{1}\binom{2}{2}u_2^2u_3$$
$$+ \binom{3}{0}\binom{3}{2}\binom{1}{0}u_1u_3^2 + \binom{3}{0}\binom{3}{2}\binom{1}{1}u_2u_3^2$$
$$+ \binom{3}{0}\binom{3}{3}\binom{0}{0}u_3^3$$

$$+\binom{3}{1}\binom{2}{0}\binom{2}{0}u_1^2u_4+\binom{3}{1}\binom{2}{0}\binom{2}{1}u_1u_2u_4+\binom{3}{1}\binom{2}{0}\binom{2}{2}u_2^2u_4$$

$$+\binom{3}{1}\binom{2}{1}\binom{1}{0}u_1u_3u_4+\binom{3}{1}\binom{2}{1}\binom{1}{1}u_2u_3u_4$$

$$+\binom{3}{1}\binom{2}{2}\binom{0}{0}u_3^2u_4$$

$$+\binom{3}{2}\binom{1}{0}\binom{1}{0}u_1u_4^2+\binom{3}{2}\binom{1}{0}\binom{1}{1}u_2u_4^2$$

$$+\binom{3}{2}\binom{1}{1}\binom{0}{0}u_3u_4^2$$

$$+\binom{3}{3}\binom{0}{0}\binom{0}{0}u_4^3.$$

The rule is: First write down the $m-1$ coefficients $\binom{n}{0}\binom{n}{0}\cdots\binom{n}{0}$. The formation of the coefficients of the first column is easily seen from the example. In each row the lower index of the last binomial coefficient increases by 1 until it is equal to the upper index of *that* binomial coefficient.

The literal factors are formed according to the rule given above.

II. The right side of the expansion of a multinomial may be used to obtain, in analogy with Leibniz's rule, an expression for the n-th derivative of a product of m factors. But then the exponents represent the orders of the derivatives, and the zero powers of the functions $u_1, u_2, \cdots$ must be considered as derivatives of order zero, i.e. the functions themselves. We use the example above to obtain $(u_1u_2u_3u_4)'''$. Calculating the binomial coefficients, performing the multiplications, and substituting derivatives for the exponents as explained above, we obtain:

$$\begin{aligned}(u_1u_2u_3u_4)''' = {} & u_1'''u_2u_3u_4+3u_1''u_2'u_3u_4+3u_1'u_2''u_3u_4+u_1u_2'''u_3u_4\\ & +3u_1''u_2u_3'u_4+6u_1'u_2'u_3'u_4+3u_1u_2''u_3'u_4+3u_1'u_2u_3''u_4\\ & +3u_1u_2'u_3''u_4+u_1u_2u_3'''u_4+3u_1''u_2u_3u_4'+6u_1'u_2'u_3u_4'\\ & +3u_1u_2''u_3u_4'+6u_1'u_2u_3'u_4'+6u_1u_2'u_3'u_4'+3u_1u_2u_3''u_4'\\ & +3u_1'u_2u_3u_4''+3u_1u_2'u_3u_4''+3u_1u_2u_3'u_4''+u_1u_2u_3u_4'''.\end{aligned}$$

The referee kindly supplied historical suggestions.

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A NEW VERSION OF THE EUCLIDEAN ALGORITHM

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Introduction. Given a set of positive integers (or elements of a Euclidean ring), $a_1, a_2, \cdots, a_n$, the Euclidean algorithm provides a method for computing the greatest common divisor, d, of these numbers. If the steps performed during the operation of the algorithm are traced back, it is possible to deduce elements $x_1, x_2, \cdots, x_n$ such that

$$d = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

In nearly all applications, e.g., for the Chinese problem of remainders (see [1]), for finding the inverse of an element of a Galois field, etc., it is desired to find the elements x_i. Although the process of untangling the steps of the algorithm to find the x_i is straightforward, anyone who has ever tried it will appreciate the difficulty of deriving the necessary formulas and carrying them out without error. This paper sets forth an algorithm which, although equivalent to the Euclidean, is much easier to visualize, is easily programmed either on paper or on a computer, and, in addition, produces the x's.

The Algorithm. We make use of the trick, well-known to the computing trade, of carrying along a matrix to keep track of the operations which have been performed. To prepare for the algorithm we first form an n by $n+1$ matrix whose first column consists of the positive integers $a_1, a_2, \cdots, a_n$ and the rest of which is the identity matrix, as follows:

$$\begin{matrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & 0 & \cdots & 0 \\ a_3 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & 0 & 0 & 0 & \cdots & 1. \end{matrix}$$

The algorithm consists of performing elementary row operations on this matrix so as to reduce all but one of the elements in the first column to zero. If we refer to the first element of a row (at any stage of the process) as the leader of that row, the algorithm may be formalized as follows:

Step 1. Select the row with the smallest nonzero leader and call it the "operator."

Step 2. Select any other row with a nonzero leader and call it the "operand." (When no operand can be found the process is completed.)

Step 3. Divide the leader of the operator into the leader of the operand, ignoring the remainder. Denote the quotient by q.

Step 4. Subtract q times the operator from the operand, recording the result as a new row and striking out the operand.

Step 5. Return to step 1.

When the process is completed (see step 2), the one remaining row whose leader is not zero will be

$$d,\ x_1,\ x_2,\ x_3,\ \cdots,\ x_n,$$

where

$$d = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

We now verify the algorithm. That the process terminates is easily seen by noting that every time step 4 is performed, a column leader decreases, but never becomes negative. Hence the sum of the column leaders is a strictly decreasing positive integer. Since it cannot decrease more than $\sum a_i$ times, the process must terminate.

We next note that elementary row operations (such as step 4) preserve the greatest common divisor of the leaders (or of any column); that is, g.c.d.$(b_1, b_2, \cdots, b_n)$ = g.c.d.$(b_1+\alpha b_j, b_2, b_3, \cdots, b_n)$ for any integer α and any $j \leqq n$ different from 1. When the last step is reached all the leaders are zero except that of the previous operand and that number must be the g.c.d. of the original set of leaders.

If we denote the matrix by (a, I) where a is a column vector consisting of the a_i, and I is the identity matrix, then since each elementary operation (including permutation of rows) is equivalent to multiplying by a nonsingular matrix, M_i, we have

$$\begin{aligned}\text{Final matrix} &= \cdots M_3M_2M_1(a, I)\\ &= (Ma, M),\end{aligned}$$

where M denotes the product of the matrices M_i. Thus the last n columns of the final matrix consist of the matrix M, and if the nonzero leader d occurs in say the jth row, and we let m_j be the jth row of M, then the above equation implies that

$$d = m_ja,$$

which was our contention. This completes the proof that the algorithm works. It is also useful to remember that a row of the matrix, at any stage of the algorithm, represents a linear equation relating the leader of that row to a linear combination of the original a's.

The following remarks, though mostly obvious, may help clarify what is going on, and make it evident that in the first column we are simply executing the Euclidean algorithm.

a. In step 1 the operator will always be the new row adjoined during the previous pass, unless its leader was zero.

b. In step 2, it will usually be most convenient to select the previous operator as the new operand.

c. In step 4, the leader of the new row will always be the remainder resulting from the division in step 3.

d. The process will still work for any selection of operand in step 1, provided only that its leader is not zero and that there is a row available with a larger leader. (Sometimes the work can be shortened by exercising judgment here).

e. In step 4, although the quotient q is the optimum multiplier to use and guarantees convergence, a different choice of multiplier does not cause an error, but perhaps lengthens the process.

An Example. We illustrate with $n=3$, $a_1=99$, $a_2=77$, $a_3=63$, and of course, $d=1$. In the following chart we have numbered the rows chronologically and show, for expository purposes, the ordinal numbers of the rows used to compute each new row. Instead of striking out old operands we have written x's in the column marked "validity."

Row	*Operator*	*Operand*	q	*Validity*				
1				x	99	1	0	0
2				x	77	0	1	0
3				x	63	0	0	1
4	3	2	1	x	14	0	1	-1
5	4	3	4	x	7	0	-4	5
6	5	4	2		0	0	9	-11
7	5	1	14		1	1	56	-70
8	7	5	7		0	-7	-396	495

Row 7 is to be interpreted to mean

$$1 = 1 \times 99 + 56 \times 77 - 70 \times 63,$$

as indeed it is.

Application to the Problem of Chinese Remainders. Given the residues $m_1, m_2, \cdots, m_n$ of a number, m, modulo a number of distinct primes, $p_1, p_2, \cdots, p_n$, respectively, it is required to find m. If we take $a_i = p_i^{-1}\Pi_1^n p_j$ and solve for the x_i as before, then $d=1$ and we have

$$1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Multiplying this equation by m and noting that $a_im = a_im_i$ (modulo Πp_j), we ob-

tain

$$m = a_1x_1m_1 + a_2x_2m_2 + \cdots + a_nx_nm_n \text{ (modulo } \Pi p_j).$$

In this case the value of x_i need only be known modulo p_i and this fact may be used throughout the algorithm by reducing all numbers in a given column modulo the appropriate prime. When this is done, an additional check on the calculations is provided in that any row with a zero leader must consist entirely of zeroes.

The example in the last section illustrates the Chinese problem of remainders when $p_1=7$, $p_2=9$, $p_3=11$. The result

$$1 = 99 + 56 \times 77 - 70 \times 63$$

implies that

$$1 = 99 + 2 \times 77 + 7 \times 63 \text{ (modulo 693)}$$

or

$$m = 99m_1 + 154m_2 + 441m_3 \text{ (modulo 693)}.$$

Now if we wished to know the smallest number m whose remainders modulo 7, 9, and 11 are, say, 6, 2, and 5, respectively, we simply substitute in the above formula:

$$m = 99 \times 6 + 154 \times 2 + 441 \times 5 = 3107 = 335 \text{ (modulo 693)}.$$

Application to Polynomial Rings. The algorithm works equally well for any Euclidean ring provided we interpret "smaller" and "larger" in terms of the ring's norm. Thus for polynomials over a field we would rephrase step 1 to read "Select the row with the leader of lowest degree and call it the operator." In addition, due to the freedom in the choice of a multiplier for step 4 (as noted in remark e), we can incorporate the division algorithm into our process, simply by replacing step 3 by "Divide the *leading term of the leader* of the operator into the *leading term of the leader* of the operand, calling the result q." Thus q will always be a monomial. If we represent a polynomial by the vector consisting of its coefficients, the algorithm goes very nicely, particularly if the coefficients are in GF (2), since step 4 will consist of merely shifting and adding.

The reader is invited to apply the algorithm to the following problem: Find the inverse of the polynomial x^3+x+1 modulo x^5+x^2+1, where the coefficients are taken modulo 2. Note that if these polynomials are relatively prime, the algorithm yields polynomials $p(x)$ and $q(x)$ such that $p(x)(x^3+x+1)+q(x)(x^5+x^2+1)=1$. Reading this modulo x^5+x^2+1 yields the interpretation

$$p(x) = (x^3 + x + 1)^{-1} \text{ modulo } x^5 + x^2 + 1.$$

Reference

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CONDITIONS THAT THE ZEROS OF A POLYNOMIAL LIE IN THE INTERVAL $[-1, 1]$ WHEN ALL ZEROS ARE REAL

HAROLD WILLIS MILNES, Land-Air, Point Mugu, California

The problem of determining conditions which the coefficients of a polynomial must satisfy if its zeroes are to lie within the unit circle in the complex plane has received considerable attention. It was solved quite generally by I. Schur, [1] and A. Cohn [2] who have given explicit relations to determine the precise number of roots which lie outside, inside and on the unit circle. These conditions apply to the problem in hand, which is less general and restricts the polynomial to one with all real zeroes. Since, however, the more general conditions depend upon the evaluation of a sequence of determinants which range in order to twice the degree of the polynomial, they are much more complicated than are really required in the simpler instance. In this note, necessary and sufficient conditions are presented for the simpler case which are, of course, equivalent to the known conditions, but which have the considerable advantage that they are linear in the coefficients of the polynomial and are likewise very easily derived. The advantages obtained more than justify this alternate treatment, since the usual conditions are formidably nonlinear and almost unusable as adjuncts to other analytical investigations.

Let

$$P(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \qquad (a_n > 0) \tag{1}$$

be the polynomial of degree n under consideration and let $r_1, r_2, \cdots, r_n$ be its zeroes, assumed to be all real. Symmetric functions $S_i(k)$ of the first k, $(k = 1, 2, \cdots, n)$ roots are defined by:

$$S_i(k) \equiv \sum (1 + r_\alpha)(1 + r_\beta) \cdots (1 + r_\lambda)(1 - r_\mu) \cdots (1 - r_\rho), \qquad (i = 0, 1, 2, \cdots, k) \tag{2}$$

where the subscripts $\alpha, \beta, \cdots, \rho$ represent rearrangements of the first k integers and where the summation extends over all possible partitions of the first k subscripts into two groups of i and $(k-i)$ numbers. Associated with the first group are i factors with a positive sign; with the second, $(k-i)$ factors with a negative sign. An example for $k=3$ may assist in clarifying the definition

$$\begin{aligned}
S_0(3) &= (1 - r_1)(r - r_2)(1 - r_3) \\
S_1(3) &= (1 + r_1)(1 - r_2)(1 - r_3) + (1 + r_2)(1 - r_3)(r - r_1) \\
&\quad + (1 + r_3)(1 - r_1)(1 - r_2) \\
S_2(3) &= (1 + r_1)(1 + r_2)(1 - r_3) + (1 + r_2)(1 + r_3)(1 - r_1) \\
&\quad + (1 + r_3)(1 + r_1)(1 - r_2) \\
S_3(3) &= (1 + r_1)(1 + r_2)(1 + r_3).
\end{aligned} \tag{3}$$

The following fundamental lemmas will be required:

LEMMA 1.

$$S_i(k) = (1 + r_k)S_{i-1}(k - 1) + (1 - r_k)S_i(k - 1), \qquad (i = 0, 1, 2, \cdots, k) \tag{4}$$

where $S_{-1}(k-1)$ *and* $S_k(k-1)$ *are defined to be zero.*

The demonstration is easily made and follows directly from the definition. If $i=0$ or $i=k$, the lemma is immediately verified simply by writing down the expressions for $S_0(k)$ or $S_k(k)$ and extracting the factor $(1-r_k)$ or $(1+r_k)$ as the case may be. For other values of i, consider first the terms in (2) involving $(1+r_k)$. Each term is seen to be composed of the symmetric sum of $(i-1)$ factors of the form $(1+r_\alpha)$ and $(k-i)$ factors of the form $(1-r_\alpha)$. The values of the subscript α belong to the integers $1, 2, \cdots, (k-1)$ and each term in the sum represents a partition of these numbers into two groups such that factors with positive sign have subscripts which belong to one group and factors with negative sign to the other. Since all possible partitions are represented, it follows that the coefficient of $(1+r_k)$ is $S_{i-1}(k-1)$. In a similar way it may be seen that the coefficient of $(1-r_k)$ is $S_i(k-1)$.

LEMMA 2.

$$\sum_{i=0}^{k} S_i(k) \equiv 2^k. \tag{5}$$

For $k=1$, the expression is

$$(1 - r_1) + (1 + r_1) = 2. \tag{6}$$

An inductive argument may be used for other values of k. Assume that lemma is true for values of the argument less than k; then by Lemma 1:

$$\begin{aligned} \sum_{i=0}^{k} S_i(k) &= (1 + r_k) \sum_{i=1}^{k} S_{i-1}(k - 1) + (1 - r_k) \sum_{i=0}^{k} S_i(k - 1) \\ &= (1 + r_k)2^{k-1} + (1 - r_k)2^{k-1} \\ &= 2^k. \end{aligned} \tag{7}$$

It is now possible to express the conditions that the zeroes of (1) all lie in the interval $[-1, 1]$ in terms of the symmetric functions (2), in accordance with the following theorem.

THEOREM. *Necessary and sufficient conditions that all zeroes* r_α, $(\alpha=1, 2, \cdots, n)$ *of the equation* (1) *should be bounded by* $-1 \leqq r_\alpha \leqq 1$, *are that the symmetric functions* (2) *satisfy the relation* $S_i(n) \geqq 0$, $(i=1, 2, \cdots, n)$.

The necessity is shown by noting that if all the zeroes are in the interval $[-1, 1]$ then the factors $(1+r_\alpha)$ and $(1-r_\alpha)$ are positive or zero. Therefore each summand in (2) is positive and $S_i(n) \geqq 0$ for all i.

To establish sufficiency, assume that $r_n > 1$ so that $(1-r_n) < 0$ and $(1+r_n) > 0$. With the aid of the lemma:

$$
\begin{aligned}
S_0(n) &= (1 - r)S_0(n-1) \geqq 0 \\
S_1(n) &= (1 + r_n)S_0(n-1) + (1 - r_n)S_1(n-1) \geqq 0 \\
S_2(n) &= (1 + r_n)S_1(n-1) + (1 - r_n)S_2(n-1) \geqq 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
S_{n-1}(n) &= (1 + r_n)S_{n-2}(n-1) + (1 - r_n)S_{n-1}(n-1) \geqq 0 \\
S_n(n) &= (1 + r_n)S_{n-1}(n-1) \geqq 0.
\end{aligned} \tag{8}
$$

Starting at the first relation of the list, we see that if the inequality is to hold, it must be that $S_0(n-1) \leqq 0$. If $S_0(n-1) < 0$, the second inequality requires $S_1(n-1) < 0$; if $S_0(n-1) = 0$, then $S_1(n-1) \leqq 0$. If $S_1(n-1) < 0$, the third inequality requires $S_2(n-1) < 0$; if $S_1(n-1) = 0$, then $S_0(n-1) = 0$ and $S_2(n-1) \leqq 0$. The same argument may be applied to each inequality in turn until the next to the last relation gives either: $S_{n-1}(n-1) < 0$ or else $S_0(n-1) = S_1(n-1) = \cdots = S_{n-2}(n-1) = 0$ with $S_{n-1}(n-1) \leqq 0$. The final relation leads to an immediate contradiction unless: $S_0(n-1) = S_1(n-1) = \cdots = S_{n-1}(n-1) = 0$. In the latter event a contradiction of Lemma 2 is obtained; therefore $r_n < 1$. Next, assume that $r_n < -1$ so that $(1-r_n) > 0$ and $(1+r_n) < 0$. Beginning at the bottom of the list of inequalities (8) and working to the top, employing similar arguments to those used previously, a second contradiction is obtained; thus $-1 \leqq r_n \leqq 1$. By merely interchanging subscripts between r_n and any other r, it follows that this r likewise lies in the same interval, since the symmetric functions $S_i(n)$ remain unchanged.

The theorem that has just been proved has slightly broader implications. For polynomials with real coefficients, the conditions $S_i(n) \geqq 0$ are sufficient to guarantee that all the real zeroes lie in the interval $[-1, 1]$. The conditions are not necessary in this case, however, as polynomials can be exhibited with all real zeroes in $[-1, 1]$, but which do not satisfy $S_i(n) \geqq 0$. If the conjugate complex zeroes satisfy $|r| \leqq 1$, then the conditions are again necessary and sufficient. Finally, if the polynomial $P^*(x)$ is constructed which has zeroes given by all possible product combinations (r_α, r_β) of the zeroes of the original polynomial, and if a set of conditions $S_j^*(n(n-1)/2) \geqq 0$, is described for this polynomial in accordance with the definition of the symmetric functions (2), then the combined relations

$$S_i(n) \geqq 0,\ S_j^*\left(\frac{n(n-1)}{2}\right) \geqq 0,\ \left(i = 0, 1, \cdots, n; j = 0, 1, 2, \cdots, \frac{n(n-1)}{2}\right)$$

ensure that both real and complex zeros of (1) will lie in the unit circle. The demonstrations will not be presented, as the simplicity of the results for the case of all real zeroes is lost and there is little true advantage over the classical theory of Schur in these cases.

The evaluation of the symmetric functions (2) in terms of the coefficients of the polynomial may be accomplished in the following way. From (2), it is first observed that if the expression is multiplied out and terms of the same degree in the products of the roots are collected together then:

$$(9)\quad S_i(k) = c_{i0} - c_{i1}\sum r_\alpha + c_{i2}\sum r_\alpha r_\beta - c_{i3}\sum r_\alpha r_\beta r_\gamma + \cdots + (-1)^n c_{in} r_\alpha r_\beta \cdots r_\rho,$$

where the c_{ij}'s are constants to be determined and where here the sums indicated extend over all possible combinations of the n subscripts: $\alpha, \beta, \cdots, \rho$, without repetitions: $1, 2, \cdots, (n-1)$ at a time. These symmetric sums are the fundamental symmetric functions of the roots of (1) and may be expressed in terms of the coefficients of (1) as

$$(10)\quad \begin{aligned} S_i(k) &\equiv c_{i0} + c_{i1}\frac{a_{n-1}}{a_n} + c_{i2}\frac{a_{n-2}}{a_n} + \cdots + c_{in}\frac{a_0}{a_n} \\ &= \frac{1}{a_n}\left[c_{i0}a_n + c_{i1}a_{n-1} + c_{i2}a_{n-2} + \cdots + c_{in}a_0\right]. \end{aligned}$$

This identity holds for all polynomials and therefore by selecting $(n+1)$ different polynomials with known roots and evaluating $S_i(k)$ and the a's numerically in each case, a set of $(n+1)$ linear equations can be constructed for determining the c's. The most appropriate choice is apparently the set of polynomials which have successively n roots all zero, $(n-1)$ roots zero and 1 root unity, $(n-1)$ roots zero and 2 roots unity, etc., and, finally, all roots unity. This reduces the coefficient matrix for determining the c's to diagonal form and likewise makes the evaluation of $S_i(k)$ easy.

For $n=2, 3, 4, 5, 6$, the following inequalities have been determined numerically:

$$(11)\quad \begin{aligned} n = 2:&\quad a_2 + a_1 + a_0 \geqq 0, \quad a_2 - a_0 \geqq 0, \quad a_2 - a_1 + a_0 \geqq 0; \\ n = 3:&\quad a_3 + a_2 + a_1 + a_0 \geqq 0, \quad 3a_3 + a_2 - a_1 - 3a_0 \geqq 0, \\ &\quad 3a_3 - a_2 - a_1 + 3a_0 \geqq 0, \quad a_3 - a_2 + a_1 - a_0 \geqq 0; \\ n = 4:&\quad a_4 + a_3 + a_2 + a_1 + a_0 \geqq 0, \quad 2a_4 + a_3 - a_1 - 2a_0 \geqq 0, \\ &\quad 3a_4 + a_2 + 3a_0 \geqq 0, \quad 2a_4 - a_3 + a_1 - 2a_0 \geqq 0, \\ &\quad a_4 - a_3 + a_2 - a_1 + a_0 \geqq 0; \end{aligned}$$

$$\begin{aligned} n=5:&\quad a_5+a_4+a_3+a_2+a_1+a_0 \geqq 0, \quad 5a_5-a_4-a_3+a_2+a_1-5a_0 \geqq 0, \\ &\quad 5a_5-3a_4+a_3+a_2-3a_1+5a_0 \geqq 0, \quad 5a_5+3a_4+a_3-a_2-3a_1-5a_0 \geqq 0, \\ &\quad 5a_5+a_4-a_3-a_2+a_1+5a_0 \geqq 0, \quad a_5-a_4+a_3-a_2+a_1-a_0 \geqq 0; \end{aligned}$$

$$\begin{aligned} n=6:&\quad a_6+a_5+a_4+a_3+a_2+a_1+a_0 \geqq 0, \quad 5a_6-a_4+a_2-5a_0 \geqq 0, \\ &\quad 15a_6-5a_5-a_4+3a_3-a_2-5a_1+15a_0 \geqq 0, \quad 3a_6-2a_5+a_4-a_2+2a_1-3a_0 \geqq 0, \\ &\quad 3a_6+2a_5+a_4-a_2-2a_1-3a_0 \geqq 0, \quad 15a_6+5a_5-a_4-3a_3-a_2+5a_1+15a_0 \geqq 0, \\ &\quad a_6-a_5+a_4-a_3+a_2-a_1+a_0 \geqq 0. \end{aligned}$$

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MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

SCHOLASTIC APTITUDE TESTS IN MATHEMATICS

EARL A. CODDINGTON, University of California, Los Angeles

Introduction. In July, 1961, I became chairman of a faculty committee whose principal responsibility is to recommend the award of all undergraduate scholarships at the University of California, Los Angeles (UCLA). Soon after, I began to wonder how to predict academic success for a group of scholarship applicants, and, in particular, how to weigh an applicant's previous academic record with his scores on the College Entrance Examination Board Scholastic Aptitude Test (SAT). Two years ago, as an admission requirement, the University of California began requiring students to submit scores on this test, which has two parts, the verbal section (SAT Verbal) and the mathematics section (SAT Math.). Although no direct use of the test scores was made for determining admission, the statewide University was evaluating these results for a possible additional admission requirement. However, since these studies were just beginning in the summer of 1961, and since I had another aim in mind, I decided to make a small investigation of my own. The results of this were so surprising to me that I was led to carry out a number of other studies. I think that the results, particularly as they concern the SAT Math., will be of interest to mathematicians.

The Results. An initial analysis was made of 77 students who entered UCLA from high schools in the fall of 1960 with scholarships. In this study an attempt was made to predict the first year UCLA grade point average (UCLA GPA),

FACILITIES FOR MATHEMATICS

Buildings and Facilities for the Mathematical Sciences is the title of a report published in June, 1963 by the Conference Board of the Mathematical Sciences (CBMS) with the financial support of the Educational Facilities Laboratories, Inc. (EFL) and the technical assistance of the Columbia University Press Printing Office. The report of over 160 pages includes photographs and line drawings and summarizes the findings and recommendations gathered in a one year study of mathematical buildings and facilities sponsored by CBMS, supported by the EFL, and directed by J. S. Frame with the assistance of McLeod and Ferrara architects, and the following advisory committee of mathematicians: Wallace Givens, Chairman, C. B. Allendoerfer, Mrs. J. G. Herriot, D. A. Johnson, G. E. Nicholson, Jr., C. E. Rickart, and E. P. Vance.

Complimentary copies of the report are being sent upon publication to a list of persons including college and university presidents and mathematics and statistics department chairmen. Other interested persons may request copies by writing Conference Board of the Mathematical Sciences, 308 Mills Building, 17th and Penna. Ave. N.W., Washington 6, D. C. Questions other than requests for copies may be addressed to the project director, Professor J. S. Frame, Mathematics Department, Michigan State University, East Lansing, Michigan.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
COLLABORATING EDITOR: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1597 (corrected). *Proposed by Ralph Greenberg, University of Pennsylvania*

Let $E_1(x_1)=x_1$, $E_2(x_1, x_2)=x_1^{x_2}$, $\cdots$, $E_n(x_1, x_2, \cdots, x_n)=x_1^{E_{n-1}(x_2,x_3,\cdots,x_n)}$, and let $a_n>a_{n-1}>\cdots>a_1>e$. Which permutation of the a_i's maximizes $E_n(x_1, x_2, \cdots, x_n)$?

E 1611. *Proposed by Peter Ungar, Courant Institute, New York University*

Magistrate: How do you know he had been speeding?

Officer: His car skidded 30 ft. with all four wheels locked, going up a 30° slope. I later made a test on the road in front of this courthouse, which is paved

with the same material. I slammed on the brakes of the defendant's car at 60 mph and it skidded to a halt also in exactly 30 ft. But this road is level. Obviously he had greatly exceeded the speed limit of 60 mph going up the hill.

Magistrate (after consulting his slide rule): The charge of speeding is dismissed.

Explain the magistrate's decision.

E 1612. *Proposed by C. S. Ogilvy, Hamilton College*

The exam question was: If $xy=4$, find d^2y/dx^2 and d^2x/dy^2. The student found d^2y/dx^2 correctly; he then wrote down its reciprocal and labeled that d^2x/dy^2—and it was right. The professor claimed that this was just luck, and that it could not happen with any other function. Was the professor right?

E 1613. *Proposed by F. M. Sioson, University of Hawaii*

Towns T_i $(i=1, 2, \cdots, n)$ are cyclically connected by a one-way road, thus forming an n-gon. There are n means of transportation, $m_1, m_2, \cdots, m_n$, available, and under the ith means of transportation a mile is travelled in a_i minutes. If one starts from town T_j and goes around the circuit using the different means of transportation $m_1, m_2, \cdots, m_n$ consecutively, changing transportations only after each town, the circuit is completed in $\sum_{i=1}^{n} a_i - 2a_j$ hours. Find the length of the circuit.

E 1614. *Proposed by R. T. Hood, Franklin College, Franklin, Indiana*

A plane is intersected by a right circular cone and its axis. The resulting conic is projected onto a plane perpendicular to the axis. Show that the axis passes through a focus of the projected conic.

E 1615. *Proposed by M. W. Pownall, Colgate University*

Define the distance between two integers i and j to be $d(i, j) = |i-j|$. Show that any one-to-one mapping of the set of integers onto itself which preserves the relation $d(i, j) \leqq k$, for some fixed positive integer k, is an isometry.

E 1616. *Proposed by Leonard Carlitz, Duke University*

Show that in an acute triangle, $h_1+h_2+h_3 \leqq 3(R+r)$, where the h_i are the altitudes, R the circumradius, and r the inradius, and show that the equality sign holds only in the case of an equilateral triangle.

E 1617. *Proposed by J. I. Nassar, Socony Mobil Oil Company, Inc., Paulsboro, New Jersey*

Let A, B, C, D be any four points in the plane, and let PQ be the line segment joining the midpoints of BC and AD. Show that $|AB-CD| \leqq 2PQ \leqq AB+CD$, where one of the equalities holds if and only if AB is parallel to (or collinear with) CD.

E 1618. *Proposed by Ralph Greenberg, University of Pennsylvania*

Find all integers n such that the $\phi(n)$ integers less than n and prime to n are: (a) relatively prime in pairs, (b) in arithmetic progression.

E 1619. *Proposed by H. Kestelman, University College, London*

Let f be any function with integer values, and A any n by n matrix whose (r, s)th element is zero whenever $f(r)+f(s)$ is even. Prove that A and $-A$ are similar.

E 1620. *Proposed by D. L. Silverman, Beverly Hills, California*

Solve $x^3=4y(xy+z^2)$ in nonzero integers x, y, z.

SOLUTIONS

Friday the 13th's

E 1541 [1962, 919]. *Proposed by G. C. Bush, Queen's University, Kingston, Canada*

Find the maximum and minimum number of "Friday the 13th's" that can occur in a year.

Solution by C. V. Heuer, University of Nebraska. Define an equivalence relation $\equiv$ on the set of the months of the year by: month $A \equiv$ month B if and only if the 1st of month A is on the same day of the week as the 1st of month B. Then for any nonleap year we have the following equivalence classes: {Jan, Oct}, {Feb, Mar, Nov}, {Apr, July}, {May}, {June}, {Aug}, {Sept, Dec}. For a leap year the equivalence classes become: {Jan, Apr, July}, {Feb, Aug}, {Mar, Nov}, {Sept, Dec}, {May}, {June}, {Oct}. From this it is clear that the maximum number of Friday the 13th's is three and the minimum is one for any given year.

Also solved by Jack Abad, R. H. Anglin, Marc Aronson, W. T. Bailey, Leon Bankoff, Merrill Barnebey, E. D. Bender, Jeanette Bickley, Walter Bluger, J. B. Bohal, D. A. Breault, Norman Brenner, Brother T. C. Wesselkamper, B. H. Brown, A. W. Brunson, Grace A. Bush, R. L. Carmichael, F. Cartuyvels, R. L. Caskey, J. R. Caspar, D. I. A. Cohen, R. J. Cormier, D. R. Daluge, E. S. Eby, S. J. Einhorn, Stephen Fisk, David Forslund, Anton Glaser, Michael Goldberg, Jay Gottesfeld, Ralph Greenberg, G. R. Hagan, Ned Harrell, Geoffrey Horrocks, A. R. Hyde, J. E. Jean, Jr., Leslie Katz, A. J. Keeping, Sidney Kravitz, Harry Langman, Betty Levine, Thomas Linton, D. C. B. Marsh, Otto Mond, Stephen Montague, P. N. Muller, J. B. Muskat, Herbert Nadler, C. C. Oursler, Bart Park, R. R. Perez, Stanton Philipp, J. L. Pietenpol, R. H. Ransom, David Rothman, S. J. Ryan, Arnold Singer, Gerald Strahs, Gary Venter, Andy Vince, Robert Winter, Samuel Wolf, and the proposer.

Bankoff pointed out that a solution to this problem occurs in connection with the solution to Problem 160, Mathematics Magazine, Sept.-Oct. 1953, pp. 55–6. Hagan remarked that if "a year" may be any 12 month period, then the maximum is four (September 1st a Sunday into a year with 28 days in February) and the minimum is zero (September 1st a Wednesday into a year with 28 days in February). Attention was called to Problem E 36, in which B. H. Brown established that the 13th of the month is more likely to be a Friday than any one of the other days of the week.

Jordan's Generalization of the Euler ϕ-Function

E 1542 [1962, 919]. *Proposed by Azriel Rosenfeld, Yeshiva University*

Find the number of n-tuples $(x_1, \cdots, x_n)$, $0 \leqq x_i < m$, such that the greatest common divisor $(x_1, \cdots, x_n, m) = 1$.

Solution by N. J. Fine, University of Pennsylvania. This number, say $N_n(m)$, is C. Jordan's generalization of the Euler ϕ-function to which it reduces for $n=1$. (See C. Jordan, *Traité des Substitutions*, Paris, 1870, pp. 95–7, or Dickson's *History*, vol. 1, p. 147.) Its value is given by

$$N_n(m) = m^n \prod_{p|m} (1 - p^{-n}), \tag{1}$$

or by the equivalent

$$N_n(m) = \sum_{d|m} \mu(d)(m/d)^n. \tag{2}$$

To prove (2), for example, we classify the m_n n-tuples according to $d = (x_1, \cdots, x_n, m)$. The number having a given d is $N_n(m/d)$, since

$$(x_1/d, \cdots, x_n/d, m/d) = 1$$

and $0 \leqq x_i/d < m/d$. Thus

$$m^n = \sum_{d|m} N_n(m/d) = \sum_{d|m} N_n(d).$$

Applying the Möbius inversion formula, we get (2).

Also solved by Marjorie Bicknell, D. I. A. Cohen, E. W. Ewing, Ralph Greenberg, A. J. Keeping, M. S. Klamkin, D. C. B. Marsh, Stephen Montague, C. P. Seguin, Dennis Travis, and the proposer.

The problem, with an unimportant change and generous hints for a solution, was located in W. J. LeVeque, *Topics in Number Theory*, vol. 1, Prob. 4, p. 89.

Covering a Rectangle with L-tetrominoes

E 1543 [1962, 920]. *Proposed by S. W. Golomb, Jet Propulsion Laboratory, California, Institute of Technology*

Find a necessary and sufficient condition that an $a \times b$ rectangle can be exactly covered (completely, and without overlaps) with "L-tetrominoes," i.e., with shapes composed of four unit squares like that formed by consecutively joining the points (0, 0), (3, 0), (3, 1), (1, 1), (1, 2), (0, 2), (0, 0) of a rectangular cartesian coordinate system.

Solution by D. A. Klarner, Humboldt State College, Arcata, California. Since 4 divides ab, a may be taken even. Let $a/2$ alternate rows of b squares each be colored black in the rectangle. Then every L-tetromino in the covering must cover three squares of one color and one square of the other. If m L-tetrominoes cover three black squares and n L-tetrominoes cover three white squares, then

$3m+n=ab/2=3n+m$; hence $m=n$. This means that the covering must use an even number of L-tetrominoes, and hence that 8 divides ab. Except for the $1\times 8k$ rectangle, every rectangle of area $8k$ can be partitioned into exhaustive, disjoint rectangles of dimensions 2×4 and/or 3×8, but both the 2×4 and 3×8 rectangle can be packed with L-tetrominoes in an obvious way. Hence, the necessary and sufficient conditions are that a and b be greater than 1 and $ab=8k$.

Also solved by S. B. Akers, Jr., Walter Bluger, Norman Brenner and D. I. A. Cohen (jointly), Michael Goldberg, A. J. Keeping, Harry Langman, J. G. Mauldon, G. W. Miller, Lewis Parker and Robert Spira (jointly), S. J. Ryan, Andy Vince, R. E. Winter, and the proposer.

Langman pointed out that this problem appears, with solution, on pp. 42–3 of his book *Play Mathematics* (Stechert-Hafner, 1961). Mauldon remarked that if the L-tetrominoes may not be turned over, then we must add the condition $a, b\neq 3$.

Editorial Note. What is the corresponding result for T-tetrominoes?

Expectation in a Solitaire Game

E 1544 [1962, 920]. *Proposed by H. W. Hickey, Alexandria, Virginia*

A solitaire player shuffles a deck of cards numbered from 1 to n inclusive, and lays them face up, one at a time. Every time he finds a card whose number is greater than any card turned up previously, he scores one point (the first card automatically gives him a point, whatever it is). What is the expectation of his score, assuming all arrangements of the deck are equiprobable?

I. *Solution by Arnold Singer, Institute of Naval Studies, Cambridge, Mass.* Let $E(n)$ be the player's expectation. Then $E(n)$ satisfies the equation

$$E(n+1) = E(n) + 1/(n+1),$$

since the $(n+1)$st card scores one point only if it is numbered "$n+1$." Repeated substitution yields

$$E(n) = 1 + 1/2 + 1/3 + \cdots + 1/n.$$

II. *Solution by D. L. Silverman, National Security Agency, Ft. Meade, Md.* Card k must precede the $n-k$ larger cards in order to be scored. This event has probability $1/(n-k+1)$, and the expected score is therefore

$$\sum_{k=1}^{n} 1/(n-k+1) = \sum_{k=1}^{n} 1/k.$$

III. *Solution by M. S. Klamkin, State University of New York at Buffalo.* Consider the set of $n!$ permutations. The number of points scored from the first card will be $n!$. The number of points scored from the second card will be the number of permutations in which the second card is greater than the first, or $n!/2$. Similarly, the number of points scored for the rth card will be $n!/r$. Thus the expected score is

$$E(n) = 1 + 1/2 + 1/3 + \cdots + 1/n.$$

Also solved by Jack Abad, A. F. Abrahamse, J. P. Ballantine, Walter Bluger, Norman Brenner, David Brillinger and Roger Pinkham (jointly), D. I. A. Cohen, N. J. Fine, F. E. Fischer, J. E. Freund, C. M. Frye, W. W. Funkenbusch, Gene Glass, Michael Goldberg, Jay Gottesfeld, Ralph Greenberg, Henry Harmeling, Jr., Geoffrey Horrocks, A. R. Hyde, R. A. Jacobson, A. J. Keeping, Harry Lass and C. B. Solloway (jointly), Brockway McMillan, D. C. B. Marsh, Leo Moser, J. B. Muskat, Bart Park, Walter Penney, J. L. Pietenpol, David Rothman, Perry Scheinok, Michael Skalsky, Kirk Stewart, Julius Vogel, and the proposer.

Editorial Note. Most solvers found the above answer, which is based upon the interpretation of "any" as "every." If the interpretation of "any" is "at least one," the answer can be shown to be

$$E(n) = 1 + \sum_{j=2}^{n} (j-1)/j = n - \sum_{j=2}^{n} 1/j.$$

A Telescoping Series

E 1545 [1962, 920]. *Proposed by Y. Matsuoka, University of St. Andrews, Scotland*

Let p be a positive integer. Prove that

$$\sum_{n=1}^{\infty}\left\{(n+p)\Big/\prod_{k=0}^{2p}(n+k)^2\right\} = 1/4p\{(2p)!\}^2.$$

Solution by N. J. Fine, University of Pennsylvania. Let A_n be the nth term of the series, and let

$$B_n = \{n^2(n+1)^2\cdots(n+2p-1)^2\}^{-1}.$$

Then for $n \geqq 1$, $B_n - B_{n+1} = 4pA_n$. Summing, we get

$$1/\{(2p)!\}^2 = B_1 = \sum_{n\geq 1}(B_n - B_{n+1}) = 4p\sum_{n\geq 1} A_n.$$

Also solved by Nadhla Abdul-Halim, A. N. Aheart, Norman Brenner and D. I. A. Cohen (jointly), Leonard Carlitz, H. L. Chow, Ralph Greenberg, S. H. Greene, F. W. Herlihy, Geoffrey Horrocks, Erwin Just and Norman Schaumberger (jointly), A. J. Keeping, M. S. Klamkin, T. J. Lee, D. C. B. Marsh, P. N. Muller, Walter Penney, Stanton Philipp, D. Ramakotaiah and D. Suryanarayana (jointly), Perry Scheinok, Michael Skalsky, F. C. Smith, J. E. Vinson, and the proposer.

A Sequence of Functions with no Convergent Subsequence

E 1546 [1962, 920]. *Proposed by N. V. Glick, North American Aviation, Inc.*

Consider the sequence of functions $f_n(x) = nx - [nx]$ defined on any nondegenerate interval $[a, b]$. Prove that no subsequence converges.

Solution by N. J. Fine, University of Pennsylvania. We obtain the result as a special case of the following

THEOREM. *Let f be a bounded integrable function of period 1, and suppose that f is not equivalent to a constant. Then it is impossible for a subsequence of $\{f(nx)\}$ to converge on a nondegenerate interval $[a, b]$.*

We have, for any interval (α, β),

$$\int_\alpha^\beta f(nx)dx = (1/n)\int_{n\alpha}^{n\beta} f(x)dx$$

$$= (1/n)\left\{\sum_{j=[n\alpha]}^{[n\beta]-1}\int_j^{j+1} f(x)dx + \int_{[n\beta]}^{n\beta} f(x)dx - \int_{[n\alpha]}^{n\alpha} f(x)dx\right\}$$

$$= (1/n)([n\beta] - [n\alpha])\int_0^1 f(x)dx + o(1)$$

$$= (\beta - \alpha)\int_0^1 f(x)dx + o(1), \qquad (n \to \infty).$$

That is,

$$\lim_{n\to\infty}\int_\alpha^\beta f(nx)dx = (\beta - \alpha)K, \tag{1}$$

where $K = \int_0^1 f(x)dx$.

Now suppose that $f(n_\nu x) \to g(x)$ for all $x \in [a, b]$. By the bounded convergence theorem, if $[\alpha, \beta] \subset [a, b]$,

$$\int_\alpha^\beta g(x)dx = \lim_{\nu\to\infty}\int_\alpha^\beta f(n_\nu x)dx = (\beta - \alpha)K.$$

Therefore

$$\int_\alpha^\beta (g(x) - K)dx = 0.$$

Because α, β are arbitrary in $[a, b]$, this implies that $g(x) = K$ a.e. in $[a, b]$. Now apply (1) to $|f(x) - K|$:

$$(b - a)\int_0^1 |f(x) - K|\,dx = \lim_{\nu\to\infty}\int_a^b |f(n_\nu x) - K|\,dx = 0,$$

again by the bounded convergence theorem. Therefore $f(x) = K$ a.e. in $[0, 1]$, and f is equivalent to a constant. This contradiction proves the theorem. The special case, of course, is $f(x) = x - [x]$.

Also solved by J. R. Caspar, Ralph Greenberg, and the proposer.

A Locus Yielding a Zeeman Quadrilateral

E 1547 [1962, 920]. *Proposed by Simon Vatriquant, Brussels, Belgium*

Find the locus of a point P such that the tangents issued from it to a given parabola ω form with the axis and a given line l a Zeeman quadrilateral (i.e., a quadrilateral one side of which is parallel to the Euler line of the triangle formed by the remaining three sides).

Solution by D. C. B. Marsh, Colorado School of Mines. Let P_1 be the point on the axis of ω one-third the way from the vertex to the directrix; let the perpendicular to l from the focus of ω intersect the directrix of ω at P_2. The line determined by P_1P_2 is the desired locus.

Proof. Referred to a rectangular coordinate system, a quadrilateral is a Zeeman quadrilateral if and only if the slopes, $m_i (i=1, 2, 3, 4)$, of its sides satisfy

$$\sum_{1 \leq i < j \leq 4} m_i m_j + 3\left(1 + \prod_{i=1}^{4} m_i\right) = 0. \tag{1}$$

If we let ω be $y^2 = 2px$ and assume l has slope λ, we compute the slopes of the tangents to ω from (x_0, y_0) and apply the above condition to obtain

$$6x_0 + 2\lambda y_0 + p = 0,$$

which is the line described above (relative to ω and l, independent of axes).

Also solved by Sister M. Stephanie, Roscoe Woods, and the proposer.

Editorial Note. For a proof of condition (1) above see Mathesis, LXX, Note 1, R. Deaux, 64.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

COLLABORATING EDITORS: L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; and A. WILANSKY, Lehigh University.

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editors. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5121. *Proposed by J. B. Kruskal, Bell Telephone Laboratories, Murray Hill, N. J.*

By definition, let $G_1 > G_2$ if group G_1 contains a subgroup isomorphic to G_2, but G_2 does not contain a subgroup isomorphic to G_1. Can an infinite descending sequence $G_1 > G_2 > G_3 > \cdots$ exist where G_1 is finitely generated?

5122. *Proposed by Harley Flanders, Purdue University*

Let k be a field of characteristic p and let $f(x)$ be an irreducible polynomial in $k[x]$ in the variables $x = (x_1, \cdots, x_r)$. Form the polynomial $g(x)$ by $\{g(x)\}^p = f(x^p)$, and the field F obtained by adjoining to k all coefficients of g. Prove that $g(x)$ is irreducible in $F[x]$.

5123. *Proposed by R. W. Newcomb, Stanford, California*

Prove that the set of infinitely differentiable functions of support bounded on the left is dense in the set of distributions of support bounded on the left.

5124. *Proposed by Fred Suvorov, Princeton University*

Give an example of a Banach algebra such that each element is a sum of products, but which also contains an element which is not a product.

5125. *Proposed by Richard Bellman, The RAND Corporation, Santa Monica, California*

It is easy to determine the minimum number of multiplications required to generate a^N from a. What is the minimum number required to generate $a^M b^N$ starting with a and b?

5126. *Proposed by Erwin Just, Bronx Community College, New York*

Prove that there exists an uncountable closed subset of the irrationals in $[0, 1]$. More generally, does every uncountable set of reals contain an uncountable closed subset?

5127. *Proposed by Alain Etcheberry, Universidad de Chile, Santiago*

Let $S=\{a_1, a_2, \cdots\}$ be a countable, infinite set. Does there exist a strictly increasing and uncountable chain of subsets of S? (The problem is suggested by a comment of J. E. Hafstrom [1962, 249].)

5128. *Proposed by E. A. Franz, Culver-Stockton College, Canton, Mo.*

For a given positive integer n consider the function $f(x)=x^n$. If the domain of f is $\{0, 1, 2, \cdots, n-1\}$ and the arithmetic is mod n, what is the range of f?

5129. *Proposed by D. S. Mitrinovic, Belgrade, Yugoslavia*

I. Let π be a plane, O a point of π, and $L_k(k=1, 2, \cdots, s)$ a finite sequence of lines passing through O and lying in π. Let $E_k=\{S_k^1, S_k^2, \cdots, S_k^{n_k}\}$ be a set of n_k points of line L_k.

(a) How many triangles are there whose vertices belong to the set

$$(1)\ E_1 \cup E_2; \qquad (2)\ E_1 \cup E_2 \cup E_3; \qquad (3)\ E_1 \cup E_2 \cup \cdots \cup E_s?$$

(b) How many triangles are there whose vertices are O and two other points belonging to two distinct lines E_k?

II. Let $L_k(k=1, 2, \cdots, s)$ be s lines passing through O such that no three lines are coplanar. Let E_k have the same significance as in I above.

(a) Determine the cardinal number of the set of tetrahedrons whose vertices belong to

$$(1) E_1 \cup E_2 \cup E_3; \qquad (2)\ E_1 \cup E_2 \cup \cdots \cup E_s \qquad (s \geqq 4).$$

(b) How many tetrahedrons are there whose vertices are O and three points belonging to three distinct E_k?

III. Generalize the problem for space of $N(\geqq 4)$ dimensions.

5130. *Proposed by B. L. Osofsky, Douglass College, Rutgers, The State University*

Let G be a finite group in which any two elements that have the same period are conjugate. If G has a nontrivial character of degree $\leqq 3$, then G is either S_2 or S_3.

SOLUTIONS

Geodesics on a Polyhedron

4982 [1961, 673]. *Proposed by G. Di Antonio, Fresno State College*

On any of the five regular solids, let two points be given, not both on the same face. Determine the geodesic between the two points.

Solution by Michael Goldberg, Washington, D. C. On the plane of a development of the polyhedron, a geodesic is a straight line which traverses one or more of the faces of the polyhedron. The development can be performed, however, in several ways. For each development, the straight line joining the two given points is one of the sought geodesics. Among these there is, in general, a shortest one.

Length of a Graph

5023 [1962, 317]. *Proposed by D. J. Newman, Yeshiva University*

Let $P(z)$ be a nonconstant polynomial and for each positive number t define $L(t)$ to be the length of the curve $|P(z)|=t$. Prove that $L(t)$ is an increasing function.

Editorial Note. No solution or discussion has been received. In reply to a query from the editor, the proposer states that he has no proof and that he now has grave doubts as to the correctness of the original notion.

Jordan's Generalized Totient

5037 [1962, 670]. *Proposed by A. Mąkowski, Warsaw, Poland*

Let $J_k(n)$ denote Jordan's generalization of Euler's totient function, and let $(x_1, \cdots, x_p)$ be the greatest common divisor of $x_1, \cdots, x_p$. Prove that

$$J_k(n) = n^k + \sum (-1)^l (a_1, \cdots, a_l, n)^k,$$

summed for all systems of positive integers $a_1, \cdots, a_l$ such that $a_1 < a_2 < \cdots < a_l < n$. ($J_k(n)$ is the number of incongruent sets $\{a_1, \cdots, a_k\}$ mod n such that $(a_1, \cdots, a_k, n) = 1$.)

Solution by Leonard Carlitz, Duke University. Let $f_r(n)$ denote the number of sets of integers $a_1, \cdots, a_r$ such that

$$a_1 < \cdots < a_r < n, \; (a_1, \cdots, a_r, n) = 1.$$

Since the number of sets of integers $a_1, \cdots, a_r$ such that $a_1 < \cdots < a_r < n$ is equal to $\binom{n-1}{r}$, it follows by a familiar argument that

$$\sum_{d|n} f_r(d) = \binom{n-1}{r},$$

so that

$$f_r(n) = \sum_{ab=n} \mu(a)\binom{b-1}{r}.$$

Now, on the other hand, it is clear that

$$\begin{aligned} n^k + \sum_{r=1}^{n-1}(-1)^r \sum_{a_1<\cdots<a_r<n} (a_1, \cdots, a_r, n)^k &\\ = n^k + \sum_{r=1}^{n-1} (-1)^r \sum_{cd=n} c^k f_r(d) &\\ = n^k + \sum_{r=1}^{n-1} (-1)^r \sum_{cd=n} c^k \sum_{ab=d} \mu(a)\binom{b-1}{r} &\\ = \sum_{r=0}^{n-1} (-1)^r \sum_{abc=n} \mu(a)\binom{b-1}{r} c^k &\\ = \sum_{abc=n} \mu(a)c^k \sum_{r=0}^{n-1} (-1)^r \binom{b-1}{r}. & \end{aligned}$$

The inner sum vanishes unless $b=1$; thus we get $\sum_{ac=n}\mu(a)c^k$, which is indeed equal to $J_k(n)$.

Also solved by A. Schinzel.

Division Ring

5038 [1962, 670]. *Proposed by G. H. Meisters, University of Nebraska*

Let Δ denote a division ring. Show that $a+b\neq 0$ is a necessary and sufficient condition that $ax+xb=c$ has a unique solution x in Δ for each c in Δ, if and only if Δ is a field.

Solution by Harlan Stevens, Pennsylvania State University. Suppose Δ is a field (commutative division ring). Then it is familiar that $ax+xb=(a+b)x=c$ has a unique solution if and only if $a+b\neq 0$. Conversely, let $a=gh$, $b=-hg$ and $c=0$, where g and h are arbitrary nonzero members of the division ring Δ. Then $x=0$ and $x=g$ are distinct solutions of $ghx-xhg=0$, which implies that $a+b=gh-hg=0$; thus Δ is commutative.

Also solved by Elizabeth Appelbaum, J. R. Baugh, Jerry W. Brown, Robert B. Brown, R. D. Byrd, P. M. Gibson, Jiang Luh, Barbara L. Osofsky, Stanton Philipp, M. F. Smiley, B. R. Toskey, Seth L. Warner, W. L. Waterhouse, J. Ernest Wilkins, Jr., and the proposer.

Normal Subgroup of a Semigroup

5039 [1962, 670]. *Proposed by Seth Warner, Duke University*

Let G be a semigroup. If H is a subgroup of G, define xR_Hy (respectively, xL_Hy) to mean x and y belong to the same right (left) coset of H. A subgroup H is called normal if the identity element of H is an identity element of G, and if $xH=Hx$ for all elements x of G. An equivalence relation R on E and a composition on E are called compatible if xRy and uRv imply $xuRyv$. Prove that R_H and L_H are equivalence relations on G compatible with the composition of G if and only if H is a normal subgroup of G.

An equivalence relation R on G is called trivial either if xRy means $x=y$ or if xRy for all elements x, y of G. Prove that if G has an identity element and satisfies the left and right cancellation laws, then every nontrivial equivalence relation on G compatible with the composition of G is defined by a normal subgroup if and only if G is a group.

Solution by B. R. Toskey, Seattle University. Suppose that R_H and L_H are equivalence relations on G compatible with the composition of G, let 1 be the identity element of H and let x be an arbitrary element of G. Since xR_Hx and xL_Hx, x belongs to a right coset Hy and a left coset zH for some y, z in G. Thus $x=h_1y=zh_2$ for some h_1, h_2 in H. Therefore $1\cdot x=1\cdot h_1y=h_1y=x$ and $x\cdot 1=zh_2\cdot 1=zh_2=x$, and hence 1 is an identity for G, and x belongs to xH and to Hx. If h is in H, then $1R_Hh$ and $1L_Hh$, so that, using the compatibility condition, we have $xR_H(hxh)$ and $xL_H(hxh)$, and therefore $hxh=h_3x=xh_4$ for some h_3, h_4 in H. From this we obtain $xh=h^{-1}h_3x$ and $hx=xh_4h^{-1}$, so that $xH=Hx$. Hence H is a normal subgroup.

Conversely, suppose that H is a normal subgroup of G. By definition we have xR_Hy implies yR_Hx. Since $x=1\cdot x$, x is in Hx for all x in G, so that xR_Hx. If xR_Hy and yR_Hz, we may let $y=h_1x=h_2z$ for some h_1, h_2 in H. Hence $x=h_1^{-1}h_2z$, so that xR_Hz. Thus R_H is reflexive, symmetric, and transitive. Finally, if xR_Hy and uR_Hv, let $x=h_3y$ and $u=h_4v$ for some h_3, h_4 in H, so that $xu=h_3(yh_4)v=h_3h_5yv$ for some h_5 in H, since H is normal, and hence xuR_Hyv. This proves that R_H is an equivalence relation on G compatible with the composition of G. In a symmetric manner, the same is proved for L_H.

We now assume that G has an identity element 1, and satisfies the left and right cancellation laws.

Suppose that G is not a group, and let x be an element of G which does not have an inverse. Let H be the subset of G consisting of 1 and the even powers of x, $(1, x^2, x^4, \cdots)$, let K be the subset of G consisting of the odd powers of x, $(x, x^3, \cdots)$, and let N be the remaining elements of G, if any. Let R be the

equivalence relation on G induced by the partition H, K, N of G, so that R is nontrivial. If y is in N, then yx^k is not a power of x, for otherwise the cancellation laws in G imply that y itself is in H or K, or that $yx^n=1$ for some integer $n\geqq 1$ which yields an inverse for x. Thus R is compatible with the composition of G. But H is the set of elements equivalent to 1 and it is not a subgroup of G. Hence R is not defined by a normal subgroup.

Conversely, suppose that G is a group, and that R is a nontrivial equivalence relation on G compatible with the composition of G. Let H be the set of elements equivalent to 1. If $xR1$ and $yR1$, then $y^{-1}Ry^{-1}$, so that $y^{-1}R1$ and hence $xy^{-1}R1$. Hence, H is a subgroup of G containing 1. If xRy, then $(yx^{-1})R1$ and $(x^{-1}y)R1$, so that y is in both xH and Hx and hence R is defined by H, and $xH=Hx$, showing that H is a normal subgroup.

Also solved by James J. Burton, Barbara L. Osofsky, and the proposer.

An Inequality

5040 [1962, 670]. *Proposed by D. J. Newman, Yeshiva University*

Suppose that a polynomial $a_0+a_1x+\cdots+a_nx^n$, $a_n\neq 0$, has all its zeros in $|x|<1$. Prove that

$$\sum_{k=0}^{n} k|a_k|^2 \Big/ \sum_{k=0}^{n} |a_k|^2 > \tfrac{1}{2}n.$$

Solution by Robert Breusch, Amherst College. Call the polynomial f; thus $f(z)=\Pi_{k=1}^{n}(z-z_k)$, $|z_k|<1$. With $z=r\cdot e^{i\theta}$,

$$f\bar{f} = \prod_{k=1}^{n}\left\{r^2+z_k\bar{z}_k - r(e^{i\theta}\bar{z}_k+e^{-i\theta}z_k)\right\},$$

$$\frac{\partial}{\partial r}(f\bar{f}) = f\bar{f}\cdot\sum_{k=1}^{n}\frac{2r^2-r(e^{i\theta}\bar{z}_k+e^{-i\theta}z_k)}{r^2+z_k\bar{z}_k-r(e^{i\theta}\bar{z}_k+e^{-i\theta}z_k)}\cdot\frac{1}{r}.$$

If $|z_k|<r\leqq 1$ $(k=1, 2, \cdots, n)$, then each term in the sum is greater than 1, and thus $(\partial/\partial r)f\bar{f}>n\cdot f\bar{f}$. Now integrate over $0\leqq\theta\leqq 2\pi$, and in the left member interchange integration and differentiation. Since

$$(1/2\pi)\int_0^{2\pi} f\bar{f}d\theta = \sum_{k=0}^{n} a_k\bar{a}_k r^{2k},$$

it follows that

$$\sum_{k=0}^{n} 2k\cdot a_k\bar{a}_k\cdot r^{2k-1} > n\cdot\sum_{k=0}^{n} a_k\bar{a}_k\cdot r^{2k}.$$

The proof is completed upon setting $r=1$.

Also solved by Irving Gerst, Simon Hellerstein, M. A. Malik, Stanton Philipp, Q. I. Rahman, T. J. Rivlin, Maurice Scheier, and the proposer.

A Lawn Sprinkler Problem

5041 [1962, 671]. *Proposed by Peter Ungar, New York University*

A lawn sprinkler with a rotating head delivers $f(r)$ units of water per unit area per unit time at distance r from the head. If the sprinkler travels at a low constant speed along a straight line, a strip of width $2R$ is watered uniformly. The outside of the strip remains dry. Find $f(r)/f(0)$.

Solution by F. P. Callahan, Jr., Upper Darby, Pa. Let the sprinkler move along the x-axis and consider the amount of water received at a point with ordinate y. It is easy to see that $f(r)$ satisfies the integral equation

$$\int_0^{(R^2-y^2)^{1/2}} f(\sqrt{(x^2+y^2)})dx = \text{constant}. \tag{1}$$

Let $x=(R^2-y^2)^{1/2}\cdot\cos\theta$. This changes (1) to

$$\int_0^{\frac{1}{2}\pi} f(\{R^2-(R^2-y^2)\sin^2\theta\}^{1/2})(R^2-y^2)^{1/2}\cdot\sin\theta d\theta = \text{const.} \tag{2}$$

Let a function $g(u)$ be defined by $g(u)=uf\{\sqrt{(R^2-u^2)}\}$ and let $\sqrt{(R^2-y^2)}=v$. Then, (2) becomes

$$\int_0^{\frac{1}{2}\pi} g(v\sin\theta)d\theta = \text{constant}.$$

This is a form of Schlömilch's equation. It is evident that $g=$constant is a solution, and the details may be tidied up as in Whittaker and Watson, *Modern Analysis*. This yields

$$f(r)/f(0) = R/(R^2-r^2)^{1/2}.$$

Also solved by P. G. Heyda and G. Gilbert, Lawrence Shepp, and the proposer.

Legendre-Jacobi Symbol

5042 [1962, 671]. *Proposed by P. Barrucand, Paris, France*

Let $(n|q)$ be the symbol of Legendre-Jacobi and let $q\equiv 7 \pmod 8$. Show that $\sum_{n=1}^{h} n(n|q)=0$, where $q=2h+1>0$.

Solution by Emma Lehmer, Berkeley, California. Write the complete sum $\sum_{n=1}^{q-1} n(n|q)$ in two ways; first as a sum over numbers less than and greater than $q/2$, and next as a sum over even and odd numbers. Since $q\equiv 7 \pmod 8$ we know that 2 is a quadratic residue, while -1 is a quadratic non-residue of q. Hence, on one hand

$$\begin{aligned}\sum_{n=1}^{q-1} n(n\mid q) &= \sum_{n=1}^{h} n(n\mid q) + \sum_{n=1}^{h}(q-n)((q-n)\mid q)\\ &= 2\sum_{n=1}^{h} n(n\mid q) - q\sum_{n=1}^{h}(n\mid q),\end{aligned}$$

while on the other hand

$$\sum_{n=1}^{q-1} n(n \mid q) = \sum_{n=1}^{h} 2n(2n \mid q) + \sum_{n=1}^{h} (q - 2n)((q - 2n) \mid q)$$

$$= 4 \sum_{n=1}^{h} n(n \mid q) - q \sum_{n=1}^{h} (n \mid q).$$

Subtracting, we obtain the desired result.

Also solved by L. Carlitz, and John B. Kelly.

Three-dimensional Poker

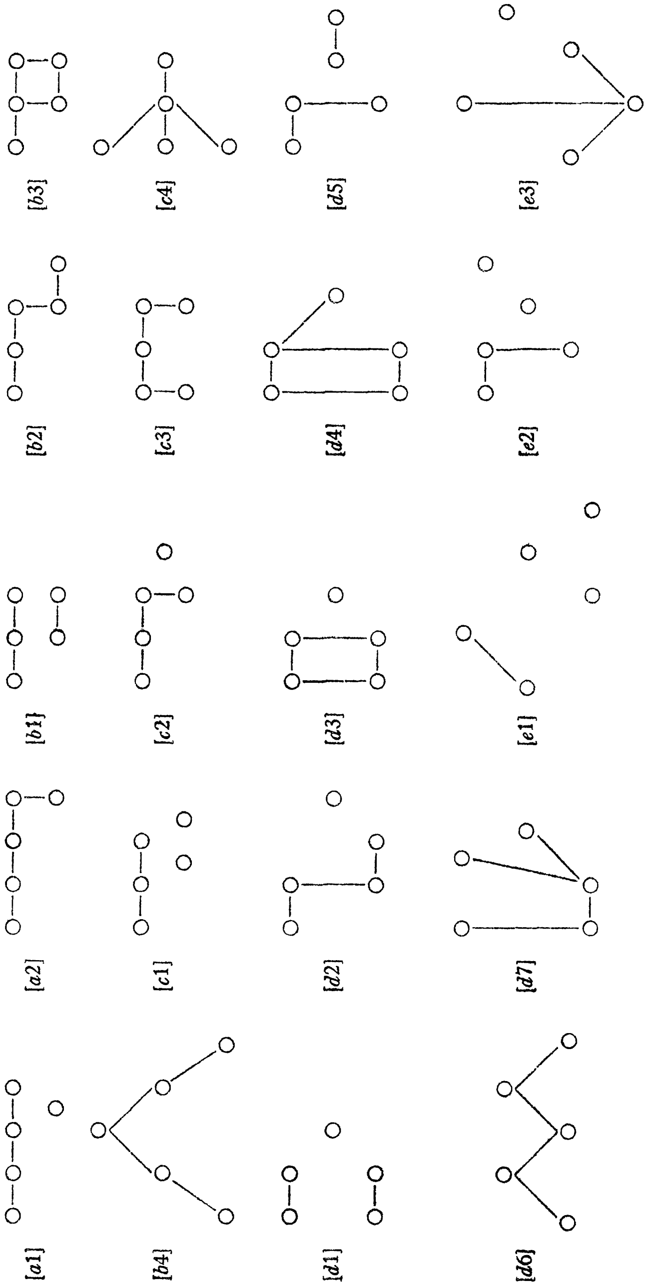
5043 [1962, 671]. *Proposed by W. W. Funkenbusch, Michigan College of Mining and Technology*

The three dimensional deck consists of 64 cards, uniquely determined by 4 suits, 4 denominations, and 4 colors. In three dimensional poker, two cards are of the same kind if they have two common coordinates. Any one particular card can be used to form only one unit. Find the numbers of five card hands of the following types:

(a) Four of a kind, (b) Full house, (c) Three of a kind,
(d) Two pair, (e) One pair, (f) No pair.

Solution by the proposer. The calculation of these numbers is more involved than for orthodox poker. There turn out to be 21 different types of hands, and care must be exercised in the analysis not to count the same hand more than once. We will omit details (calculations were actually performed on an I.B.M. 704) and merely tabulate results. To check any individual item is an exercise in elementary combinatorial analysis. The figure gives the different types of hands (a line connecting two points (cards) indicates that they form a pair). The table shows the numbers of distinct hands of each type.

(a) Four of a kind:	1,728 [type $a1$]+1,152 [type $a2$]	= 2,880
(b) Full house:	28,800 [type $b1$]+13,824[$b2$]+3,456[$b3$]+1,728 [$b4$]	= 47,808
(c) Three of a kind:	136,512[$c1$]+131,328[$c2$]+17,280[$c3$]+5,184 [$c4$]	= 290,304
(d) Two pair:	888,192[$d1$]+328,320[$d2$]+17,280[$d3$] +5,184[$d4$]+260,928[$d5$]+43,200[$d6$] +20,736[$d7$]	=1,563,840
(e) One pair:	2,915,136[$e1$]+1,299,456[$e2$]+67,392[$e3$]	=4,281,984
(f) No pair:		=1,437,696
Total $=C(64,5)$		=7,624,512



Problem 5043—Figure.

Roots of Unity

5044 [1962, 812]. *Proposed by P. M. Cohn, Manchester, England*

Show that an algebraic integer whose conjugates are all of absolute value one is necessarily a root of unity. Does the conclusion hold for any algebraic integer of absolute value one?

Solution by P. T. Bateman, University of Illinois. Both results have been given previously. For the first part see Harry Pollard, *The Theory of Algebraic Numbers*, Carus Monograph 9, 1950, Lemmas 10.10 and 11.5. A negative answer to the second part is given in problem 4656 [1956, 732]; in fact the roots $\sqrt{2}-1\pm i(2\sqrt{2}-2)^{1/2}$ of the equation $(x+1)^4-8x^2=0$ provide a ready counterexample.

Also solved by Robert Breusch, F. P. Callahan, Jr., L. Carlitz, S. Chowla and S. L. Segal, A. E. Currier, and the proposer. Later solution by George Bergman.

Homeomorphisms of the Circle

5045 [1962, 812]. *Proposed by J. V. Whittaker, University of British Columbia*

Show that no group G of homeomorphisms of the circle C has the property that, for any $u, v, x, y \in C$ with $u \neq v$ and $x \neq y$, there is exactly one $g \in G$ such that $g(u)=x$ and $g(v)=y$.

Solution by D. A. Moran, University of Chicago. Let $U, V, X, Y \in C$, where the pair (X, V) separates the pair (U, Y), and suppose that $g(U)=X$ and $g(V)=Y$. We distinguish two cases:

(1) g is orientation-reversing. Then g has two fixed points, say P and Q. (This is an easy result which appears, for instance, as an exercise in S. S. Cairns, *Introductory Topology*.) But the identity also fixes P and Q, which contradicts the uniqueness of an element of G possessing this property. (Take $u=x=P$, $v=y=Q$ in the statement of the problem.)

(2) g is orientation-preserving. Then g maps the arc $[UXYV]$ onto the arc $[XY]$. By the Brouwer fixed point theorem, g has a fixed point P in $[XY]$. Similarly g^{-1} can be shown to have a fixed point in arc $[UV]$, say Q, whence g also fixes Q. Since g has two fixed points, we again see that G cannot have the stated uniqueness property.

Also solved by F. P. Callahan, Jr., Ben Carter, Dennis Couzin, A. W. Currier, D. H. Taylor, and the proposer.

bersome because restricted to procedures of secondary school mathematics, but they are clearly and accurately presented. The book is well organized and the printing is excellent. The few misprints noted are insignificant.

The book should be in every high school (and college) library and should be required work for every mathematics teacher, particularly for those who deal with gifted students. It will prove a source of excellent problems for competitions and material to be assigned to the better students for individual study and report to the class or the school mathematics club.

E. P. STARKE, Bloomfield College

The Theory of Probability. By B. V. Gnedenko. Translated from the Russian by B. D. Seckler. Chelsea, New York, 1962. 459 pp. $8.75.

The book under review is an English translation of an amplified second Russian edition of B. V. Gnedenko's *Kurs teorii veroyatnostei.* The reviewer is of the opinion that any book by the distinguished Soviet probabilist Gnedenko would be a welcome and meaningful addition to the literature of probability theory, and that an English translation of his basic text on probability is a particularly outstanding addition to the literature. This text is suitable for a senior or first-year graduate, nonmeasure-theoretic, course in probability, although some results from measure theory are used in the text.

The chapter headings are as follows: 1. The concept of probability; 2. Sequences of independent trials; 3. Markov chains; 4. Random variables and distribution functions; 5. Numerical characteristics of random variables; 6. The law of large numbers; 7. Characteristic functions; 8. The classical limit theorem; 9. The theory of infinitely divisible distribution laws; 10. The theory of stochastic processes; 11. Elements of statistics. A good bibliography is provided, which gives references to books, monographs, and journal articles. Of the chapters listed above, Chapters 9 and 10 are of especial interest because of the topics treated and the masterful way in which they are presented. The reviewer feels that Chapter 11 is the only weak chapter in the book, and also feels that it should have been omitted. The last statement is a reflection of the reviewer's view that a basic course in probability theory should be devoted to the concepts and structures of probability theory and an introduction to the analytical study of the formulation and properties of mathematical representations of random phenomena, and should not include a discussion of statistical inference.

It is of interest to compare Gnedenko's book with the books of Feller *An Introduction to Probability Theory and Its Applications, Volume 1* and Parzen *Modern Probability Theory and Its Applications.* Feller, in his now classical work, restricts his attention to discrete sample spaces, and in doing so is able to give a more detailed treatment of a limited field, and to introduce the reader to the study of Markov chains and processes with a discrete state space. Gnedenko's book covers both the discrete and continuous cases and is, therefore, able to introduce additional topics, such as Markov processes of the diffusion type.

While Feller and Gnedenko are closer in spirit and style, Gnedenko and Parzen are closer in content, one important difference being the treatment of infinitely divisible distributions and stochastic processes in Gnedenko.

The publisher and translator are to be congratulated for making this lucid and authoritative text available to students and teachers of probability who do not know the Russian language.

A. T. BHARUCHA-REID, Wayne State University

Ordinary Differential Equations. By G. Birkhoff and G.-C. Rota. Ginn and Company, Boston, 1962. 318 pp. $8.50.

As stated in their preface, one of the chief purposes of the authors in writing this text is to aid the student in making the transition from the elementary theory of differential equations to the study of advanced methods and techniques. The reader is assumed to have some experience with the formal manipulation of elementary differential equations and to have a knowledge of advanced calculus and elementary complex variable theory. Fortunately, the second printing corrects many of the slips and errors which tended to undercut the achievement of the authors' purpose in the first printing. The terse style of exposition combined with errors confound the reader in many places as if in an obstacle course through which he can barely move, and thus unusual fortitude on the part of the reader becomes an additional prerequisite.

Apart from such difficulties there is much to admire in this book. The scope of the book and the coverage of basic topics are excellent. Besides the standard existence and uniqueness theorems employing the Lipschitz condition, some of the less well-known ones are included. Power series solutions are given very thorough treatment and the method of majorants is used for a rigorous discussion of convergence. Sturm-Liouville systems are admirably treated with considerable emphasis on asymptotic formulas. There are two chapters in which numerical methods of integration and related topics are discussed in some detail and the last chapter provides a good introduction to the theory of eigenfunction expansions.

Throughout the book classical results and modern methods are well blended, as illustrated by the advantageous use of differential inequalities in various proofs. There is an ample supply of exercises, both the manipulative type and those requiring considerable mastery of the theory, but too few answers are provided. In view of the authors' stated purpose one must applaud the fact that many topics are not presented in complete generality but primarily in the context of second order equations or in the context of a system of two first order equations. Linear equations and related topics predominate, however, and the stability of critical points is about the only topic of nonlinear theory included. Compensating for some of the erroneous diagrams in this part is an interesting example showing that all solutions of an autonomous system can tend to a critical point as $t \to +\infty$ and yet the critical point need not be asymptotically stable.

The kind of difficulty caused by the excess of errors in this text is well illustrated in the discussion of transfer functions. Inconsistencies begin to appear on page 87, but the real culprit seems to be in the definition of phase lag on page 85. The very concise exposition throughout this book makes such demands on the reader that he might almost as well begin with some of the "more advanced and systematic treatments" for which the text "can be used as an introduction." At any rate, one may expect that in such systematic treatments conciseness will be accompanied by a precision of statement and completeness of reasoning, which is too often lacking in this work. Not only are many of the guideposts barely adequate to point out the path of reasoning, but some could tend to encourage the less sophisticated reader to supply invalid arguments as, for example, the word "differentiating" on page 234. This reviewer seriously doubts the adequacy of this hint to suggest quickly the sort of steps required for a correct verification that (at points where $|V(r)|\neq 0$) the inequality $|d|V(r)|/dr| \leqq |dV(r)/dr|$ follows from the inequality $|\int_a^r V'(t)dt| \leqq \int_a^r |V'(t)|dt$.

One welcomes a book which includes so many diverse but basic topics, but one should be wary of the formidable challenge this text presents to the relative novice in the theory of differential equations. Unfortunately, the challenge is frequently attributable more to faulty composition than to difficulty of the material. This reviewer, for one, hopes that a careful revision of this book will not be too long in making its appearance.

C. E. LANGENHOP, Southern Illinois University

Logic and Boolean Algebra. By B. H. Arnold. Prentice-Hall, Englewood Cliffs, N. J., 1962. 144 pp. $6.75.

There are eight chapters: I Some Intuitive Concepts of Logic, II Boolean Functions, III Ordered Sets, IV Lattices, V Boolean Algebras, VI Boolean Rings, VII Normal Forms, Duality, and VIII Some Applications of Boolean Algebra. Each chapter consists of an introduction and three or four sections. A list of comprehensive and thought-provoking problems is included in each section. There are more than 200 problems in all. In spite of the chapter titles, the material contains nothing more specialized than, for example, a characterization of lattices and Boolean algebras in terms of binary operations and a proof of the representation theorem for finite Boolean algebras.

The objectionable features are few and not significant. At the beginning, the distinction between object language and metalanguage is not entirely clear. The technique of proof is somewhat slighted. After a careful treatment of a method of proof which involves only a comparison of two truth tables, there is an abrupt leap into more complex arguments for which the earlier examination is inadequate. Some readers may not like the use of the real numbers 0 and 1 as Boolean constants so that, for example, $f(\xi, \eta)=\xi+\eta$ fails to be a Boolean function. The atrocious grammar of two problems in Section 3–5 is not in keeping with the carefully chosen language in the rest of the book. It is not

apparent whether this is intentional, but it seems to serve no useful purpose. The definition of ring is redundant.

The book is intended for upper-division undergraduate students. The material is entirely abstract, of the type that is an essential part of the curriculum of any student who will pursue courses in advanced mathematics. The graphical representation of finite ordered sets should make this topic more understandable to the beginner. The use of a blank space to represent an order relation in Section 4–3 is novel, to say the least, but it may have instructional value. The applications are simple and closely dependent upon the theoretical treatment. An adequate list of the standard elementary references is included. The author's style is clear and concise, but an undergraduate class will nevertheless need a competent and conscientious instructor or some of the more valuable points will be missed. Both teacher and student should be pleased with this text.

CHARLES H. CUNKLE, Clarkson College of Technology

Statistical Theory. By B. W. Lindgren. Macmillan, New York, 1962. xii+427 pp. $7.95.

This book, as stated, in the preface, is intended as a text for a year's course in the theory of statistics for students who have had at least a year of calculus. It is the reviewer's opinion that additional background in analysis, linear algebra and matrices, and indeed even an introduction to probability and statistics would be most helpful before a student studies this book. It differs from most of the others that have appeared in recent years, in that the statistical problems of estimation and testing are presented within the framework of decision theory.

The first four chapters of this book present the usual probability theory ending with the Central Limit Theorem. This material, although presenting nothing new, is written carefully and precisely within the limits of the mathematical background assumed. Chapters 5, 6, and 7 introduce decision theory with particular application to the problems of estimation and testing of hypotheses. Chapter 8 presents an introduction to the theory of order statistics and the necessary distribution theory for the sampling problems to follow. The three remaining chapters apply the theory already developed to various particular statistical problems. There is a very good set of tables at the end of the book together with a set of answers to the exercises.

The book contains the usual number of typographical errors for a first printing. Most of these are of the obvious type and should not cause the reader any difficulty. One somewhat confusing error occurs on page 254, example 7–16. The likelihood ratio should be inverted so as to read $L(\mu_0)/L(\overline{X})$. The conclusion at the top of page 255 does not follow from the argument as given.

The author makes no use of "real" problems for his illustrative examples nor does his adequately large collection of exercises contain many "real" problems. Although this avoids the possibility of losing the point of an argument in a mass of computation, it does not contribute to the motivation which is so necessary for many of the students at this level.

Mathematics and Industry. By John Crank. Oxford University Press, London, 1962. vi+91 pp. $2.00.

It should be obvious that such a tiny book could not possibly deliver what one might expect of a book with such a ponderous title, but therein lies its interest. Mr. Crank grinds his axe in a refined and charming way: his purpose is to charge us to use mathematics as a language to communicate with problems, not merely as a bludgeon to overpower them. Unfortunately, his examples are perhaps more reminiscent of long-forgotten lectures in physics than they are germane to what one could consider to be relevant, up-to-date, and general industrial problems. But then this book must be written for the mathematician, and not for the manager, who would probably understand neither its substance nor its purpose. The book shows us how mathematical ingenuity can be applied in the formulation of useful models of the real world; if it is true as the author says, however, that such formulations comprise an "art . . . learnt only from experience," then we must question the purpose of a text of this type, as well as the exposition of well-known general models (such as Linear Programming) as art forms.

F. PARKER FOWLER, JR., Colorado State University

Introduction to Topology. By Bert Mendelson. Allyn and Bacon, Boston, 1962. 217 pp. $8.50.

This is a text for a one-semester elementary topology course. Chapters 1, 2, and 3 deal with set theory and functions, metric spaces, and topological spaces. Much of Chapter 3 is a repetition of Chapter 2, but in the setting of topological spaces. Product spaces with only finitely many factors are introduced. In Chapter 4, on connectedness, the Brouwer fixed-point theorem and the Borsuk-Ulam antipodal-point theorem are stated without proof. Homotopic paths and simple connectedness are introduced. Chapter 5, on compactness, includes an excellent discussion of the relation between compactness and the Bolzano-Weierstrass property, both in metric spaces and in spaces satisfying the second axiom of countability. Chapter 5 ends with a glance at some surfaces obtained as identification spaces.

In the main, the author's choice of material is acceptable. It seems curious to omit completely any mention of T_1, regular, and normal spaces. The overlap between Chapters 2 and 3 appears to be excessively large.

This book has two major weaknesses as a text. First, the level of writing in the proofs is excessively low. The author is not adept in the use of common mathematical language, and the proofs contain an excessive number of ambiguous statements. Second, there are far too few examples. A majority of the exercises are merely extensions of theory. The development of the major ideas is poorly motivated. The result is rather unfortunate: the book is little more than a theorem list.

STEVE ARMENTROUT, State University of Iowa

BRIEF MENTIONS

Encyclopaedic Dictionary of Physics. L. Thewlis, editor-in-chief. Pergamon Press, New York, 1962. $298.00 per set.

Analytic Geometry. By William L. Schaaf. Doubleday, New York, 1962. 378 pp. $1.95 (paperback).

Introduction to Mathematical Statistics, 3rd ed. By Paul G. Hoel. Wiley, New York, 1962. xi+427 pp. $6.95.

"A rather modest modification of the second edition" of this well-known text-book. A chapter on the elementary parts of decision methods has been added.

Quantum Statistical Mechanics: Green's Function Methods in Equilibrium and Non-equilibrium Problems, from the Series *Frontiers in Physics.* By Leo Kadanoff and Gordon Baym. W. A. Benjamin Co., New York, 1962. xi+203 pp. $6.95 (clothbound).

Confidence Interval Lengths for Small Numbers of Replicates. By Raymond H. Pierson, published by the author, U. S. Naval Ordnance Test Station, China Lake, Calif. v+37 pp.

Higher Education in Engineering and Science. Edited by Herman Estrin. McGraw-Hill, New York, 1963. xv+548 pp. $7.95.

An anthology consisting of a large number of short articles on various phases of the subject.

Studies in Mathematical Analysis and Related Topics. Essays in Honor of George Polya. Stanford University Press, Stanford, 1962. xxi+447 pp. $10.00.

This book contains sixty research papers, published for the first time by leading American and European mathematicians, treating an extremely wide range of topics.

Intelligent Machines, An Introduction to Cybernetics. By D. A. Bell. Blaisdell, New York, 1962. 98 pp. $2.95.

Modern Algebra and Trigonometry. By Mary P. Dolciani, Simon L. Berman and William Wooton. Houghton Mifflin, Boston, 1963. xi+658 pp. $5.20.

Geometry: Plane; Solid; Coordinate. By Frank M. Morgan and Jane Zartman. Houghton Mifflin, Boston, 1963. vii+615 pp. $4.96.

Relativity in Illustrations. By Jacob T. Schwartz. New York University Press, 1962. 117 pp. $5.00.

Elements of Finite Mathematics. By Francis J. Scheid. Addison Wesley, Reading, Mass., 1962. vii+279 pp. $6.75.

Analytic Geometry, 2nd ed. By Paul K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1963. xi+275 pp. $6.95.

An Introduction to Projective Geometry. By R. M. Winger. Dover, New York, 1962. xiii+443 pp. $2.00.

Modern Geometry; Structure and Method. By Ray Jurgensen, Alfred Donnelly and Mary Dolciani. Houghton Mifflin, Boston, 1963. xv+574 pp. $4.96.

Rational Approximations to Irrational Numbers. By K. F. Roth. H. K. Lewis and Co., London, 1962. 13 pp. 3s.6d.

A very interesting summary of results and unsolved problems in this field.

Gravitation: An Introduction to Current Research. By Louis Witten. Wiley, New York, 1962. x+481 pp. $15.00.

Contributions of seventeen scientists from five countries for readers who have some knowledge of Einstein's gravitational theory.

Calculus. By L. R. Ford, Sr., and L. R. Ford, Jr., McGraw-Hill, New York, 1963. vii+468 pp. $7.95.

Mathematical Diversions. By J. A. H. Hunter and J. S. Madachy. Van Nostrand, Princeton, N. J., 1963. vii+178 pp. $4.95.

Play Mathematics. By Harry Langman. Hafner, New York, 1963. 216 pp. $4.95.

Uses of Infinity. By Leo Zippin. Random House, New York, 1963. vii+151 pp. $1.95 (paperback).

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor James N. Eastham, Queensborough Community College, represented the Association at the Academic Convocation on the occasion of the Charter Centenary of Manhattan College on April 20, 1963.

Professor Louis O. Kattsoff, Boston College, represented the Association at the Centennial Convocation of Boston College on April 20, 1963.

Central Washington State College: Associate Professor B. A. Robinson has been appointed Chairman of the Mathematics Department; Assistant Professor B. L. Martin is on leave of absence for 1962–63 and 1963–64 for graduate study at Oregon State University.

Loras College: Rev. R. J. Collins and Rev. J. C. Friedell have been promoted to Assistant Professors.

Professor Mario Benedicty, University of Pittsburgh, has been appointed Chairman of the Mathematics Department.

Mr. J. M. Cameron, Assistant Chief of the Statistical Engineering Laboratory at the National Bureau of Standards, has been awarded the Gold Medal for Exceptional Service from the U. S. Department of Commerce.

Dr. C. E. Falbo, University of Texas, has been appointed Assistant Professor at Fresno State College.

Dr. Phyllis Fox, Rutgers, The State University, has been appointed Associate Professor and Assistant Director of the Computing Center of the Newark College of Engineering.

Assistant Professor R. C. Gilbert, University of California, Riverside, and University of Wisconsin, has been appointed Associate Professor at Orange State College.

Mr. D. P. Hayes, Florida State University, has accepted a position with E. I. du Pont de Nemours and Company, Old Hickory, Tennessee.

Assistant Professor H. S. Hayashi, University of Hawaii, has accepted a position as Engineering Specialist with Nortronics, Palos Verdes Estates, California.

Associate Professor J. M. Horvath, on leave from the University of Maryland, has been appointed Visiting Professor at the University of Nancy, France, for the academic year 1963–64.

Mr. P. A. Hulka, Kent State University, has accepted a position as Computing Analyst with Douglas Aircraft Company, Santa Monica, California.

Mr. D. W. Jared, Thiokol Chemical Corporation, Brigham City, Utah, has accepted a position as Project Engineer with Amphenol-Borg Electronics Corporation, Chatsworth, California.

Mr. Jüri Kalviste, Boeing Airplane Company, Seattle, Washington, has accepted a position as Senior Research Engineer with the A. C. Spark Plug Company, El Segunda, California.

Dr. R. B. Kelman, Remington Rand UNIVAC, Washington, D. C., has been appointed Research Fellow at the Institute of Fluid Dynamics and Applied Mathematics of the University of Maryland.

Mr. D. G. Kuehner, University of Bridgeport, has been promoted to Assistant Professor.

Dr. A. T. Lauria, Linde Company, Tonawanda, New York, has accepted a position as Head of the Computing Department in the Parma Research Center of the Union Carbide Corporation, Cleveland, Ohio.

Assistant Professor E. B. Leach, Case Institute of Technology, has been appointed Visiting Associate Professor at the Indian Institute of Technology, Kampur, India.

Mr. L. D. Levine, Hollis, New York, has accepted a position as Programmer with International Business Machines, New York, New York.

Mr. J. D. Lifsey, Air Weather Service, Asheville, North Carolina, has accepted a position as Mathematician with the National Aeronautics and Space Administration, Huntsville, Alabama.

Dr. Knox Millsaps, Holloman Air Development Center, Washington, D. C., has been appointed Research Professor of Aerospace Engineering at the University of Florida.

Dr. M. J. Norris, Sandia Corporation, Albuquerque, New Mexico, has accepted a position as Head of the Analysis Department of Bellcomm, Washington, D. C.

Dr. J. M. Ortega, Sandia Corporation, Albuquerque, New Mexico, has accepted a position as staff member with Bellcomm, Washington, D. C.

Mr. Arthur Reetz, Jr., General Atomic, San Diego, California, has accepted a position as Aerospace Technologist with the National Aeronautics and Space Administration, Washington, D. C.

Dr. C. E. Rieck, Jr., Tufts University, has accepted a position as Senior Scientist with Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Lt. Comdr. U. N. Robinson, U. S. Naval Academy, has been promoted to Captain.

Mr. Norbert Schlomiuk, Dalhousie University, has been promoted to Professor.

Dr. M. A. Shader, International Business Machines, San Jose, California, has accepted a position as Director of Scientific and Academic Programs with the World Trade Corporation, New York, New York.

Dr. J. B. Tysver, Boeing Airplane Company, Seattle, Washington, has accepted a position as Research Mathematician with the Stanford Research Institute, Menlo Park, California.

Professor Emeritus Norman Anning, University of Michigan, died on March 1, 1963 at the age of seventy. He was a charter member of the Association.

Dr. F. T. H'Doubler, Springfield, Missouri, died on June 18, 1962. He was a member of the Association for thirty-four years.

Associate Professor Emeritus Emma Hyde, Kansas State University, died on February 15, 1963. She was a charter member of the Association.

Professor H. R. Kingston, retired from the University of Western Ontario, died on February 10, 1963. He was a charter member of the Association.

Professor T. M. Simpson, University of Florida, died on February 6, 1963. He was a charter member of the Association.

Sister Anna Concillo, College of St. Elizabeth, died on March 5, 1963. She was a member of the Association for eleven years.

Professor D. M. Smith, Georgia Institute of Technology, died on November 26, 1962. He was a charter member of the Association.

Professor Emeritus I. W. Smith, North Dakota Agricultural College, died on January 23, 1962. He was a member of the Association for forty-three years.

Associate Professor Emeritus Louise A. Wolf, University of Wisconsin, Milwaukee, died on November 14, 1962. She was a member of the Association for twenty-two years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for the three year term July 1, 1963 to June 30, 1966 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	Evan Johnson, Jr., Pennsylvania State University
Indiana	J. E. Yarnelle, Hanover College
Kentucky	A. W. Goodman, University of Kentucky
Metropolitan New York	J. N. Eastham, Queensborough Community College
Nebraska	L. K. Jackson, University of Nebraska
Northern California	Irving Sussman, University of Santa Clara
Oklahoma	Gene Levy, University of Oklahoma
Rocky Mountain	W. E. Briggs, University of Colorado
Wisconsin	J. V. Finch, Beloit College

In the Nebraska Section 54% of the members voted—less than 50% voted in the remaining Sections.

HARRY M. GEHMAN, *Executive Director*

CUPM: NEW DIRECTORS AND CHAIRMAN; NEW ADDRESS

On September 1, Associate Professor Alfred B. Willcox of Amherst College became Director of the Association's Committee on the Undergraduate Program in Mathematics. He replaced Dr. Robert J. Wisner who has resigned to become Associate Professor of Mathematics at New Mexico State University.

Also on September 1, Professor B. E. Rhoades of Lafayette College replaced Dr. Bernard Jacobson as Associate Director. Dr. Jacobson has returned to his position as Associate Professor at Franklin and Marshall College.

On June 1, Professor W. L. Duren, Jr. of the University of Virginia became Chairman of CUPM in place of Professor R. C. Buck of the University of Wisconsin who continues as a member of the committee.

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CONTENTS

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THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York 53, N. Y.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

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FIXED POINTS AND ANTIPODAL POINTS

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Introduction. We present a chapter from fixed point theory which has become "classic." In fact, it seems to be so classic as almost to be forgotten in the present growth of topology. It remains important, and we seek to give here an elementary presentation, and show this, for example in the proof of Theorem 3. We begin with several basic theorems on the homotopy of maps into spheres. Theorem 1 is not only first but fundamental, and its proof, as you may observe, is trivial. A large and remarkable set of famous corollaries follows almost directly, although we have by no means exhausted the mine. In general, we do not attempt illustration of the results for we think they speak for themselves, and a truly adequate exposition of their significance would prolong the discussion beyond our intended brief arrangement.

The guiding notion is coincidence of maps, and thereafter fixed points and antipodal points. It is to be expected that the consideration of antipodal points should lead to, and culminate in theorems about real projective spaces. Thus the significance of antipode-preserving and antipode-collapsing maps of spheres is explained by Theorem 5: these are precisely the maps which correspond to continuous maps of projective space.

Preliminary. We review some basic concepts.

If X and Y are spaces and the mappings $f,\ g\colon X\rightarrow Y$ are continuous, then f and g are *homotopic*, $f\sim g$ (or $f\simeq g$) if there is a continuous map $F\colon X\times I\rightarrow Y$ (I=closed interval, $[0,\ 1]$ say) such that $F(x,\ 0)=f(x)$, $F(x,\ 1)=g(x)$. This is an equivalence relation and the equivalence classes of maps $X\rightarrow Y$ are *homotopy* classes. The recognition, or determination of these classes is a central problem of topology.

There are abelian groups, $H_n(X)$, n any integer, associated with any space X, the singular homology groups, which are topologically invariant. When $X=S^n=n$-sphere, $H_n(S^n)$ is an infinite cyclic group. A continuous map $f\colon X\rightarrow Y$ induces a homomorphism $f_*\colon H_n(X)\rightarrow H_n(Y)$, obeying the rule $(f\circ g)_*=f_*\circ g_*$. Homotopic maps induce the same homomorphism, but not conversely, in general. For the case $X=Y=S^n$, however, the converse is true. If one uses the additive group of integers as exemplar of an infinite cyclic group, it is apparent that an endomorphism is characterized by an integer, the image of 1. For a map $f\colon S^n\rightarrow S^n$, the integer so characterizing $f_*\colon H_n(S^n)\rightarrow H_n(S^n)$ is the *degree* of f, $\deg f$. Thus $\deg f\circ g=\deg f \deg g$, for $f,\ g\colon S^n\rightarrow S^n$; $f\sim g$ iff $\deg f=\deg g$.

The notions of point, segment, triangle, tetrahedron, generalize readily in analytic geometry to that of a *simplex* of dimension n, or n-simplex. The simplexes of varying dimension k, $0\leqq k\leqq n-1$, which make up the boundary of an n-simplex are its *k-faces*. The n-simplex itself is thrown in for good measure as an (improper) n-face of itself. The spaces we consider are all constructible from simplexes as follows. A finite collection of simplexes in euclidean space constitute a (geometric) *complex* K, if every face of every simplex in K is in K and two

simplexes meet in a common face or are disjoint. The subspace determined by K is its *polyhedron*, $|K|$. A homeomorphism of $|K|$ onto some space Y carries the simplicial partition of $|K|$ over to a partition of Y, and Y is said to be *triangulated*. We are thereby freed of the rigidity of the rectilinear decomposition K, but, on the other hand, entangled in the complications of a curvilinear partition.

A space may or may not admit of triangulation, but if it does, there are many ways in which such decompositions are feasible. In particular, a given triangulation can be made very fine, i.e., each simplex can be partitioned into further simplexes so that the diameters of all simplexes are $<\epsilon$, ϵ any given positive number. One of the most effective instruments of topology is *simplicial approximation*, the invention of Brouwer, who used it to show that homotopic maps of n-spheres have the same degree. The simplicial approximation theorem asserts that any continuous map f of one complex in another is homotopic to a map g (provided the partition of the domain is fine enough) which sends each simplex affinely on a simplex ("piecewise linearly"). It asserts, moreover, that during the homotopy the image of a point x never leaves the simplex where it was under f.

It was a big step when Hopf reversed the above implication and showed that maps of the same degree are homotopic. The proof is not trivial. (See Lefschetz's *Introduction to Topology*, p. 132, or Whitney's article, *On the maps of an n-sphere into another n-sphere*, Duke Journal, 1937, p. 46.)

We require the homotopy extension theorem, which states that if $f\colon X\to Y$ is a continuous map of complexes and if f is deformed homotopically on some subcomplex of X, then the homotopy can be extended to f on all of X.

A *covering space* X of a space Y consists of X, Y and a continuous map p of X on Y such that a family of open neighborhoods $\{N(y)\}$, $y\in Y$, exists, and each component of $f^{-1}N(y)$ is open, and if $x\in f^{-1}(y)$, $N(x)$ the component containing x, $f\colon N(x)\to N(f(x))$ is a homeomorphism.

A *closed path* in a space X, based at x, is a continuous map of a closed interval in X so that the ends go onto x. One multiplies such paths by joining them end to end. An equivalence relation is obtained among these maps by admitting homotopies which leave the base point fixed, and the classes form a group under the induced multiplication, the *first homotopy group* $\pi_1(X, x)$ or *fundamental group*, or *Poincaré group*. This may be regarded as the group of maps of a 1-sphere into X, the advantage of using a closed interval and identifying the ends being a simplification of definitions and proofs of the various properties of the multiplication involved. In two dimensions, a similar situation arises by mapping a rectangle into X so that the boundary goes into x. Multiplication occurs by joining rectangles end to end, and homotopies are admitted which leave the base point x fixed. The homotopy classes have an induced multiplication, and the resulting group is denoted by $\pi_2(X, x)$, the *second homotopy group* of X. Similarly there is the *n-th homotopy group* $\pi_n(X, x)$, of X, the group of classes of maps of S^n into X (based at x).

A continuous map $f: X \to Y$ induces a homomorphism $\pi_n(X, x) \to \pi_n(Y, f(x))$. This occurs, in particular, when $f: X \to Y$ is a covering. We regard real projective n-space, P^n, as covered by the n-sphere, S^n, by identification of antipodal points and denote the projection by $p: S^n \to P^n$. p induces an isomorphism: $\pi_n(S^n) \approx \pi_n(P^n)$.

If $f: X \to Y$ and $g: Y \to Z$ are continuous, so is $h = g \circ f: X \to Z$. If g and h are given, f *lifts* h to Y, or f *covers* h. If $H: X \times I \to Z$ is a homotopy of h, then, similarly, a homotopy $F: X \times I \to Y$ such that $g \circ F = H$ *covers* H and F *lifts* to H. If $g: Y \to Z$ is a covering, X, Y, Z complexes, then h and H can always be lifted to Y.

The oriented n-sphere $S^n = \{x \mid |x| = 1\}$ in euclidean $(n+1)$-space R^{n+1} serves as our model. Reflection in a central hyperplane reverses the orientation of S^n; hence reflection in a central k-plane, being a composition of $(n+1)-k$ hyperplane reflections has degree $(-1)^{n+1-k}$, so a central reflection, or what is the same, negation, $-$, or the antipodal map, a, has degree $(-1)^{n+1}$. Hence, if n is odd, a is homotopic to the identity, I.

A map g of a subset $A \subset R^{n+1}$ into R^{n+1} is a *vector field on* A. A zero of g is also called a *singularity* of g. From g one obtains another field, f, by the prescription that $g(x) = x + f(x)$. The significance of the relationship is clear at once if we observe that the fixed points of g are precisely the singularities of f, and the singularities of g are precisely the points negated by f; i.e. g coincides with the identity (0-map) precisely where f coincides with 0 (negation). If A is the n-sphere, S^n, and f is nonsingular, then the map h specified by $h(x) = f(x)/|f(x)|$ is a map of S^n into S^n and so has a degree called the *index* of f.

THEOREM 1. *If f, g: X cont $\to S^n$ so that $f(x)$ and $g(x)$ are never antipodal ($f(x) \neq -g(x)$) then f and g are homotopic. If $X = S^n$, then* $\deg f = \deg g$. *In terms of vector fields on S^n, this may be stated: two continuous nonsingular vector fields on S^n which are never opposite have the same index.*

Proof. Since $f(x) \neq -g(x)$,

$$F(x, t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$

provides the necessary homotopy.

COROLLARY 1. *If f, g: X cont$\to S^n$ and f and g never coincide ($f(x) \neq g(x)$) then f and $-g$ ($= a \circ g$) are homotopic, i.e. $f_* = (-1)^{n+1} g_*$. If $X = S^n$, then* $\deg f = (-1)^{n+1} \deg g$. *For vector fields: if f and g are two continuous nonsingular vector fields on S^n which never have the same direction, then index $f = (-1)^{n+1}$ $\cdot$index g.*

Proof. Under the circumstances, $f(x)$ and $-g(x)$ are never opposite, so f and $-g$ are homotopic, i.e. $f_* = (a \circ g)_* = a_* \circ g_* = (-1)^{n+1} g_*$.

THEOREM 2. *If f: S^n* cont$\to S^n$ *so that antipodal points never have antipodal images* ($f(-x) \neq -f(x)$), *in particular, if antipodal points are collapsed, then* $\deg f = (1+(-1)^{n+1})m$, *for some integer m; that is* $\deg f$ *is* 0 *or* $2m$ *according as n is even or odd. If n is odd, m is arbitrary, i.e. there exists, for odd n, and any m, an antipode-collapsing map of S^n in itself of degree $2m$. In terms of vector fields: if f is a continuous nonsingular vector field on S^n such that vectors at antipodal points are never opposite, then* index $f = (1+(-1)^{n+1})m$.

Proof. Since $f(x) \neq -f(-x)$, then f and $g = f \circ a$ are homotopic, by Theorem 1, and using the homotopy given there, to the map

$$F(x, \tfrac{1}{2}) = \frac{f(x) + f(-x)}{|f(x) + f(-x)|} = F(-x, \tfrac{1}{2}),$$

i.e. to a map collapsing antipodal points. So we suppose f collapses antipodal points.

$$\begin{tikzcd} S^n \arrow[d, "p"'] \arrow[dr, "f"] & \\ P^n \arrow[r, "\bar{f}"'] & S^n \end{tikzcd}$$

DIAGRAM 1

If f collapses antipodal points, then f splits through P^n (see diagram 1). But then $f_* = \bar{f}_* p_*$; if n is even $p_* = 0$, since $H_n(P^n) = 0$; if n is odd $H_n(P^n)$ is cyclic infinite, and if c_n is a fundamental n-cycle of S^n, d_n a suitable one in P^n, then $c_n \to 2d_n$, hence $f_*(c_n) = (1+(-1)^{n+1})f_*(d_n)$.

Now let n be odd, m any integer. Partition S^n, as domain, into a descending sequence $S^n \supset S^{n-1} \supset \cdots \supset S^1 \supset S^0 (= A \cup B)$ of great spheres; partition S^n as range into B and $S^n - B$. Refine the partition of the domain central-symmetrically, so that there are at least m n-cells on each hemisphere of S^n. Send the $(n-1)$-skeleton into B, m of the n-cells on the "lower" hemisphereon to $S^n - B$ with degree 1, all others on the lower hemisphere into B. Reflect this map centrally to the "upper" hemisphere. The resulting map of S^n has degree $2m$.

THEOREM 3. *If f: S^n* cont$\to S^n$ *so that antipodal points have distinct images, in particular, if antipodal points have antipodal images, then* $\deg f$ *is odd, and hence f is onto. Moreover, the degree is arbitrary: if $2k+1$ is any odd integer, there is an antipode-preserving map of S^n on itself of degree $2k+1$. For vector fields: if f is a continuous nonsingular vector field on S^n such that vectors at antipodal points never have the same direction, then index f is odd, and hence f points in every direction.*

Proof. Since $f(x) \neq f(-x) = f \circ a(x)$, f and $a \circ f \circ a$ are never opposite, so are homotopic; in fact,

$$F(x, t) = \frac{(1-t)f(x) + t(-f(-x))}{|(1-t)f(x) + t(-f(-x))|}$$

provides a homotopy. They are both homotopic, therefore, to

$$F(x, \tfrac{1}{2}) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} = -F(-x, \tfrac{1}{2}),$$

i.e. f is homotopic to an antipode-preserving map.

Suppose that f is antipode-preserving. By applying a suitable rotation after f, we obtain a new antipode-preserving map of the same degree as f and having a fixed point B. Then the antipode of B is also fixed. We call this new map f again.

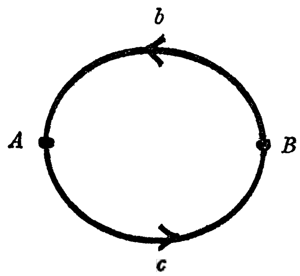

DIAGRAM 2

We proceed by induction on n. Split S^1 into semicircles, b, c using B and its antipode A. Refine this partition central-symmetrically until cellular approximation applies, and consider $f|b$. Deform $f|b$ relative to A, B until the deformed map is an edge-path: either $(bc)^k b$ or $(c^{-1}b^{-1})^k c^{-1}$. If $F(x, t)$ denotes this homotopy of f on b, then $H(-x, t) = -F(x, t)$ deforms $f|c$ relative to A and B, respectively to $c(bc)^k$, $b^{-1}(c^{-1}b^{-1})^k$. We have, therefore, deformed f relative to A and B to an edge-path on S^1: $(bc)^{2k+1}$ resp. $(c^{-1}b^{-1})^{2k+1}$, i.e. f has odd degree on S^1.

Suppose the assertion true for $n-1$, and f: $S^n \rightarrow S^n$, and f to be antipode-preserving, and having antipodal fixed points A and B. Give S^n, as range, using A and B, the octahedral subdivision, and refine this subdivision of S^n, as domain, barycentrically, and hence, central-symmetrically, until f admits a simplicial approximation, g. Select a great oriented S^{n-1} of the octahedral subdivision, containing A and B, let S^n_+ and S^n_- denote the caps determined by it, and suppose orientations so that $\partial S^n_+ = S^{n-1}$, $\partial S^n_- = -S^{n-1}$. The mode of construction of a simplicial approximation permits us to suppose that g leaves A and B fixed and that g is itself antipode-preserving. Since g is simplicial, some interior point P of an n-simplex of S^n is not covered by $g|S^{n-1}$. Then by a deformation retrac-

tion, g on $S^{n-1} \cap g^{-1}(S^n)$ can be deformed to a map whose range on that set is in S^{n-1}. By a central reflection of the deformation, we may suppose the same for S^n_-, i.e. we may suppose $g|S^{n-1}$ deformed to a map sending S^{n-1} into S^{n-1} and in antipode-preserving fashion. Extend the homotopy to $g|S^n_+$. Then reflect this homotopy centrally to S^n_-. We have deformed g to a map which is antipode-preserving and sends S^{n-1} in S^{n-1}. Call this map g again. Again by simplicial approximation, we may suppose g is simplicial, antipode-preserving and sends S^{n-1} in S^{n-1}. Then g has odd degree on S^{n-1}. The image of S^n_+ has boundary in S^{n-1} (and it is an odd multiple of S^{n-1}, since deg g is odd on S^{n-1}). But then the image of S^n_+ is a chain of the form $rS^n_+ + tS^n_-$, r, t integers, and $\partial(rS^n_+ + tS^n_-) = (r-t)S^{n-1}$, $r-t$ an odd integer. So one of r, t is odd, the other even. Since g is antipode-preserving, the image of S^n_- is $tS^n_+ + rS^n_-$. So the image of S^n is $(r+t)S^n$, i.e. deg $g = r+t$, which is odd, since one of r, t is odd, the other even.

To show that the degree can be any odd integer $2k+1$, it is clear that for S^1 we can construct an antipode-preserving map of $S^1 \to S^1$ of degree $2k+1$ (see the proof of case $n=1$ above, for example). Regarding S^1 as a great circle on S^2 we can extend this map to S^2 by joining to the identity of the north and south poles (a "suspension"). The resulting map is antipode-preserving $S^2 \to S^2$, and has degree $2k+1$. Successive continuation to S^n by a sequence of decomposing great spheres $S^n \supset S^{n-1} \supset \cdots \supset S^1$ completes the construction.

We can now obtain many corollaries. We first recall a few facts. An n-disk, $D^n(=n\text{-ball})$ may be obtained from an annulus, $S^n \times I$, by identifying the points of a base, say those in $S^n \times \{1\}$. Thus to define a continuous function on a disk D^n amounts to saying that the function $f|\partial D^n(=S^{n-1})$ on its boundary is homotopic to a constant; that is, a continuous function on S^{n-1} is homotopic to a constant iff it extends to D^n. For vector fields this means that a continuous nonsingular vector field on S^{n-1}(in R^n) has index zero iff the field can be extended without singularity to D^n. So a continuous vector field on D^n in R^n either has singularities or it has index 0 on the boundary.

Note. If f is a continuous map of a disk D^n into the surrounding R^n and sends ∂D^n into ∂D^n with degree $\neq 0$, then range $f \supset D^n$.

Proof. Otherwise f could be deformed to a map of D^n into ∂D^n, hence would have deg 0 on the boundary.

COROLLARY 2. *A continuous map f of D^n in S^{n-1} collapses a pair of antipodal points on the boundary of D^n. For vector fields: if f is a continuous nonsingular vector field on D^n (in R^n), there is a pair of antipodal points of the boundary at which the vectors have the same direction.*

Proof. $f|\partial D^n$ is homotopic to a constant, so has degree 0. If $f|\partial D^n$ preserved the distinction of antipodes, then deg $f|\partial D^n$ would be odd.

In particular, we get

COROLLARY 3. *A continuous map of a disk D^n in S^{n-1} cannot have degree ± 1 on the boundary, so cannot be a homeomorphism on the boundary, so cannot be the*

identity or the antipodal map, so cannot leave every point on the boundary fixed. So S^{n-1} *is not a retract of* D^n. *For vector fields: there is no continuous nonsingular vector field on a disk* D^n *(in* R^n*) which has index* ± 1 *on the boundary, e.g. it cannot be everywhere outward normal or everywhere inward normal on the boundary.*

The following corollary is an immediate consequence of Theorem 1.

COROLLARY 4. *If* f: $S^n \to S^n$,

(a) *and* f *sends no point in its antipode, then* f *is homotopic to the identity, i.e.* $\deg f = 1$; *(hence)*

(b) *and* $\deg f \neq 1$, *then* f *sends some point in its antipode;*

(c) *and* f *has no fixed point, then* f *is homotopic to the antipodal map, i.e.* $\deg f = (-1)^{n+1}$; *(hence)*

(d) *and* $\deg f \neq (-1)^{n+1}$, *then* f *has a fixed point;*

(e) *and* n *is odd and* $\deg f \neq 1$, *then* f *has a fixed point and sends some point in its antipode;*

(f) *and* n *is even and* $\deg f \neq \pm 1$, *then* f *has a fixed point and sends some point in its antipode;*

(g) *and* n *is even, then either* f *has a fixed point or sends some point in its antipode.*

In terms of vector fields: *if* f *is a continuous nonsingular vector field on* S^n *in* R^{n+1},

(a′) *and has no inward normal, then* f *can be deformed to a field which is outward normal everywhere, i.e. index* $f = 1$;

(b′) *and index* $f \neq 1$, *then* f *has an inward normal;*

(c′) *and* f *has no outward normal, then* f *can be deformed to a field which is inward normal everywhere, i.e. index* $f = (-1)^{n+1}$;

(d′) *and index* $f \neq (-1)^{n+1}$, *then* f *has an outward normal;*

(e′) *and* n *is odd and index* $f \neq 1$, *then* f *has an outward and an inward normal;*

(f′) *and* n *is even and index* $f \neq \pm 1$, *then* f *has an outward and an inward normal;*

(g′) *and* n *is even, then* f *has a normal.*

Hence by Corollary 4(g′) and 4(e′):

COROLLARY 5. *There is no continuous nonsingular tangent vector field on an even-dimensional sphere. A continuous nonsingular tangent vector field on an odd-dimensional sphere has index* 1.

From Corollary 4(d), (e), (d′), (e′):

COROLLARY 6. *A continuous map of* S^n *in itself which is antipode-preserving and has degree* $\neq (-1)^{n+1}$ *has two fixed antipodes. So:*

(a) *A rotation of an even-dimensional sphere has a fixed axis.*

(b) *Any orthogonal transformation* ($\in 0^n$) *of* R^n *whose* det $\neq (-1)^n$ *has a point-wise invariant subspace* ($\neq 0$), *hence has eigenvalues.*

(c) *That is, any isometry of* S^n *of* $\det \neq (-1)^{n+1}$ *has a fixed axis.*

(d) *Any automorphism f of* R^n *such that* $\operatorname{sgn} \det f \neq (-1)^n$ *has an invariant subspace* $\neq 0$ *and so has eigenvalues.*

(e) *A continuous nonsingular vector field on* S^n *in* R^{n+1} *which has opposite directions at antipodal points and index* $\neq (-1)^{n+1}$ *has outward normals at some pair of antipodes.*

COROLLARY 7. *If* f: $R^{2n+1} \text{ cont}\rightarrow R^{2n+1}$, *there exists* $x \neq 0$ *and a real number* λ *such that* $f(x) = \lambda x$. *Hence, in particular, an endomorphism of an odd-dimensional euclidean space has eigenvalues.*

Proof. If $f(x) = 0$ for some $x \neq 0$, take $\lambda = 0$. If $f(x)$ is never 0 then $f(x)/|f(x)|$ defines a continuous map of S^{2n} in itself, so there exists an $x \in S^{2n}$ such that $f(x)/|f(x)| = \pm x$, i.e. $f(x) = \pm |f(x)| x$.

Comparing Corollary 6 and Corollary 7, we observe:

COROLLARY 8. *An endomorphism of an even-dimensional euclidean space with* $\det \leqq 0$ *has eigenvalues.*

We say that a vector field f on S^n in R^{n+1} is *directed inward* (*outward*) at a point x of S^n if $f(x)$ lies on the same (opposite) side of the tangent hyperplane to S^n at x, as S^n itself, and $f(x) \neq 0$. Now a vector field f on S^n in R^{n+1} which is everywhere directed inward (outward) by negation can be turned into one which is everywhere directed outward (inward).

If f is a continuous vector field on a disk D^n in R^n either it has singularities, or else it has index 0 on the boundary. So if f is everywhere directed inward (outward) on the boundary, $(\partial D^n = S^{n-1})$ its index on the boundary, by Corollary 4(c′) is $(-1)^n$ (respectively, 1, by Corollary 4(a′)), so f has singularities. Now, as noted earlier, the field f determines the map $g(x) = x + f(x)$ and f has singularities precisely where g has fixed points. Vice versa, g determines f. So:

COROLLARY 9. *A continuous vector field on a disk* D^n *in* R^n *which is everywhere directed inward* (*outward*) *on the boundary has index* $(-1)^n$ (*respectively,* 1) *on the boundary and has singularities in the interior of* D^n. *A continuous map f of* D^n (*in* R^n) *into* R^n *which sends* ∂D^n *into* D^n, *has fixed points. A continuous map of a disk into itself has a fixed point.* (*Brouwer*).

COROLLARY 10. *If* f: $S^n \text{ cont}\rightarrow S^n$, *there is a pair of antipodes with common or antipodal images. If f is a continuous nonsingular vector field on* S^n *in* R^{n+1}, *there is a pair of antipodes at which the vectors are parallel.*

Proof. If antipodes have distinct images, deg f is odd; if antipodes never have antipodal images, deg f is even. Both cannot hold.

COROLLARY 11. *A continuous map f of an n-disk into its boundary has a fixed point on the boundary and sends some point of the boundary into its antipode. A continuous nonsingular vector field on an n-disk in* R^n *is somewhere on the boundary inward normal and somewhere on the boundary outward normal.*

Proof. By Corollary 10, f has a fixed point (necessarily on the boundary); likewise $-f$.

COROLLARY 12. (*Borsuk-Ulam.*) *A continuous map of S^n into R^n collapses a pair of antipodal points. A continuous nonsingular vector field on a sphere S^n in R^{n+1} which does not point in every direction, must have the same direction at some pair of antipodal points.*

Proof. f: S^n cont$\rightarrow R^n$ may be regarded as a map of S^n into a punctured S^n, so deg $f=0$. f cannot send antipodal points into distinct points, since then its degree would be odd.

This may be phrased: if f is a continuous map of S^n into R^{n+1} which never vanishes and has distinct values at antipodal points, range f meets every ray from the origin.

COROLLARY 13. *S^n cannot be embedded in R^n.*

Proof. A continuous map of S^n into R^n cannot be 1-1, by Corollary 12.

COROLLARY 14. *A continuous map of a sphere into a sphere of lower dimension collapses some pair of antipodal points, so is not* 1-1. *Hence spheres of different dimension are not homeomorphic.*

Proof. Since f: S^n cont$\rightarrow S^m \subset R^n$, $m<n$.

COROLLARY 15. *Euclidean spaces of different dimension are not homeomorphic. Hence cells of different dimension are not homeomorphic.*

Proof. A homeomorphism of R^n with R^m would extend to a homeomorphism of their 1-point compactifications, i.e. of S^n with S^m.

COROLLARY 16. *An m-disk cannot be embedded in R^n, $m>n$.*

Proof. If this were not true, its boundary S^{m-1} could be embedded in R^n, $m-1 \geqq n$.

COROLLARY 17. *A continuous map of an even-dimensional real projective space in itself has a fixed point. A continuous map of an odd-dimensional real projective space in itself of* deg $\neq 1$, *has a fixed point.*

$$\begin{tikzcd} S^n \arrow[d, "p"'] \arrow[dr, "f \circ p"] & S^n \arrow[d, "p"] \\ P^n \arrow[r, "f"'] & P^n \end{tikzcd}$$

DIAGRAM 3

Proof. Suppose f: P^n cont$\to P^n$, and p: $S^n \to P^n$ identifies antipodes. Then S^n is a covering of P^n. We have a commutative diagram 3, and p induces an isomorphism of $\pi_n(S^n)$ with $\pi_n(P^n)$. That is $f \circ p$ is homotopic to a map $p \circ g$, where g: S^n cont$\to S^n$ and, by the covering homotopy property, we may suppose $f \circ p = p \circ g$, (see diagram 4). If n is even, g either has a fixed point x, or sends some point in its antipode, i.e. some x into $-x$. In the former case, $fp(x) = pg(x) = p(x)$. In the latter case, $fp(-x) = fp(x) = pg(x) = p(-x)$. If n is odd, obviously deg g = deg f; but deg $g \neq 1 \Rightarrow g$ has a fixed point$\Rightarrow$ (as above) f has a fixed point.

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, "f \circ p = p \circ g"] & S^w \arrow[d, "p"] \\ P^n \arrow[r, "f"'] & P^n \end{tikzcd}$$

DIAGRAM 4

COROLLARY 18. *If f, g: S^{2n} cont$\to S^{2n}$ one of f, g, fg has a fixed point. Hence the composition of maps without fixed points has a fixed point.*

Proof. Since deg fg = deg f deg g, at least one has degree $\neq -1$.

COROLLARY 19. *If f: S^{2n} cont$\to S^{2n}$, then either f or f^2 has a fixed point; either f has a fixed point or interchanges a pair of points.*

Proof. Take $f = g$ in Corollary 18. If f^2 has a fixed point x, $ff(x) = x$, i.e. $x \to f(x)$, $f(x) \to x$.

COROLLARY 20. *If f: S^n cont$\to R^m$, $m \leqq n$ and $f(-x) = -f(x)$ or only*

$$\frac{f(-x)}{|f(-x)|} = -\frac{f(x)}{|f(x)|},$$

i.e. *$f(x)$ and $f(-x)$ have opposite directions when $\neq 0$, then f vanishes somewhere.*

Proof. Otherwise $f(x)/|f(x)|$: $S^n \to S^{m-1}$, $m-1<n$, would have distinct values at antipodal points, contradicting Corollary 14. From Corollary 18 follows

COROLLARY 21. *If f and g are two orthogonal transformations of R^{2n+1} then one of f, g, fg has a pointwise invariant subspace ($\neq 0$).*

A point on an oriented sphere S^n determines an oriented axis, so an oriented orthogonal central hyperplane, so an equatorial oriented great S^{n-1}; its antipode determines the oppositely oriented S^{n-1}. Vice versa, an oriented great S^{n-1}

(oriented "equator") determines an oriented central hyperplane, and orthogonal oriented axis, so a north pole corresponding. That is, there is a 1-1 correspondence between the points of S^n and the oriented great S^{n-1} in S^n. Thus theorems about fixed or antipodal points on S^n lead to theorems about such equators (and so to hyperplanes in P^n). We shall see examples. In a similar way, there is a 1-1 correspondence between the oriented great S^k in S^n and the oriented great S^{n-k-1} each orthogonal to its correspondent, and whose join is S^n.

COROLLARY 22. *There is no continuous map f of an even-dimensional sphere in itself which sends every point into its equator, i.e. it is impossible that $f(x)\perp x$, for every $x\in S^{2n}$. A continuous map of an odd-dimensional sphere which sends every point into its equator must have degree* 1, *and for a circle it must be a* $\pm 90°$ *rotation.*

Proof. A continuous map of S^{2n} in itself has a fixed point or sends some point in its antipode. For f: $S^{2n+1}\to S^{2n+1}$, $\deg f\neq 1\Rightarrow f$ has a fixed point and sends some point in its antipode. As for the circle, if f: $S^1\to S^1$ and sends every point into its equator, suppose, for example, point $p\to f(p)$. A rotation r of $\pm 90°$ after $f\Rightarrow r\circ f$ has a fixed point. In fact, $r\circ f$ either leaves a point fixed or sends it into its antipode. If F is the set of fixed points, F is closed. Then S^1-F is the set of points sent into their antipodes. It is the set of fixed points of $-r\circ f$, so is closed. Since S^1 is connected, one of F, S^1-F is empty, i.e. $r\circ f=$identity, since F is not empty, i.e. $f=r^{-1}$.

One can easily construct maps of S^{2n+1} in itself (rotations) which send every point in its equator; e.g. if $a_1,\cdots,a_{2n+2}$ is an orthonormal basis of R^{2n+2}, define a rotation by the prescription:

$$a_1\to a_2,\ a_3\to a_4,\ \cdots,\ a_{2n+1}\to a_{2n+2}$$

$$a_2\to -a_1,\ a_4\to -a_3,\ \cdots,\ a_{2n+2}\to -a_{2n+1}.$$

This is clearly a rotation with the required property that $f(x)\perp x$, for every x.

By Corollary 22 or Corollary 7:

COROLLARY 23. *There is no automorphism f of R^{2n+1} such that $f(x)\perp x$ for every x.*

COROLLARY 24. *If f, g: X*cont $\to S^n$ *such that $f(x)\perp g(x)$, for every x, then $f\sim g$. Hence if f, g: S^n* cont$\to S^n$, $\deg f=\deg g$.

Proof. This follows because $f(x)$ and $g(x)$ are never antipodal.

THEOREM 4. *If f, g: X* cont$\to S^n$ *such that $f(x)\neq\pm g(x)$, then there exists h: X* cont$\to S^n$ *such that $g\sim h$ and $f(x)\perp h(x)$, for every x.*

Proof. Take

$$h(x)=\frac{g(x)-(g(x)\cdot f(x))f(x)}{|\,g(x)-(g(x)\cdot f(x))f(x)\,|};$$

then

$$F(x, t) = \frac{(1 - t)g(x) + th(x)}{|(1 - t)g(x) + th(x)|}$$

provides the homotopy.

Theorem 4 says that two maps into S^n which never coincide and are never opposite can be deformed to orthogonal maps.

COROLLARY 25. *If f, g: X(connected) cont$\rightarrow S^n$ such that $f(x)$ not$\perp g(x)$, for every x, then $f \sim \pm g$ according as $f(x) \cdot g(x) \gtrless 0$. Hence if $X = S^n$,* $\deg f = \deg g$ *or* $(-1)^{n+1} \deg g$ *according as $f(x) \cdot g(x) \gtrless 0$.*

Proof. $f(x) \cdot g(x)$ is always >0 or always <0. If always >0, then $f(x) \neq -g(x)$, so $f \sim g$ by Theorem 1; if always <0, then $f(x) \neq g(x)$, for every x, then $f \sim -g$ by Corollary 1.

COROLLARY 26. *If f: S^n* cont$\rightarrow S^n$ *and $f(x)$ not$\perp x$, for every x, then* $\deg f = 1$ *or* $\deg f = (-1)^{n+1}$, *according as $f(x) \cdot x \gtrless 0$.*

COROLLARY 27. *If f: S^n* cont$\rightarrow S^n$ *and $f(x) \neq \pm x$, for every x, then f is homotopic to a continuous map h: $S^n \rightarrow S^n$ such that $h(x) \perp x$, for every x. Hence (Corollary* 22) $\deg f = 1$ *and n is odd.*

COROLLARY 28. *If f: S^n* cont$\rightarrow S^n$ *and $f(x) \perp -x$, for every x, then* $\deg f = 1$.

Proof. Under these conditions $f(x) \perp x$, and n must be odd.

For even n, no continuous map of S^n in itself is orthogonal to the identity, hence is not orthogonal to the antipodal map $(f(x) \perp -x \Leftrightarrow f(x) \perp x)$.

A continuous map of real P^n in itself may be regarded as a continuous map of the family of diameters of S^n in itself. So we can say that two points x, y of P^n are *orthogonal*, $x \perp y$ or $x \cdot y = 0$, if the antecedent diameters in S^n (or subspaces of R^{n+1}) are orthogonal. More generally, we can define $x \cdot y = |\bar{x} \cdot \bar{y}|$ where $p(\bar{x}) = x$, $p(\bar{y}) = y$, $\bar{x}$, $\bar{y} \in S^n$.

As we have earlier observed, for any continuous map f: $P^n \rightarrow P^n$ there is a lifting $\bar{f}$: $S^n \rightarrow S^n$ such that the diagram 5 is commutative. But the commuta-

$$\begin{tikzcd} S^n \arrow[r, "\bar{f}"] \arrow[d, "p"'] & S^n \arrow[d, "p"] \\ P^n \arrow[r, "f"'] & P^n \end{tikzcd}$$

DIAGRAM 5

tivity implies that $\tilde{f}$ *is either antipode-preserving or antipode-collapsing*. Namely,

$$\begin{array}{ccc} f(p(x)) & = & f(p(-x)) \\ \| & & \| \\ p\tilde{f}(x) & & p\tilde{f}(-x). \end{array}$$

So $\tilde{f}(x) = \pm\tilde{f}(-x)$. So $\tilde{f}$ sends antipodes into antipodes or collapses antipodes. It cannot do both, of course, for if

$$A = \{x \mid \tilde{f}(x) = \tilde{f}(-x)\} = \text{set of collapsed antipodes,}$$
$$B = \{x \mid \tilde{f}(x) = -\tilde{f}(-x)\} = \text{set of antipodes preserved.}$$

A and B are obviously disjoint and cover S^n; but A is the counterimage in S^n of the diagonal set in $S^n \times S^n$ under the continuous map $x \to (\tilde{f}(x), \tilde{f}(-x))$, so is closed. Similarly B is closed (use $x \to (\tilde{f}(x), -\tilde{f}(-x))$). So one of A, B is $\varnothing$.

Thus deg $\tilde{f}$ is odd or even according as $\tilde{f}$ preserves or collapses antipodes. Conversely, any map $\tilde{f}$: S^n cont$\to S^n$ which is antipode-preserving or antipode-collapsing, induces a continuous map f: $P^n \to P^n$ so that the above diagram is commutative. So for odd n, every map S^n cont$\to S^n$ is homotopic to a map which induces a map of P^n in itself, and every map of odd P^n is so induced; for even P^n, this holds only for the homotopy classes of S^n of odd degree or degree 0. Continuous maps $S^{2n} \to S^{2n}$ of even degree $\neq 0$ do not induce maps of P^{2n} into P^{2n}.

If f: P^n cont$\to P^n$ lifts to $\tilde{f}$: S^n cont$\to S^n$, f also lifts to $-\tilde{f}$. No other maps S^n cont$\to S^n$ induce f. For if $p\tilde{f}(x) = fp(x)$, then $p(-\tilde{f}(x)) = p\tilde{f}(x)) = fp(x)$, so $\pm\tilde{f}$ both induce f. If $\tilde{g}$ induces f also, then $p\tilde{g}(x) = fp(x) = p\tilde{f}(x)$, then $\tilde{g}(x) = \pm\tilde{f}(x)$ for every $x \in S^n$. The sets $\{x \mid \tilde{g}(x) = \tilde{f}(x)\}$, $\{x \mid \tilde{g}(x) = -\tilde{f}(x)\}$ are both closed, disjoint, and cover S^n, so one of them is empty. So $\tilde{g} = \tilde{f}$ or $-\tilde{f}$. So:

THEOREM 5. *There is a* 1-1 *correspondence between the continuous maps* f: $P^n \to P^n$ *and the pairs* $\{\pm\tilde{f}\}$ *of continuous maps* $S^n \to S^n$ *which collapse or preserve antipodes*, f *lifting to* $\pm\tilde{f}$, $\pm\tilde{f}$ *inducing* f.

$$\begin{tikzcd} S^n \times I \arrow[r, "\tilde{H}"] \arrow[d, "p^1 = p \times 1"'] & S^n \arrow[d, "p"] \\ P^n \times I \arrow[r, "H"'] & P^n \end{tikzcd}$$

DIAGRAM 6

Suppose f, g: P^n cont$\to P^n$ and H: $P^n \times I \to P^n$ a homotopy of f and g. Then H lifts to $S^n \times I \to S^n$ so that the diagram 6 is commutative. Namely, $H \circ p^1$ is a homotopy of $f \circ p$ with $g \circ p$. Since p: $S^n \to P^n$ is a covering, if $\tilde{f}$ is a lifting of $f(p \circ \tilde{f} = f \circ p)$, there is a covering homotopy, $\tilde{H}$ such that $p \circ \tilde{H} = H \circ p^1$.

It is not true, of course, that if $\tilde{f}$ and $\tilde{g}$ are liftings of f, g resp., then f homotopic to $g \Rightarrow \tilde{f}$ homotopic to $\tilde{g}$. For example, the identity map of P^{2n} lifts to both the identity and the antipodal map of S^{2n}, and the latter, having degree 1 and -1 resp., are not homotopic.

THEOREM 6. *If two continuous maps f, g: $P^n \to P^n$ are never orthogonal $(f(x) \cdot g(x) \neq 0)$, then f and g are homotopic.*

Proof. Lift f, g to $\tilde{f}$, $\tilde{g}$: S^n cont$\to S^n$. Then $\tilde{f}(x) \cdot \tilde{g}(x) \neq 0$ since $\tilde{f}(x) \cdot \tilde{g}(x) = 0 \Rightarrow p\tilde{f}(x) \cdot p\tilde{g}(x) = 0 = fp(x) \cdot gp(x)$, i.e. f and g orthogonal at $p(x)$.

By Corollary 25, $\tilde{f}(x) \cdot \tilde{g}(x) > 0$ $(<0) \Rightarrow \tilde{f}$ homotopic to $\tilde{g}(-\tilde{g})$ and the homotopy is given by $F(x, t)$ in Theorem 1, which implies that if $\tilde{f}$ and $\tilde{g}$ are both antipode-preserving or both antipode-collapsing, so is $F(x, t)$ for each t. Hence $p \circ F(x, t)$ factors through $P^n \times I$, i.e. there is a homotopy F^1: $P^n \times I \to P^n$ *such that* $F^1 \circ p^1 = p \circ F$, i.e. f is homotopic to g.

THEOREM 7. *Two continuous maps of P^n in itself which never coincide may be deformed to orthogonal maps, i.e. if f, g: P^n* cont$\to$*P^n and $f(y) \neq g(y)$, for every y, then g is homotopic to h such that $f(y) \cdot h(y) = 0$, for every y.*

Proof. Lift f and g to $\tilde{f}$, $\tilde{g}$: $S^n \to S^n$, respectively, $f(y) \neq g(y)$, *for every* $y \Rightarrow \tilde{f}(x) \neq \pm \tilde{g}(x)$, so $\tilde{g}$, by Theorem 4, is homotopic to a map $\tilde{h}$ such that $\tilde{h} \perp \tilde{f}$. Moreover, the construction there shows that if g is antipode-preserving (-collapsing), $\tilde{h}$ is also antipode-preserving (-collapsing); for $\tilde{f}$ is also antipode-preserving or collapsing, and the homotopy there given shows that $F(x, t)$ has, for each t, the same property, and so, as in the proof above, induces a homotopy of g with the map h induced by $\tilde{h}$, and $\tilde{h} \perp \tilde{f} \Rightarrow h \perp f$.

THEOREM 8. f, g: P^n cont$\to$$P^n$ *and* $f \perp g \Rightarrow f \sim g$.

Proof. Let $\tilde{f}$ and $\tilde{g}$ be liftings of f and g to $S^n \to S^n$. Then $\tilde{f} \perp \tilde{g}$. $\therefore \tilde{f} \sim \tilde{g}$ (Corollary 24); moreover, the homotopy is that of Theorem 1 and by the same reasoning as in the theorems above, there is an induced homotopy of f and g.

COROLLARY 29. f, g: P^n cont$\to$$P^n$ *and* $f(x) \neq g(x) \Rightarrow f \sim g$.

Proof. g is homotopic to h such that $h \perp f$, by theorem above, but $h \perp f \Rightarrow h \sim f \Rightarrow g \sim f$.

COROLLARY 30. *If f: P^n* cont$\to$*P^n and $f(x)$ not $\perp x$ or $f(x) \perp x$ or $f(x) \neq x$ (for all x), then f is homotopic to the identity.*

COROLLARY 31. *There is no continuous map of P^{2n} in itself which is orthogonal to the identity.*

Proof. The same applies to a lifting to S^{2n} (Corollary 22).

THEOREM 9. *There is no continuous family of tangent lines to an even dimensional sphere.*

Proof. Suppose the contrary. Then the tangent line at $x \in S^{2n}$ uniquely and

continuously determines the parallel diameter, i.e. a continuous map $f: S^{2n}\to P^{2n}$. By the same arguments as before, f lifts to a map f^1: $S^{2n}\to S^{2n}$ such that the diagram 7 is commutative. $f^1(x)=\pm x$ for some x. If $f^1(x)=x$, we have $f(x)=pf^1(x)=p(x)$, i.e. the diameter through x is parallel to the tangent line at x, an absurdity. $f^1(x)=-x$ produces a similar contradiction.

$$\begin{tikzcd} S^{2n} \arrow[r, "f^1"] \arrow[dr, "f"'] & S^{2n} \arrow[d, "p"] \\ & P^{2n} \end{tikzcd}$$

DIAGRAM 7

COROLLARY 32. *There is no continuous family of central hyperplanes to S^{2n}, i.e. at each $x\in S^{2n}$, a central hyperplane.*

COROLLARY 33. *There is no continuous family of lines through the points of S^{2n} not through the origin.*

Proof. A projection from O on the tangent hyperplane at x of the line produces a continuous family of tangent lines, a contradiction.

COROLLARY 34. *There is no continuous map f of P^{2n} into the family of hyperplanes in P^{2n} such that $x\in f(x)$. By duality, there is no continuous map f of the family of hyperplanes of P^{2n} into P^{2n} such that $f(x)\in x$.*

Proof. Otherwise assign to $f(x)$ the unique point $g(f(x))$ orthogonal to it. This is a continuous map gf: $P^{2n}\to P^{2n}$ such that $gf(x)\perp x$, for every x, and this is a contradiction.

COROLLARY 35. *A continuous map of the hyperplanes of P^{2n} into themselves, i.e. in hyperplanes, carries some hyperplane into itself.*

Proof. This is the dual of the fixed point theorem for P^{2n} and this implies that a continuous map of the family of great S^{2n-1} of S^{2n} into itself carries some great S^{2n-1} into itself.

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ON LOCALLY RECURRENT FUNCTIONS

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All functions used in this note are real functions of a real variable.

1. Following K. A. Bush, we shall say that a function f is locally recurrent at x_0 if every deleted neighbourhood of x_0, $N(x_0)$, contains an element x such that $f(x)=f(x_0)$. In this note, we shall give a simple characterization of the locally recurrent functions. As a corollary, we obtain the answer to the following question raised and solved in another way in [**3**]:

(A) Does there exist an everywhere locally recurrent function which is nonconstant and continuous?

The second part of this note contains several propositions concerning the set of points where a continuous function is locally recurrent. We shall give also certain properties of locally recurrent discontinuous functions. I am indebted to the referee for helpful remarks.

2. We shall use the following definitions: If f is a function defined on $[a, b]$, a set $E_t(f)=\{x; f(x)=t\}$ is a *level set* of f on $[a, b]$. If, for a given function f, $E_t(f)$ is closed for each t, then f is *closed*.

LEMMA. *A function f defined on $[a, b]$ is locally recurrent everywhere on $[a, b]$ if and only if each level set of f is dense in itself on $[a, b]$.*

Proof. Let f be locally recurrent everywhere on $[a, b]$. Given an arbitrary level set $E_t(f)$ and an element $y\in E_t(f)$, y is, by hypothesis, an accumulation point of $E_t(f)$ and hence $E_t(f)$ is dense in itself. Now let f be such that every level set $E_t(f)$ is dense in itself. It follows that for an arbitrary $y\in[a, b]$, y is an accumulation point of $E_t(f)$, where $t=f(y)$. Hence, f is locally recurrent at y.

PROPOSITION 1. *A function f, defined and closed on $[a, b]$, is locally recurrent everywhere on $[a, b]$ if and only if each level set of f is perfect.*

Proof. Since f is closed, each level set is closed. In view of the lemma, and since a set is perfect if and only if it is closed and dense in itself, the proposition follows.

Remark. There exist closed functions which are not continuous, since every strictly monotone function is closed. It follows from the theorem of [**14**], p. 285, that a closed function is continuous on $[a, b]$ if and only if f has the Darboux property on $[a, b]$ (i.e. if x_1 and x_2 are two points of $[a, b]$, $f(x)$ takes each value between $f(x_1)$ and $f(x_2)$ in the interval (x_1, x_2)).

PROPOSITION 2 (affirmative answer to the question (A)). *There exists a continuous nonconstant function which is locally recurrent everywhere on $[a, b]$.*

Proof. J. Gillis has proved the existence of a real function f, defined, non-

constant and continuous on $[a, b]$, such that each level set $E_t(f)$ is perfect (see [6]). In view of the Proposition 1, this function is locally recurrent everywhere on $[a, b]$.

A function f is said to be *locally recurrent from the right* (*left*) at x_0 if every deleted right (left) neighbourhood of x_0 contains an element x such that $f(x)=f(x_0)$.

PROPOSITION 3. *Let f be a continuous, nonconstant, locally recurrent function on $[a, b]$. Denote by A the set of points in $[a, b]$ where f is not locally recurrent from right. Then, for each open interval $I\subset[a, b]$, such that f is not constant in I, the set $A\cap I$ has the power of the continuum.*

Proof. Since f is not constant, it has at least two values on I. Since f is continuous, it has the Darboux property on I. Therefore its values on I form a set V_I having the power of the continuum. Let $t\in V_I$. Since $E_t(f)$ is closed and $I\cap E_t(f)$ is properly contained in I, there is an open interval (α, β) contiguous to $E_t(f)$ (i.e. $(\alpha, \beta)\cap E_t(f)=0$, $\alpha\in E_t(f)$, $\beta\in E_t(f)$) such that $(\alpha, \beta)\subset I$. Since $\alpha\in E_t(f)$ and f is not locally recurrent from the right at α, it follows that for each $t\in V_I$, the set $A\cap I\cap E_t(f)$ is not void.

Remark. The proof of Proposition 3 is an exploitation of the reasoning of R. P. Dilworth ([3], p. 205), by which one obtains a negative answer to the following question raised in [3]:

(B) Does there exist a function, nonconstant and continuous, locally recurrent everywhere from the right?

By generalization of the reasoning used in the proofs of the Lemma and of Proposition 1, we obtain the following two propositions:

PROPOSITION 4. *For an arbitrary closed function f, defined on $[a, b]$, a level set $E_t(f)$ is infinite if and only if it contains a point where f is locally recurrent.*

PROPOSITION 5. *For an arbitrary closed function f, defined on $[a, b]$, a level set $E_t(f)$ is perfect if and only if at all its points f is locally recurrent.*

PROPOSITION 6. *Let f be a continuous function which has a finite derivative at no point of $[a, b]$. There exists a set H of measure zero such that, for each $t\notin H$, $E_t(f)$ contains a nondenumerable subset of points where f is locally recurrent from the left and from the right.*

Proof. By the theorem of [12], p. 31, there is a set H of measure zero such that, for $t\notin H$, $E_t(f)$ is nondenumerable. Since $E_t(f)$ is closed, it contains a nonvoid perfect set P_t. At each point of bilateral accumulation of P_t, f is locally recurrent from the left and from the right.

PROPOSITION 7. *Let f be a continuous function which is monotone in no subinterval of $[a, b]$. There is a set K of the first category, such that each level set $E_t(f)$ with $t\notin K$ contains a nondenumerable set of points where f is locally recurrent from the right and from the left.*

Proof. By the Theorem 1 of [**13**], p. 839, the values t for which $E_t(f)$ is at most denumerable, form a set K of first category. Thus, for $t \notin K$, $E_t(f)$ contains a nonvoid perfect set P_t. Evidently, f is locally recurrent at each point of P_t. By neglecting a denumerable subset of P_t, we obtain a subset Π_t whose points are points of bilateral accumulation of P_t. At each point of Π_t, f is locally recurrent from the right and from the left.

We say that a continuous nondecreasing function f is *singular of the first type on* $[a, b]$ if (i) every subinterval of $[a, b]$ contains an interval where f is constant; (ii) the intervals of constancy form a set of measure $b-a$; and (iii) $f(a) \neq f(b)$. An example of such a function is the Cantor function (see, for instance, [**8**], chapter 4, section 19).

PROPOSITION 8. *If a continuous nondecreasing function f is singular of the first type on $[a, b]$, then f is locally recurrent from the right and from the left almost everywhere on $[a, b]$.*

Proof. Let $I \subset [a, b]$ be an interval where f is constant. Evidently f is locally recurrent from the right and from the left at each point of I. Thus Proposition 8 is proved.

Remark. Proposition 8 contains an affirmative answer to the following question raised and solved in [**3**] in another way:

(C) Does there exist a function, nonconstant and continuous, which is locally recurrent almost everywhere from the right?

Following S. Banach [**1**], we say that a function f has the property T_1 if the set $\{t;\ E_t(f) \text{ is infinite}\}$ is of measure zero.

PROPOSITION 9. *If f is a continuous function of bounded variation on $[a, b]$ and M is the set of points where f is locally recurrent, then the set $f(M)$ is of measure zero.*

Proof. S. Banach has proved [**2**] that every continuous function of bounded variation has the property T_1. In view of Proposition 4, the required property follows.

PROPOSITION 10. *If f has at each point of $[a, b]$ a finite derivative and M is the same set as in Proposition 9, then $f(M)$ is of measure zero.*

Proof. S. Saks has proved [**15**] that every function with a finite derivative everywhere has the property T_1. Thus, in view of Proposition 4, Proposition 10 follows.

PROPOSITION 11. *If f is a continuous function which is locally recurrent at no point of $[a, b]$, then each subinterval of $[a, b]$ contains an interval where f is strictly monotone.*

Proof. In view of Proposition 4, every level set $E_t(f)$ is finite. Therefore by the continuity of f and a theorem of E. Čech [**4**], the desired property follows.

A function f is said to have property D if, for every set H of first category, $f(H)$ is also of first category (see [**11**]).

PROPOSITION 12. *If a continuous function f has property D and M is the same set as in Proposition 9, then $f(M)$ is of first category.*

Proof. By Theorem 5 of [**10**], the set $\{t; E_t(f) \text{ is infinite}\}$ is of first category. Hence, in view of Proposition 4, $f(M)$ is of first category.

PROPOSITION 13. *There exists a monotone function which is locally recurrent from the right everywhere on* $(0, 1)$.

Proof. Define f on $(0, 1)$ by setting $f(x) = 1/n$ if $1/n \leqq x < 1/(n-1)$ $(1 \leqq n < \infty)$. Then f is the required function.

PROPOSITION 14. *Let f be a function with the Darboux property on (a, b). Denote by $\underline{f}_l(x)$ and $\bar{f}_l(x)$ ($\underline{f}_r(x)$ and $\bar{f}_r(x)$) the left lower limit and the left upper limit (the right lower limit and the right upper limit) of f at the point $x \in (a, b)$. If $\underline{f}_l(x) < f(x) < \bar{f}_l(x)$ ($\underline{f}_r(x) < f(x) < \bar{f}_r(x)$), then f is locally recurrent at x from the left (right).*

Proof. By a known theorem (see [**7**] p. 207, [**5**], and [**9**] p. 703), we have, for each $x \in (a, b)$ and for each $\epsilon > 0$,

$$(\underline{f}_l(x), f_l((x)) \subset f((x - \epsilon, x)), \qquad (\underline{f}_r(x), f_r(x)) \subset f((x, x + \epsilon)).$$

Proposition 14 follows immediately.

PROPOSITION 15. *Every function f defined on $(-\infty, \infty)$ is the sum of two functions locally recurrent everywhere from the left and from the right.*

Proof. By a theorem of Sierpiński ([**16**], p. 73; see also [**10**], p. 490) we have $f = \phi + \psi$, where ϕ and ψ have the Darboux property and are such that $\underline{\phi}_l(x) = \underline{\phi}_r(x) = \underline{\psi}_r(x) = \underline{\psi}_l(x) = -\infty, \bar{\phi}_l(x) = \bar{\phi}_r(x) = \bar{\psi}_l(x) = \bar{\psi}_r(x) = +\infty$ for each x. By Proposition 14, ϕ and ψ are locally recurrent everywhere, from the left and from the right.

PROPOSITION 16. *If a function f of the first class of Baire is locally recurrent everywhere in (a, b) from the left and from the right, then f has the Darboux property on (a, b).*

Proof. W. H. Young [**17**] has proved the following theorem: *If f is of the first class of Baire and if for each $x \in (a, b)$ there exist two sequences $x_1 < x_2 < \cdots < x_n < \cdots < x < \cdots < y_m < \cdots y_2 < y_1$ such that* $\lim x_n = \lim y_n = x$ *and* $\lim f(x_n) = \lim f(y_n) = f(x)$ *then f has the Darboux property on (a, b).* Proposition 16 follows immediately.

PROPOSITION 17. *If f has the Darboux property on (a, b) and is left (right) semicontinuous at no point of (a, b), then f is locally recurrent from the left (right) everywhere except on a denumerable subset of (a, b).*

Proof. We use the notation introduced in Proposition 14. By a known result, the points x for which either $f(x)<\underline{f}_l(x)$ or $f(x)>\bar{f}_l(x)$ form a denumerable set B (resp., the points x for which either $f(x)<\underline{f}_r(x)$ or $f(x)>\bar{f}_r(x)$ form a denumerable set D). Since we have by hypothesis $f(x)\neq\underline{f}_l(x)$, $f(x)\neq\bar{f}_l(x)$ (resp. $f(x)\neq\underline{f}_r(x)$ and $f(x)\neq\bar{f}_r(x)$) for each $x\notin B(D)$, it follows, in view of Proposition 14, that f is locally recurrent at x from the left (right) for each $x\notin B$ (resp., $x\notin D$).

Unsolved problems. 1) Does there exist a nonconstant derivative function which is everywhere locally recurrent from the right?

2) If the answer is negative, does there exist a nonconstant function of the first class of Baire, and having the Darboux property, which is everywhere locally recurrent from the right?

3) Does there exist a nonconstant continuous function which possesses a dense set of points of differentiability, and which is locally recurrent everywhere?

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LOCAL NEAR-HOMOGENEITY

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During a seminar on invertibility [5] held during the spring and summer of 1962, several simple but apparently new topological concepts arose. Since these concepts provide a ready source of easy problems in elementary topology, it seems fitting that they be reported in this journal. Also it should be mentioned that the other participants in this seminar, Professor P. H. Doyle and Mr. R. P. Osborne, gave valuable assistance at every stage in the preparation of this paper. The investigations reported here were partially supported by the National Science Foundation under Grant GP-31.

In [1] brief mention is made of local near-homogeneity. We will repeat some of the results of that paper and then carry on with new material.

The symbol $\mathcal{G}(S)$ is used to denote the set of all homeomorphisms of the topological space S *onto* itself. (We make no use of the fact that $\mathcal{G}(S)$ may be given a group structure or a topological structure.)

DEFINITION 1. *A space S is near-homogeneous at a point $p\in S$ if, for each open neighborhood U of p and each point $x\in S$, there exists $h\in\mathcal{G}(S)$ such that $h(x)\in U$. The set of all points of S at which S is near-homogeneous is denoted by $N(S)$.*

It is clear that the space S is near-homogeneous (see [2]) if and only if $N(S)=S$. Also it should be noted that the property defined above is obviously a topological invariant of the space S. That is, if S is near-homogeneous at p and if $h(S)$ is a homeomorphic image of S, then $h(S)$ is near-homogeneous at the point $h(p)$.

The first four results below are mentioned in [1]. We give proofs here for the sake of completeness.

THEOREM 1. *For any space S, the set $N(S)$ is carried onto itself by each $h\in\mathcal{G}(S)$.*

Proof. This follows from the remark above to the effect that being a point of near-homogeneity is invariant under homeomorphisms.

THEOREM 2. *The set $N(S)$ is closed in S.*

Proof. Let p be a limit point of $N(S)$ and let U be any open neighborhood of p. By definition of limit point, there is a point $q\neq p$ in $N(S)\cap U$. Thus for any point $x\in S$, there exists a homeomorphism $h\in\mathcal{G}(S)$ such that $h(x)\in U$. By definition, then, we have $p\in N(S)$.

THEOREM 3. *The set $N(S)$ is a near-homogeneous subspace of S.*

Proof. Let p and q be any two points in $N(S)$ and let O be an open neighborhood of p in the subspace topology of $N(S)$. By definition of the subspace topology, there is an open neighborhood U of p (open in S) such that $U\cap N(S)=O$. Now there exists $h\in\mathcal{G}(S)$ such that $h(q)\in U$. But by Theorem 1,

$h[N(S)] = N(S)$ and hence $h|N(S)$, h restricted to $N(S)$, is a homeomorphism of $N(S)$ onto itself which clearly carries the point q into $U \cap N(S) = O$.

Some examples are in order at this point. For instance we may note that each point on the boundary Bd I^n of the closed n-cube is a point of near-homogeneity of I^n, i.e. $N(I^n) \supset \text{Bd } I^n$. An example in which $N(S)$ consists of a single point is pictured in Figure 1. It consists of intervals of length $1/n$, each bisected by the origin, where the intervals of length $1/n$ lie along the lines with angles of inclination $k\pi/2^{n-1}$, $k=1, 3, 5, \cdots, 2n-1$. Of course, the origin is the set $N(S)$.

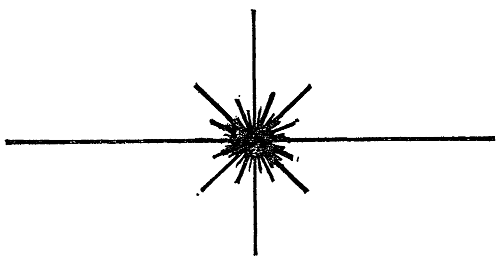

FIG. 1

In Figure 2 we picture an example for which $N(S)$ consists of exactly two points. This consists of the circle $x^2+y^2=1/4$ and the circles $(x \pm (1-3\cdot 2^{-n}))^2 + y^2 = 2^{-2n}$, $n=3, 4, 5, \cdots$ together with the limit points $(1, 0)$ and $(-1, 0)$. These two limit points constitute the set $N(S)$.

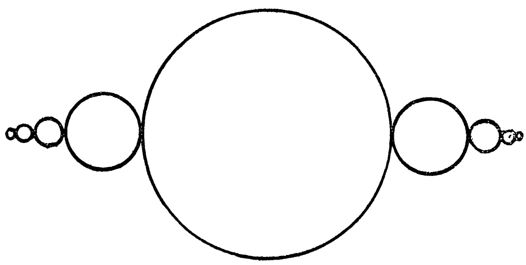

FIG. 2

It may be left as an exercise for the reader to construct examples in which the set $N(S)$ has exactly n points, $n>2$. (As a hint, it is possible to construct one generic example of this nature.)

THEOREM 4. *If $N(S)$ contains a nonempty open subset of S, then S is near-homogeneous.*

Proof. Suppose that $N(S)$ contains an open set U. Then any point $x \in S$ can be carried into U by some element $h \in \mathcal{G}(S)$, which implies that $h(x)$ is in $N(S)$. Thus $x \in N(S)$ and $N(S) = S$.

There is a general principle here by means of which any number of easily proved theorems can be generated. It may be stated as follows: If the space S is near-homogeneous at a point p and if there exists an open neighborhood of p having a certain property α as a subspace, then each point of S has a neighborhood having property α. If, in particular, the property α is in some sense hereditary, then S has the property α locally.

Examples of such properties α are the T_i separation axioms, $i=0, 1, 2$, compactness, separability and the property of being locally euclidean at a point. (We shall use this last property shortly.) Two typical results obtained through application of this principle are stated here without proof. The slight modification of the hypotheses exhibited are also typical of the results so obtained.

THEOREM 5. *If the space S is near-homogeneous and locally compact at a point, then S is locally compact.*

THEOREM 6. *If the space S is near-homogeneous at a point p and if p has an open neighborhood which is a locally connected subspace, then S is locally connected.*

In order to introduce another form of local near-homogeneity, we will first explore a few of the consequences of a restricted form of near-homogeneity which itself seems to be a new concept.

By an *isotopy* of a space S onto itself we mean a continuous mapping $h_t\colon S\times[0, 1]\to S$ of the product of S with the unit interval $[0, 1]$ onto S such that, for each t, $0\leqq t\leqq 1$, h_t is a homeomorphism of S onto itself, and h_0 is the identity mapping. The set of all images of a point x under isotopies of S onto itself is called the *continuous orbit* of x and is denoted by P_x. Note that if $y\in P_x$ (i.e. there is an isotopy carrying x onto y), then the isotopy path $h_t(x\times[0, 1])$ of x is a continuum containing x and y. Therefore P_x is a union of continuums, each containing the point x, and hence P_x is a connected subset of S.

DEFINITION 2. *A space S is continuously near-homogeneous if, for each point $x\in S$ and each nonempty open set U in S, there is an isotopy h_t of S onto itself such that $h_1(x)\in U$.*

THEOREM 7. *A continuously near-homogeneous space S is connected.*

Proof. Since the continuous orbit P_x of any point $x\in S$ contains a point in every open subset of S, P_x is dense in S. Since P_x is connected, so is $\overline{P}_x$ (the closure of P_x) and $\overline{P}_x=S$.

THEOREM 8. *If the connected T_1-space C is continuously near-homogeneous and if C has a cutpoint q, then $C=P_q$.*

Proof. Suppose that $C-q=U\cup V$, where U and V are disjoint nonempty open subsets of C. Any point $x\in U$ can be carried into V by an isotopy of C onto itself. But then we must have $q\in P_x$. This relation is an equivalence so we also have $x\in P_q$. An identical argument holds for each point $y\in V$.

COROLLARY 1. *No continuously near-homogeneous T_1 continuum has a cutpoint.*

Proof. It is well known that every T_1 continuum has at least two non-cutpoints. (See Theorem 2–18 of [3].) Thus the existence of a cutpoint in a continuously near-homogeneous T_1 continuum would contradict Theorem 8.

COROLLARY 2. *Every continuously near-homogeneous Peano continuum C contains a simple closed curve.*

Proof. By Corollary 1, C has no cutpoints. Hence by the well-known Cyclic Connectivity Theorem, every two points of C lie on a simple closed curve in C.

Before proving the main theorem of this section we need a simple but perhaps surprising result.

THEOREM 9. *Let the space S be continuously near-homogeneous and let S be locally euclidean of dimension n at some point p. Then S is an n-manifold without boundary.* (S may be either compact or noncompact, of course.)

Proof. S is connected by Theorem 7. Let x be any point of S, and U be an open n-cell neighborhood of the point p. There is an isotopy h_t of S onto itself such that $h_1(x)\in U$. Therefore $h_1^{-1}(U)$ is an open n-cell neighborhood of the point x.

The next result is yet another characterization of simple closed curves in the plane and it constitutes some justification for introducing continuous near-homogeneity.

THEOREM 10. *The only continuously near-homogeneous plane Peano continua are the simple closed curves.*

Proof. Let C be such a continuum. By Corollary 2 of Theorem 8, C contains a simple closed curve S. Suppose that $C-S$ is not empty. The open 2-cell bounded by S is not entirely contained in C because then Theorem 9 would apply to give us the impossible conclusion that C is a compact 2-manifold without boundary embedded in the plane E^2. Hence, without loss of generality, we assume that there is a point $p\in C-S$ in the bounded component of E^2-S. Let q be any point on S and let U be an open neighborhood of p such that $S\cap U$ is empty. Then there is an isotopy h_t of C onto itself such that $h_1(q)\in U$.

Now consider the intersection V of the bounded component of E^2-S and the unbounded component of $E^2-h_1(S)$. Clearly, V is not empty. Either V lies entirely in the isotopy path $h_t(S)$ of S or there is a point $x\in V$ not covered by $h_t(S)$. In the latter case, the point x and some point y in the unbounded component of E^2-S constitute a zero-sphere linking S which fails to link the isotopic image $h_1(S)$. This is known to be impossible. On the other hand, if V lies entirely in $h_t(S)$, then V lies in C and hence C contains an open 2-cell again which we know by Theorem 9 to be impossible. Having been led to a contradiction in either case, we must conclude that the point p is nonexistent and hence $C=S$.

NOTE. Any similarity between the development culminating in Theorem 10 above and the corresponding material found in [4] is purely intentional: It was this paper which inspired Definition 2.

Combining the ideas found in Definitions 1 and 2, we obtain another topological property.

DEFINITION 3. *A space S is continuously near-homogeneous at a point p if, for each point $x \in S$ and each open neighborhood U of p, there is an isotopy h_t of S onto itself such that $h_1(x) \in U$. The set of all points of S at which S is continuously near-homogeneous is denoted by $CN(S)$.*

The consequences of this definition are all analogous to the previous theorems in some fashion. We list some results below, giving a proof only in those cases which differ from the analogous theorem above.

THEOREM 11. *For any space S, the set $CN(S)$ is carried onto itself by isotopies of S onto itself.*

THEOREM 12. *The set $CN(S)$ is closed in S.*

THEOREM 13. *The set $CN(S)$ is a continuously near-homogeneous subspace of S.*

THEOREM 14. *If $CN(S)$ contains a nonempty open subset of S, then S is continuously near-homogeneous.*

In order to have an example or two on hand, we note that, for the n-cube I^n, $CN(I^n) = \mathrm{Bd}\ I^n$. Another example is the plane continuum consisting of two tangent circles. Here the point of tangency is the only point in $CN(S)$. In view of the next result it is fruitless to look for metric examples in which $CN(S)$ has a finite number of points $n > 1$.

THEOREM 15. *If S is a Hausdorff space, then $CN(S)$ has* 0,1 *or uncountably infinitely many points.*

Proof. Suppose that there are two points p and q in $CN(S)$. Let U be an open neighborhood of p not containing q. An isotopy of S onto itself carries q into U. The isotopy path of q is then a nondegenerate continuous image of the unit interval in a Hausdorff space and, by the well-known Hahn-Mazurkiewicz theorem, this isotopy path is a Peano continuum. Every nondegenerate Peano continuum contains uncountably many points. Finally, it is evident that each point in this isotopy path is in $CN(S)$.

In analogy to Theorem 7 above we have two easy results.

THEOREM 16. *If $CN(S)$ is nonempty, then S is connected.*

Proof. Let $p \in CN(S)$. Since the continuous orbit P_x of each point $x \in S$ contains a point in every open neighborhood of p, it follows that $p \in \overline{P}_x$ (the closure of P_x). Therefore $S = \bigcup \overline{P}_x$ is a union of connected sets, each containing the point p.

THEOREM 17. *The set $CN(S)$ is connected.*

Proof. This follows immediately from Theorems 7 and 13.

There is also a theorem on cutpoints analogous to Theorem 8.

THEOREM 18. *Let S be a T_1 continuum which is continuously near-homogeneous at a point p. Then p is the only possible cutpoint of S.*

(Note: The example of the two tangent circles may be kept in mind.)

Proof. Suppose that $q \neq p$ is a cutpoint of S and hence that $X - q = U \cup V$, where U and V are disjoint nonempty open sets. We may assume that $p \in U$. For each point $x \in V$, there is an isotopy h_t of S onto itself such that $h_1(x) \in U$. But then there must be some t_0, $0 < t_0 < 1$, such that $h_{t_0}(x) = q$. This implies that x is a cutpoint of S too. Since every point of V is a cutpoint of S, it follows that $\overline{V} = V \cup q$ is a subcontinuum of S having at most the one point q as a non-cutpoint. This is impossible.

COROLLARY. *If a Peano continuum C is continuously near-homogeneous at a non-cutpoint p, then C contains a simple closed curve.*

Proof. By Theorem 18, C can have no cutpoints. Hence the Cyclic Connectivity Theorem applies.

As a final result, our Theorem 9 can be rephrased.

THEOREM 19. *Let the space S be continuously near-homogeneous at a point p. If p has an open n-cell neighborhood in S, then S is an n-manifold without boundary. If p has an open neighborhood whose closure in an n-cell, then S is an n-manifold (possibly) with a connected boundary.*

Proof. We need only prove the second conclusion. To do so we first note that if p is not on the boundary of the closed n-cell neighborhood, we are back to the first case again. Hence we assume that p does lie on the boundary of the closed n-cell neighborhood. Then certainly p can be moved by an isotopy of S onto itself onto any point in a sufficiently small open $(n-1)$-cell in the boundary of this closed n-cell. (This can be done by an isotopy leaving the complement of the n-cell pointwise fixed.) Thus $CN(S)$ contains an open $(n-1)$-cell and hence is an $(n-1)$-manifold. The fact that the boundary of S has but one component follows from the fact that an isotopy must carry boundary components onto themselves.

In closing we might add an example which may prevent a few false conjectures. The plane continuum X pictured in Figure 3 has a simple closed curve S as the set $CN(X)$. Our purpose in presenting this example is to show why one of our conjectures was wrong. The conjecture was "If X is a continuum in E^{n+1} and $CN(X)$ contains an n-sphere, then X is an n-sphere."

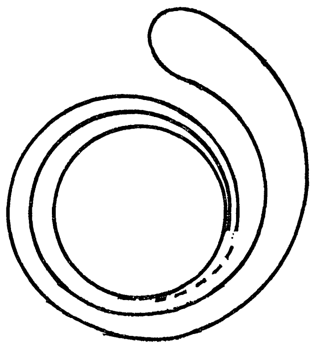

FIG. 3 (A loop spiraling down toward a circle.)

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SYMMETRIC BOOLEAN FUNCTIONS

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1. Introduction. A classical theorem in the theory of equations (see [1] or [3]) is that any polynomial symmetric in n variables over a field can be written as a polynomial in the n elementary symmetric functions. In this paper an even stronger result (Theorem 4.1) for Boolean functions is proved. A *Boolean function* is one which can be expressed by a finite number of joins, meets, and complementations. In a Boolean algebra, the operation of join will be denoted by the symbol $+$, meet by juxtaposition, and 0 and 1 will represent the first and last elements. Following the notation of [4], x^0 means the complement of x, and, when convenient, x^1 will be used in place of x. These exponents are not to be confused with the first and last elements, as they will often be treated in summations as real integers. The exponent or subscript e_j will assume only the values 0 and 1.

2. Elementary symmetric functions. As in the theory of polynomials, the kth *elementary symmetric function* of n variables, denoted by s_k, is defined as the sum (join) of all possible products (meets) of k distinct variables. Thus $s_k = \sum x_{i_1} \cdots x_{i_k}$, where all subscripts are, of course, taken from the first n positive integers. In order to simplify notation, it will be understood that the subscripts are in strictly ascending order and that the summation is taken over all subsets containing k positive integers, none exceeding n. Some relations between these functions will be considered.

THEOREM 2.1. $s_k + s_{k+1} = s_k$.

Proof. $s_k = \sum x_{i_1} \cdots x_{i_k}$ and $s_{k+1} = \sum x_{i_1} \cdots x_{i_{k+1}}$. Now each term in the sum s_{k+1} has $k+1$ distinct factors, and the product of the first k of these factors is exactly one of the terms in the sum s_k. Hence each term in s_{k+1} is a multiple of some term in s_k and can thus be combined with it by the absorption principle ($a + ab = a$ for each a, b), so that $s_k + s_{k+1} = s_k$.

The next theorem follows by induction.

THEOREM 2.2. $s_k + s_{k+j} = s_k$.

In a Boolean algebra, the properties $a + b = a$ and $ab = b$ are equivalent. Thus the next result follows from Theorem 2.2.

THEOREM 2.3. $s_k s_{k+j} = s_{k+j}$.

COROLLARY 2.3.1. $s_k^0 s_{k+j} = 0$.

COROLLARY 2.3.2. $s_k^0 s_{k+j}^0 = s_k^0$.

COROLLARY 2.3.3. $s_k^0 + s_{k+j}^0 = s_{k+j}^0$.

THEOREM 2.4. $s_k = \sum s_{i_1} \cdots s_{i_k}$.

Proof. Using induction on Theorem 2.3, $s_{i_1} \cdots s_{i_k} = s_{i_k}$, since the subscripts are arranged in ascending order. But $k \leqq i_k \leqq n$, so that

$$\sum s_{i_1} \cdots s_{i_k} = \sum_{i_k = k}^{n} s_{i_k} = s_k,$$

using induction on Theorem 2.2.

The kth *coelementary symmetric function*, c_k, is defined as $c_k = \sum x_{i_1}^0 \cdots x_{i_k}^0$. The elementary and coelementary functions are closely related.

THEOREM 2.5. $s_k^0 = c_{n-k+1}$.

Proof. Applying DeMorgan's formula,

$$\begin{aligned} s_k^0 &= \left(\sum x_{i_1} \cdots x_{i_k}\right)^0 = \prod \left(x_{i_1}^0 + \cdots + x_{i_k}^0\right) \\ &= \left(x_1^0 + \cdots + x_k^0\right) \cdots \left(x_{n-k+1}^0 + \cdots + x_n^0\right). \end{aligned}$$

This expansion is convenient for establishing several properties. First, no term

contains less than $n-k+1$ factors. For if there were a term of m factors with $m<n-k+1$, then, because of the symmetry of the variables, every term of m factors, and thus $x_1^0 \cdots x_m^0$, would appear; but this is impossible, since it is clear that one of $x_{n-k+1}^0, \cdots, x_n^0$ must be a factor of each term. Next, since one term appearing is $x_1^0 \cdots x_{n-k+1}^0$, every possible term of $n-k+1$ factors must appear also because of the symmetry. Finally, any term of more than $n-k+1$ factors would be a multiple of some term of $n-k+1$ factors and could be omitted by the absorption principle. Thus s_k^0 can be written as the sum of all possible products of $n-k+1$ factors taken from

$$\overset{0}{x_1}, \cdots, \overset{0}{x_n}, \quad \text{or} \quad \overset{0}{s_k} = \sum \overset{0}{x_{i_1}} \cdots \overset{0}{x_{i_{n-k+1}}} = c_{n-k+1}.$$

COROLLARY 2.5.1. $c_k^0 = s_{n-k+1}$.

The result of Theorem 2.5 is a convenient tool in the proof of the next theorem. As a further convenience, the symbol x^1 will now be used in place of x. The notation $\sum_{e,k}$ denotes that the summation is over all permutations of 0's and 1's for the e's such that their sum is k.

THEOREM 2.6. $s_k s_{k+1}^0 = \sum_{e,k} x_1^{e_1} \cdots x_n^{e_n}$.

Proof. $s_k s_{k+1}^0 = s_k c_{n-k} = (\sum x_{i_1}^1 \cdots x_{i_k}^1)(\sum x_{i_1}^0 \cdots x_{i_{n-k}}^0)$. Consider a term in the expansion, that is, the product of a term from the first sum by a term from the second sum. It has $k+(n-k)=n$ factors. In the case where less than n variables are represented, some variable is multiplied by its complement, and the term vanishes. The only terms not vanishing are those with exactly n variables, of which k are not complemented, with exponent 1, and $n-k$ are complemented, with exponent 0. In each such term the sum of the exponents is k. The theorem follows.

3. Functions of elementary symmetric functions. The theorems in this section reveal a striking similarity between the elementary symmetric functions and the variables themselves.

THEOREM 3.1. $\sum_{e,k} s_1^{e_1} \cdots s_n^{e_n} = s_k s_{k+1}^0$.

Proof. In the sum, each term, with one exception, will have a factor $s_j^0 s_{j+1}^1$ and, by Corollary 2.3.1, will vanish. This leaves only the term

$$(\overset{1}{s_1} \cdots \overset{1}{s_k})(\overset{0}{s_{k+1}} \cdots \overset{0}{s_n}) = \overset{1}{s_k}\overset{0}{s_{k+1}},$$

and the theorem is proved.

The next theorem is an immediate consequence of the preceding and Theorem 2.6.

THEOREM 3.2. $\sum_{e,k} s_1^{e_1} \cdots s_n^{e_n} = \sum_{e,k} x_1^{e_1} \cdots x_n^{e_n}$.

4. More general symmetric functions. In Section 1, a Boolean function was described as one which can be expressed by a finite number of joins, meets, and

complementations. It is shown in [4] that any function of n variables can be written as

$$f(x_1, \cdots, x_n) = \sum_{e_j=0}^{1} f(e_1, \cdots, e_n) x_1^{e_1} \cdots x_n^{e_n}.$$

It is now possible to prove the main theorem on symmetric functions.

THEOREM 4.1. *If f is symmetric, then $f(x_1, \cdots, x_n) = f(s_1, \cdots, s_n)$.*

Proof. It is readily verified that f is symmetric if and only if $f(e_1, \cdots, e_n)$ is constant for all choices of the e's whose sum is constant. Let $a_k = f(e_1, \cdots, e_n)$, where $e_1 = \cdots = e_k = 1$, $e_{k+1} = \cdots = e_n = 0$. Then

$$\begin{aligned} f(x_1, \cdots, x_n) &= \sum_{k=0}^{n} \sum_{e,k} a_k\, x_1^{e_1} \cdots x_n^{e_n}. \\ &= \sum_{k=0}^{n} a_k \sum_{e,k} x_1^{e_1} \cdots x_n^{e_n} = \sum_{k=0}^{n} a_k \sum_{e,k} s_1^{e_1} \cdots s_n^{e_n} \\ &= \sum_{k=0}^{n} \sum_{e,k} a_k s_1^{e_1} \cdots s_n^{e_n} \\ &= \sum_{k=0}^{n} \sum_{e,k} f(e_1, \cdots, e_n) s_1^{e_1} \cdots s_n^{e_n} = f(s_1, \cdots, s_n). \end{aligned}$$

It has thus been proved that any symmetric function of n variables can be written as the same function of the n elementary symmetric functions.

Further information on symmetric functions may be found in [2] and [5]. The author is indebted to the referee for suggesting these references.

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A LINEAR CONGRUENCE WITH SIDE CONDITIONS

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For positive integral r and n, and integral m, we denote by $\phi_r(n, m)$ the number of distinct solutions of the congruence

$$(1)\qquad \begin{cases} x_1 + x_2 + \cdots + x_r \equiv m \pmod{n} \text{ having} \\ (x_1, n) = (x_2, n) = \cdots = (x_r, n) = 1. \end{cases}$$

By establishing a connection between $\phi_r(n, m)$ and Ramanujan's exponential sum $C_n(m)$, we shall obtain for the former a product formula which reveals a striking sensitivity in the behavior of (1) to the parities of r, n and m.

First we observe several elementary properties of the function $\phi_r(n, m)$. For example, we have

$$(2)\qquad \sum_{m \bmod n} \phi_r(n, m) = \phi^r(n),$$

the right side being the rth power of the Euler ϕ-function. Here the sum on m is extended over any complete residue system modulo n. Later we shall use an asterisk to indicate summation over any *reduced* residue system. Equation (2) may be interpreted as saying that, for fixed r and n, the average number of solutions of (1) is $\phi^r(n)/n$.

To verify (2) we note that the left side represents the number of ways of forming r-tuples $(x_1, \cdots, x_r)$ with each x_i relatively prime to n and the sum $x_1 + \cdots + x_r$ congruent to any number m whatever. Since there are $\phi(n)$ possible values for each x_i, the number of such r-tuples is $\phi^r(n)$, as asserted by (2).

By similar reasoning we establish the following identity, which amounts to saying that the "cross-correlation function" of $\phi_r(n, -)$ and $\phi_s(n, -)$ is $\phi_{r+s}(n, -)$:

$$(3)\qquad \sum_{m \bmod n} \phi_r(n, m)\phi_s(n, m + k) = \phi_{r+s}(n, k).$$

For each value of $m \pmod{n}$ inside the sum there are $\phi_r(n, m)$ solutions of the congruence (1), and $\phi_s(n, m+k)$ solutions of

$$y_1 + \cdots + y_s \equiv m + k \pmod{n}$$

with

$$(y_1, n) = (y_2, n) = \cdots = (y_s, n) = 1.$$

There are accordingly $\phi_r(n, m)\phi_s(n, m+k)$ solutions of the simultaneous congruences

$$\begin{aligned} x_1 + \cdots + x_r &\equiv m \pmod{n} \\ y_1 + \cdots + y_s - x_1 - \cdots - x_r &\equiv k \pmod{n} \end{aligned}$$

with each x_i and y_j relatively prime to n. After summation on m, the left side of

(3) represents simply the number of solutions of $y_1+\cdots+y_s-x_1-\cdots-x_r \equiv k \pmod{n}$ with all variables relatively prime to n, and this number is $\phi_{r+s}(n, k)$.

An interesting special case of (3) occurs when $r=1$. Since $\phi_1(n, m)=1$ if $(m, n)=1$ and is zero otherwise, we have

$$\phi_{s+1}(n, k) = \sum_{m \bmod n}^{*} \phi_s(n, m+k), \tag{4}$$

a recurrence formula by means of which values of $\phi_r(n, m)$ can be computed systematically, starting with the function $\phi_1(n, m)$.

To obtain an explicit formula for $\phi_r(n, m)$, we shall begin by showing that it and the r-th power of Ramanujan's sum $C_n(m)$ form a "Fourier transform pair," namely:

$$C_n^r(m) = \sum_{k \bmod n} \phi_r(n, k)e(km/n) \tag{5}$$

$$\phi_r(n, m) = \frac{1}{n} \sum_{k \bmod n} C_n^r(k)e(km/n). \tag{6}$$

Here we have written $e(x)$ for $\exp(2\pi ix)$. Equation (5) thus asserts that $\phi_r(n, k)$ is the k-th coefficient in the expansion of $C_n^r(m)$ in a finite Fourier series. The formula (6) for $\phi_r(n, m)$ follows from (5) by the familiar orthogonality method of determining Fourier coefficients.

Ramanujan's sum $C_n(m)$ is defined by

$$C_n(m) = \sum_{k \bmod n}^{*} e(km/n).$$

Raising it to the r-th power, we have

$$\begin{aligned} C_n^r(m) &= \sum_{x_1 \bmod n}^{*} e(x_1m/n) \sum_{x_2 \bmod n}^{*} e(x_2m/n) \cdots \sum_{x_r \bmod n}^{*} e(x_rm/n) \\ &= \sum_{x_1 \bmod n}^{*} \sum_{x_2 \bmod n}^{*} \cdots \sum_{x_r \bmod n}^{*} e(m(x_1+x_2+\cdots+x_r)/n) \\ &= \sum_{k \bmod n} e(km/n) \sum^{*}_{\substack{x_1 \bmod n \\ x_2 \bmod n \\ \vdots \\ x_r \bmod n}} 1 \quad \Big\}\, x_1+x_2+\cdots+x_r \equiv k \pmod{n} \\ &= \sum_{k \bmod n} e(km/n)\phi_r(n, k), \end{aligned}$$

thus establishing (5). Since $C_n(m)$ reduces to $\phi(n)$ when $m=0$, we see that (2) is a special case of (5).

From (6) we may obtain an evaluation of $\phi_r(n, m)$ as a divisor sum, namely:

$$\phi_r(n, m) = \frac{\phi^r(n)}{n} \sum_{d|n} \frac{\mu^r(d)}{\phi^r(d)} C_d(m), \tag{7}$$

where $\mu(n)$ is the Möbius function. The proof is based on the formula

$$C_n(m) = \frac{\phi(n)\mu(n/(n, m))}{\phi(n/(n, m))} \tag{8}$$

for Ramanujan's sum given by Anderson and Apostol [1] and others. Substitution of (8) into (6) yields

$$\phi_r(n, m) = \frac{\phi^r(n)}{n} \sum_{k \bmod n} \frac{\mu^r(n/(n, k))}{\phi^r(n/(n, k))} e(km/n).$$

The order of summation is now changed, running first over all k (mod n) such that $(n, k)=d$ and finally over all divisors d of n.

$$\phi_r(n, m) = \frac{\phi^r(n)}{n} \sum_{d|n} \frac{\mu^r(n/d)}{\phi^r(n/d)} \sum_{\substack{k \bmod n \\ (n,k)=d}} e\left(\frac{(km/d)}{(n/d)}\right).$$

Since d also divides k, we may put $k=dj$, obtaining

$$\phi_r(n, m) = \frac{\phi^r(n)}{n} \sum_{d|n} \frac{\mu^r(n/d)}{\phi^r(n/d)} \sum_{\substack{j \bmod n/d \\ (n/d,j)=1}} e\left(\frac{jm}{(n/d)}\right).$$

The inside sum is $C_{n/d}(m)$, yielding (7) with the sum on d running backward.

Since $C_n(m)$ is a multiplicative function of n, as are $\phi(n)$ and $\mu(n)$, it is clear from the form of (7) that $\phi_r(n, m)$ is also multiplicative in the variable n. This enables us to convert the divisor sum (7) into an Euler product. To accomplish this, we note that for any multiplicative function $f(n)$, the sum

$$\sum_{d|n} \mu^r(d)f(d)$$

has the Euler product

$$\prod_{p|n} (1 + (-1)^r f(p))$$

extended over all prime divisors p of n. For fixed m and r, if we let $f(d) = C_d(m)/\phi^r(d)$, equation (7) yields

$$\phi_r(n, m) = \frac{\phi^r(n)}{n} \prod_{p|n} \left(1 + (-1)^r \frac{C_p(m)}{(p - 1)^r}\right).$$

From (8) we find that $C_p(m)=p-1$ or -1 according as $p|m$ or $p\nmid m$, and substitution of these values gives our final result:

THEOREM

$$\phi_r(n, m) = \frac{\phi^r(n)}{n} \prod_{p|(n,m)} \left(1 - \frac{(-1)^{r-1}}{(p-1)^{r-1}}\right) \prod_{\substack{p|n \\ p\nmid m}} \left(1 - \frac{(-1)^r}{(p-1)^r}\right). \tag{9}$$

Inspection of this formula reveals several interesting facts. First of all, the product vanishes if, and only if, n is even and r and m have opposite parity, and therefore these are precisely the conditions under which the congruence (1) is unsolvable. Clearly, under these conditions there can be no solution, for all r of the x_i would have to be odd, and their sum must then have the same parity as r. In any other case the congruence (1) possesses solutions.

We also note from (9) a basic difference between the cases in which the number r of variables is even, and those in which r is odd. For fixed modulus n if r is even, the number of solutions of (1) is relatively large for values of m having many small prime factors in common with n, and relatively small otherwise. In fact, $\phi_r(n, m)$ is maximum when $m=0$ and minimum (zero in the case of even n) when $m=1$. If r is odd, exactly the reverse is true.

We suppose now that r is *large* (either even or odd) and consider the effect of the parity of n on the number of solutions of (1). We have seen that the average number of solutions, for fixed n, is $\phi^r(n)/n$. The theorem shows that for n odd, the number of solutions is nearly equal to this average value for all m, whereas for n even, the number of solutions is zero half the time (when m is of opposite parity from r) and about twice the average value for the remaining values of m. This is so because for large r each factor of the product in (9) is approximately equal to 1 except if $p=2$, when the corresponding factor is zero or 2.

In the case of two variables, the function $\phi_2(n, m)$ is readily verified to be equal to the function $\phi(n, m)$ defined by Alder [2] as the number of integer solutions (x, y) of the equation

$$x + y = n + m$$

satisfying $1 \leqq x \leqq n$ and $(x, n) = (y, n) = 1$. When $r=2$, equation (9) reduces to

$$\phi_2(n, m) = n \prod_{p|(n,m)} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|n \\ p\nmid m}} \left(1 - \frac{2}{p}\right), \tag{10}$$

a result given by Alder.

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A GENERATING FUNCTION FOR A CLASS OF ARITHMETIC FUNCTIONS

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We propose to generalize here the results given by V. C. Harris and L. T. Warren [1] concerning the function $\sigma_k(n)$ which stands for the sum of the kth powers of the divisors of n.

Let $g(n)$ be any multiplicative function of n, so that we have $g(mn) = g(m)g(n)$ whenever m and n are mutually prime. Let $h(n) = \sum g(d)$, summed over the divisors of n, so that $h(n)$ is also a multiplicative function of n. Let a be any positive integer and r the largest divisor of a which is prime to n. Let $a = rs$. Then, as a generalization of Theorem 1 of [1], we have

THEOREM 1. *Let $f(a, n)$ be the arithmetic function defined by the relation*

$$\sum_{n=1}^{\infty} \frac{f(a, n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} h(an)x^n.$$

Then $f(a, n) = h(r)g(sn)$.

Proof. We have $h(an) = \sum_{d|n} f(a, d)$, so that

$$f(a, n) = \sum_{d|n} h(an/d)\mu(d) = \sum_{d|ns} h(an/d)\mu(d) = h(r)\sum_{d|ns} h(sn/d)\mu(d), \tag{1.1}$$

$\mu(n)$ being the Möbius function. Also $h(n) = \sum_{d|n} g(d)$, so that by the Möbius Inversion formula,

$$g(ns) = \sum_{d|ns} h(sn/d)\mu(d).$$

The theorem now follows on combining this result with (1.1).

THEOREM 2. *Let $g(n)$ be a positive valued and unconditionally multiplicative function of n, so that $g(mn) = g(m)g(n)$ for all positive integers m and n. Then the function $k(a, n)$ given by $k(a, n) = f(a, n)/g(n)$ is periodic in n with least period P, where P is the product of the distinct prime factors of a.*

Proof. If b is any factor of a such that $(b, n) = 1$, then $(b, n+P) = 1$ and conversely. Hence r and s are unaltered by replacing n by $n+P$. Now

$$\begin{aligned} f(a, n)/g(n) &= h(r)g(sn)/g(n) \\ &= h(r)g(s)g(n)/g(n) = h(r)g(s). \end{aligned}$$

Hence $k(a, n)$ has period P in n.

We can easily show that P is the least period. For if R is the least period we have $k(a, n) = k(a, n+R)$ for all n. Taking $n = a$ we get $h(1)g(a) = h(t)g(u)$ where t is the largest factor of a such that $(t, a+R) = 1$ and $a = tu$. Since $g(n)$ is positive and unconditionally multiplicative it follows that $h(n)$ is positive and

multiplicative in n, and in particular $h(1)=1$, since $g(1)=1$. Thus $h(t)g(u)=g(a)=g(ut)=g(u)g(t)$, giving $h(t)=g(t)$. Since $h(t)=\sum_{d|t} g(d)$ it follows that $t=1$. This shows that every prime factor of a is a prime factor of R, proving that P is the least period of $k(a, n)$.

Remarks. A large number of arithmetic functions are of the form $h(n)$. Thus taking $g(n)=1$, $h(n)$ becomes $\tau(n)$, the number of divisors of n. If $g(n)=n^r$ we get $h(n)=\sigma_r(n)$, and thus obtain the results of [1].

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MATHEMATICAL NOTES

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ON SIMULTANEOUS HERMITIAN CONGRUENCE TRANSFORMATIONS OF MATRICES

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We shall establish here a few results regarding simultaneous transformation of two matrices. The transpose and conjugate transpose of a square matrix P will be denoted respectively by P' and P^*; similarly for a vector; and $|P|$ will denote the determinant of P. We shall use the notation

$$\begin{bmatrix} P & \\ & Q \end{bmatrix}$$

to denote the direct sum of two square matrices P and Q. A triangular matrix is one in which the elements below the main diagonal are 0.

Our first result is given in the following theorem.

THEOREM 1. *If A and B are square matrices of the same size and are such that for no column vector ξ with complex elements $\xi^*A\xi=\xi^*B\xi=0$, then there exists a nonsingular matrix C such that C^*AC and C^*BC are both triangular matrices.*

Proof. Let A and B be $n \times n$ matrices. Let λ be a root of the equation $|A-\lambda B|=0$. Take a nonnull vector ξ_1 written as a column vector, such that $A\xi_1=\lambda B\xi_1$. Choose a set of linearly independent column vectors $\xi_2, \xi_3, \cdots, \xi_n$ satisfying $\xi_i^*A\xi_1=0$ or $\xi_i^*B\xi_1=0$, $i=2, 3, \cdots, n$ according as $\lambda\neq 0$ or $\lambda=0$.

Let D be a matrix whose columns are $\xi_1, \xi_2, \cdots, \xi_n$. Then, noting that $\xi_i^* A\xi_1 = \xi_i^* B\xi_1 = 0$, $i = 2, 3, \cdots, n$, we get

$$
(1) \qquad
\begin{aligned}
D^*AD &= \begin{bmatrix} \xi_1^*A\xi_1 & \xi_1^*A\xi_2 \cdots \xi_1^*A\xi_n \\ 0 & \\ 0 & A_1 \\ \vdots & \\ 0 & \\ 0 & \end{bmatrix} \\
D^*BD &= \begin{bmatrix} \xi_1^*B\xi_1 & \xi_1^*B\xi_2 \cdots \xi_1^*B\xi_n \\ 0 & \\ 0 & B_1 \\ \vdots & \\ 0 & \end{bmatrix}
\end{aligned}
$$

where A_1 and B_1 are $(n-1)\times(n-1)$ matrices.

The vector ξ_1 is linearly independent of $\xi_2, \xi_3, \cdots, \xi_n$. For, if not, let $\xi_1 = e_2\xi_2 + e_3\xi_3 + \cdots + e_n\xi_n$ for suitable e's. We now have

$$
(2) \qquad
\begin{aligned}
\xi_1^*A\xi_1 &= (\bar{e}_2\xi_2^* + \bar{e}_3\xi_3^* + \cdots + \bar{e}_n\xi_n^*)A\xi_1 = 0, \\
\xi_1^*B\xi_1 &= (\bar{e}_2\xi_2^* + \bar{e}_3\xi_3^* + \cdots + \bar{e}_n\xi_n^*)B\xi_1 = 0.
\end{aligned}
$$

So $\xi_1^*A\xi_1 = \xi_1^*B\xi_1 = 0$ and this contradicts the hypothesis. Hence $\xi_1, \xi_2, \cdots, \xi_n$ are linearly independent and thus D is nonsingular.

Now A_1 and B_1 satisfy the conditions of the theorem. For if there exists a nonnull column vector η_1 with complex elements $z_2, z_3, \cdots, z_n$, not all 0, such that $\eta_1^*A_1\eta_1 = \eta_1^*B_1\eta_1 = 0$, then clearly

$$
(3) \qquad
\begin{aligned}
0 &= \eta_1^*A_1\eta_1 = \eta_1^*D_1^*AD_1\eta_1 = \eta^*A\eta \\
0 &= \eta_1^*B_1\eta_1 = \eta_1^*D_1^*BD_1\eta_1 = \eta^*B\eta,
\end{aligned}
$$

where D_1 is an $n\times(n-1)$ matrix whose columns are $\xi_2, \xi_3, \cdots, \xi_n$ and $\eta = \sum_2^n z_i\xi_i$. Since $\xi_2, \xi_3, \cdots, \xi_n$ are linearly independent, η cannot be the null vector and it satisfies the relation $\eta^*A\eta = \eta^*B\eta = 0$, contradicting the hypothesis.

In case A and B are 2×2 matrices the above shows that D^*AD and D^*BD are triangular matrices while D is nonsingular. Thus the theorem holds for 2×2 matrices. For the purpose of mathematical induction let us assume that the theorem holds for $(n-1)\times(n-1)$ matrices.

We then have a nonsingular $(n-1)\times(n-1)$ matrix E such that E^*A_1E and E^*B_1E are both triangular. If

$$
F = \begin{bmatrix} 1 & \\ & E \end{bmatrix} \quad \text{and} \quad C = DF,
$$

it is obvious that C^*AC and C^*BC are both triangular. So the theorem holds for $n \times n$ matrices. This completes the proof.

Supposing A and B to be hermitian, we infer Theorem 2 from Theorem 1 by using the fact that now $C^*AC=(C^*AC)^*$ and $C^*BC=(C^*BC)^*$.

THEOREM 2. *If A and B are hermitian matrices (of the same size) and are such that for no nonnull column vector ξ, $\xi^*A\xi=\xi^*B\xi=0$, then there exists a nonsingular matrix C such that C^*AC and C^*BC are both diagonal matrices with real elements.*

If A and B are real symmetric matrices then the roots of the equation $|A-\lambda B|=0$ are real. Therefore ξ_1 and the vectors $\xi_2, \xi_3, \cdots, \xi_n$ in the proof of Theorem 1 can be chosen to be real. Hence the following theorem holds.

THEOREM 3. *If A and B are real symmetric matrices of the same size and are such that for no nonnull column vector ξ with complex elements $\xi^*A\xi=\xi^*B\xi=0$, then there exists a real nonsingular matrix C with the property that $C'AC$ and $C'BC$ are both diagonal matrices.*

This theorem fails to hold if we replace the condition that for no nonnull column vector ξ with complex elements $\xi^*A\xi=\xi^*B\xi=0$ by the condition that for no nonnull column vector ξ with real elements $\xi'A\xi=\xi'B\xi=0$. To see this take

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}.$$

Theorem 3 is a generalization of the known result that if A and B are real symmetric matrices of same size and one of them is positive definite, then there exists a nonsingular real matrix C for which $C'AC$ and $C'BC$ are both diagonal matrices. Theorem 3 has been given in [1].

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A NOTE ON COMPLETE RESIDUE SYSTEMS

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The following question seems to be of significance in connection with the design of computing machines: Given integers $m_1, \cdots, m_n$, each larger than 1, and with product $m_1 \cdots m_n=m$, and integers $a_1, \cdots, a_n$, under what circumstances is the set of m integers

$$A\colon a_1q_1+\cdots+a_nq_n, \quad q_1=0,1,\cdots,m_1-1,\cdots,q_n=0,1,\cdots,m_n-1$$

a complete residue system (mod m)? Since this question does not appear to be amenable to a simple direct combinatorial argument, the following theorem and its proof may be of some interest.

THEOREM. *The set A is a complete residue system* (mod m) *if and only if the pairs m_j, a_j can be renumbered in such fashion that for suitable integers $c_1, \cdots, c_n$,*

$$
\begin{array}{ll}
a_n = c_n m_{n-1} \cdots m_1, & (c_n, m_n) = 1, \\
a_{n-1} = c_{n-1} m_{n-2} \cdots m_1, & (c_{n-1}, m_{n-1}) = 1, \\
\cdots\cdots\cdots & \\
a_2 = c_2 m_1, & (c_2, m_2) = 1, \\
a_1 = c_1, & (c_1, m_1) = 1.
\end{array} \tag{1}
$$

In proving the theorem, it is convenient to rename A as A_n, since we shall presently reduce the problem to one in which there are $n-1$ pairs m_j, a_j. If A_n is a complete residue system (mod m), then every m-th root of unity is to be found exactly once among the numbers $e^{2\pi i a/m}$, $a \in A_n$, and hence

$$\sum_{\substack{0 \leq q_j < m_j \\ 1 \leq j \leq n}} \exp\left[2\pi i(a_1 q_1 + \cdots + a_n q_n)/m\right] = 0,$$

or

$$\prod_{j=1}^{n} \sum_{q_j=0}^{m_j-1} \exp(2\pi i a_j q_j/m) = 0.$$

It follows that for some j,

$$\sum_{q_j=0}^{m_j-1} \exp(2\pi i a_j q_j/m) = \frac{1 - \exp(2\pi i a_j m_j/m)}{1 - \exp(2\pi i a_j/m)} = 0,$$

and hence that $m \mid a_j m_j$. By reordering, if necessary, we may suppose that $j=n$, and thus that $m \mid a_n m_n$. Put $d_n = (m, a_n)$. If $m/d_n < m_n$, then the congruence

$$\sum_{k=1}^{n} 0 \cdot a_k \equiv 0 \cdot a_1 + \cdots + 0 \cdot a_{n-1} + \frac{m}{d_n} \cdot a_n \pmod{m}$$

shows that the zero residue class is represented twice in A_n, contrary to hypothesis. Hence it must be true that

$$\frac{m}{d_n} \geqq m_n. \tag{2}$$

On the other hand, since

$$\frac{m}{d_n} \,\Big|\, \frac{a_n}{d_n} \cdot m_n \quad \text{and} \quad \left(\frac{m}{d_n}, \frac{a_n}{d_n}\right) = 1,$$

we have $m/d_n \mid m_n$, and hence, by (2), $m/d_n = m_n$. By the definition of m, we have $d_n = m_{n-1} \cdots m_1$, and by the definition of d_n, $a_n = c_n m_{n-1} \cdots m_1$, where $(c_n, m_n) = 1$.

Since A_n is a complete residue system (mod m), the congruence

$$a_1q_1 + \cdots + a_nq_n \equiv b \pmod{m}$$

is solvable for every b, and hence the congruence

$$a_1q_1 + \cdots + a_{n-1}q_{n-1} \equiv b \pmod{d_n}$$

is solvable for every b. Since the number of integers in the set A_{n-1} of integers on the left hand side of this last congruence is $m/m_n = d_n$, the set A_{n-1} must be a complete residue system (mod d_n), that is, (mod $m_{n-1} \cdots m_1$).

Proceeding inductively, we see that the first $n-1$ of the relations (1) must hold and, furthermore, that the set A_1 of numbers a_1q_1, $q_1 = 0, 1, \cdots, m_1 - 1$, must be a complete residue system (mod m_1). It is well known that A_1 has this property if and only if a_1 is relatively prime to m_1, giving the last of the relations (1).

Conversely, suppose that $a_1, \cdots, a_n$ are as in (1). Then if A is not a complete residue system (mod m), two elements of A must be congruent (mod m). Suppose that these elements arise from the n-tuples $q_1, \cdots, q_n$ and $q_1', \cdots, q_n'$. Putting $x_j = q_j - q_j'$, for $j = 0, 1, \cdots, n$, we see that

$$c_1x_1 + c_2m_1x_2 + c_3m_1m_2x_2 + \cdots + c_nm_1 \cdots m_{n-1}x_n \equiv 0 \pmod{m}, \tag{3}$$

with $|x_j| < m_j$ for $j = 0, 1, \cdots, n$. Replacing the modulus m by m_1 in (3), we obtain $x_1 = 0$; replacing m by m_1m_2 in (3), we obtain $x_2 = 0$, etc. This leads to the contradiction $q_1 = q_1', \cdots, q_n = q_n'$, which completes the proof.

The theorem proved here may be regarded as a special case of a conjecture of Minkowski (see [2]), for which a proof was given by Hajós [1]. Its deduction from Hajós's theorem is not completely trivial, however.

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ANOTHER PROOF OF CAYLEY'S FORMULA FOR COUNTING TREES

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Let $T(p)$ denote the number of trees with p labelled points. Many proofs have been given of Cayley's result that $T(p) = p^{p-2}$. (For definitions and references see, for example, [2].) The following derivation requires very little other than the following well-known and easily established identity (see [1]):

$$\sum_{i=0}^{p} (-1)^i \binom{p}{i} (p-i)^k = 0, \tag{1}$$

where k is a positive integer less than p.

Let $G(p, q)$ be the number of connected linear graphs with p labelled points and q edges; let $F(p, q)$ be the number of these which have no endpoints. An endpoint, by definition, is adjacent to precisely one other point and this other point cannot also be an endpoint if the graph is connected and has more than two points. Thus we find by the sieve method that

$$F(p, q) = \sum_{i=0}^{p} (-1)^i \binom{p}{i} G(p - i, q - i)(p - i)^i, \qquad \text{if } p > 2. \tag{2}$$

We may specialize (2) to trees by supposing that $q = p - 1$. In this case $F(p, p-1) = 0$ if $p > 1$, since every nontrivial tree has at least two endpoints. The general result that $T(p) = G(p, p-1) = p^{p-2}$ now follows immediately by induction when we replace k by $p-2$ in (1).

In closing we remark that it is a simple exercise to extend (2) to provide a direct and alternate proof of the result of Rényi [3] that the number $T(p, r)$ of trees with p labelled points of which exactly r are endpoints is given by the formula

$$T(p, r) = \frac{p!}{r!} S(p - 2, p - r), \tag{3}$$

where $S(n, k)$ denotes a Stirling number of the second kind.

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ON PERIODIC SOLUTIONS OF A FUNCTIONAL EQUATION

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In the present paper we are concerned with the functional equation

$$\phi[f(x)] = G(x, \phi(x)), \tag{1}$$

where $\phi(x)$ is the unknown function and $f(x)$ and $G(x, y)$ are given functions. We shall prove (under suitable conditions) the uniqueness of the continuous periodic solutions of equation (1). The very interesting and much more delicate problem of the existence of periodic solutions of (1) (comp. e.g. [1]) will not be dealt with. But even such a pure uniqueness theorem may be of interest in some cases, especially when a solution is known to exist, as is the case in [3]. In fact, Theorem 1 from [3] is a particular case of our Theorem 2 below.

1. In the sequel we assume that the function $f(x)$ fulfills the following conditions:

(i) $f(x)$ is continuous and strictly increasing in $[0, \infty)$,

(ii) $0<f(x)<x$ for $x \in (0, \infty)$,

(iii) $\lim_{x\to\infty} f(x)=\infty$.

It follows from the above conditions that the iterates $f^n(x)$ of the function $f(x)$:

$$f^0(x) = x, \quad f^{n+1}(x) = f(f^n(x)), \qquad x \in [0, \infty),\ n = 0, \pm 1, \pm 2, \cdots$$

are defined in the whole of $[0, \infty)$ for all integers n.

LEMMA 1. *If $f(x)$ satisfies* (i), (ii) *and* (iii) *then for every $x \in (0, \infty)$ the sequence $f^n(x)$ for $n>0$, decreases to* 0 *and the sequence $f^{-n}(x)$ for $n>0$, increases to $+\infty$.*

A proof of this lemma is to be found e.g. in [2].

Further we assume that

(iv) For every a, b, $0<a<b$, $\limsup_{n\to\infty} [f^{-n}(b)-f^{-n}(a)]>h>0$.

Condition (iv) will be fulfilled in particular if the function $f(x)$ satisfies for large x, a Lipschitz condition with constant $c<1$. Namely we have

LEMMA 2. *If the function $f(x)$ satisfies conditions* (i) *through* (iii) *and if there exists a number $t>0$ such that for every $x_1, x_2 \in (t, \infty)$*

$$|f(x_1) - f(x_2)| \leqq c|x_1 - x_2|, \tag{2}$$

where

$$0 < c < 1 \tag{3}$$

then for every a, b, $0<a<b$, we have

$$\lim_{n\to\infty} [f^{-n}(b) - f^{-n}(a)] = \infty. \tag{4}$$

Proof. We fix a, b, $0<a<b$. According to Lemma 1 there exists an N such that for $n \geqq N$, $f^{-n}(a)>t$. On account of (i) the functions $f^{-n}(x)$ are increasing; consequently $t<f^{-n}(a)<f^{-n}(b)$ for $n \geqq N$. We have moreover by (2)

$$f^{-n}(b) - f^{-n}(a) \leqq c[f^{-n-1}(b) - f^{-n-1}(a)] \qquad \text{for } n \geqq N,$$

hence, by iteration,

$$f^{-n}(b) - f^{-n}(a) \leqq c^k[f^{-n-k}(b) - f^{-n-k}(a)] \quad \text{for } n \geqq N,\ k \geqq 0,$$

and so, replacing n by N, multiplying through by c^{-k}, and letting $k\to\infty$, (4) results in view of (3).

Our considerations will be based on the following

THEOREM 1. *If the function $f(x)$ satisfies conditions* (i) *through* (iv), *then for every $x \in (0, \infty)$ the set of points $f^n(x+kh)$, $n, k=0, 1, \cdots$ is dense in* $(0, \infty)$.

Proof. We fix arbitrarily $x, a, b \in (0, \infty)$, $a<b$. By Lemma 1 there exists an integer $N_1 \geqq 0$ such that $f^{-n}(a)>x$ for $n \geqq N_1$. By (iv) there exists an integer

$N \geqq N_1$ such that $f^{-N}(b) - f^{-N}(a) > h$. Consequently there exists an integer $K \geqq 0$ such that $f^{-N}(a) < x + Kh < f^{-N}(b)$, whence by (i)

$$a < f^N(x + Kh) < b.$$

Since a and b have been arbitrary, this proves that the set of points $f^n(x+kh)$ is dense in $(0, \infty)$.

2. If a function $\phi(x)$ satisfies equation (1) in $[0, \infty)$, the value $d = \phi(0)$ cannot be quite arbitrary. It follows from (i) and (ii) that $f(0) = 0$, whence we obtain on setting $x = 0$ in (1) that d must be a root of the equation

$$d = G(0, d). \tag{5}$$

THEOREM 2. *If the function $f(x)$ satisfies conditions* (i) *through* (iv), *then for every value d satisfying* (5) *there exists at most one solution $\phi(x)$ of equation* (1) *which is continuous in* $(0, \infty)$ *and periodic with period h and such that*

$$\phi(0) = d. \tag{6}$$

Proof. Let $\phi(x)$ be a solution of equation (1) in $[0, \infty)$, continuous in $(0, \infty)$, periodic with period h and satisfying (6). Thus $\phi(kh) = d$, for $k = 0, 1, 2, \cdots$. By equation (1) the values of $\phi(x)$ are uniquely determined on the set E of the points $f^n(kh)$, $n, k = 0, 1, 2, \cdots$. On account of Theorem 1 the set E is dense in $(0, \infty)$ (one takes $x = h$) and thus $\phi(x)$, being continuous, is uniquely determined in the whole of $[0, \infty)$. (The periodicity condition allows us to extend $\phi(x)$ uniquely onto the whole real axis.) This completes the proof.

If the function $z = G(x, y)$ has the inverse (with respect to the second variable) $y = H(x, z)$, then in Theorem 2 it is enough to suppose that $\phi(x)$ is continuous in a right neighbourhood R of 0 (and not necessarily in the whole interval $(0, \infty)$). In fact, the above argument shows that then $\phi(x)$ is uniquely determined in R and according to Lemma 1 it is determined in the whole of $(0, \infty)$ by the equation

$$\phi(x) = H(x, \phi[f(x)]).$$

But it is quite essential that $\phi(x)$ be defined in the whole of $[0, \infty)$. For instance, it is impossible to apply the argument of Theorem 2 to the equation

$$\phi\left(\frac{x}{2}\right) = \frac{\phi(x)}{1 - s(x)\sqrt{(1 + [\phi(x)]^2)}}, \tag{7}$$

where

$$s(x) = \begin{cases} +1 & \text{for } x \in \left[(4n-3)\dfrac{\pi}{2},\ (4n-1)\dfrac{\pi}{2}\right) \\ -1 & \text{for } x \in \left[(4n-1)\dfrac{\pi}{2},\ (4n+1)\dfrac{\pi}{2}\right) \end{cases} \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Equation (7) and the condition

$$\phi(x+\pi) = \phi(x) \tag{8}$$

do not allow us to determine the function $\phi(x)$ for x other than $k\pi$. Thus if we want to characterize the function $\phi(x)=\tan x$ as the solution of equation (7) continuous in a neighbourhood of 0 and satisfying (8) (the only real solution of equation (5) in this case is $d=0$) we must assume further, e.g., that

$$\phi\left(\frac{\pi}{4}\right) = 1, \tag{9}$$

or that $\phi(x)$ is strictly increasing in a neighbourhood of 0 and continuous at $x=\pi/4$. In the latter case condition (9) can be proved.

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A MULTIPLE-POINT BOUNDARY CONDITION FOR LINEAR DIFFERENTIAL SYSTEMS

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I. Introduction We desire a unique solution of the differential system

$$y_i' = \sum_{j=1}^{n} a_{ij}(x)y_j + b_i(x), \qquad i = 1, \cdots, n; n \geqq 2 \tag{1}$$

satisfying the boundary conditions

$$y_1(\alpha_1) = \beta_1, \qquad y_m(\alpha_2) = \beta_m, \qquad y_n(\alpha_3) = \beta_n, \qquad m = 2, \cdots, n-1 \tag{2}$$

where the $a_{ij}(x)$, $b_i(x)$ are continuous over an interval $[a, b]$, where α_1, α_2, α_3 are required only to belong to $[a, b]$, and where β_i are arbitrarily assigned real numbers. It is easily seen that conditions must be imposed on the $a_{ij}(x)$ if the α_i are to remain arbitrary. Such conditions are established in the present paper.

Similar results are found in [1]. It is easily shown that these conditions neither imply nor are implied by the conditions of the present paper.

II. The Boundary Condition. For points α_1, α_2, α_3 of $[a, b]$ such that $\alpha_1 \leqq \alpha_2 \leqq \alpha_3$, with one equality holding in $n=2$, we impose the following conditions:

A. $a_{ij}(x) \equiv 0$, $i, j=1, \cdots, n$, $i \geqq j$, except for $a_{n1}(x)$.
For $k=1, \cdots, n-2$ and $m=2, \cdots, n-1$,

B. $a_{in}(x),\ a_{km}(x)\begin{cases}\leqq 0 & x\in(\alpha_1,\alpha_2] \\ \geqq 0 & x\in[\alpha_2,\alpha_3)\end{cases}$ if i and $k+m$ are even, and
$a_{n1}(x),\ a_{in}(x),\ a_{km}(x)\geqq 0$ if i and $k+m$ are odd.

C. For each fixed i, $1\leqq i\leqq n$, there exist segments $(\alpha_2-\delta_i,\alpha_2)$ and $(\alpha_2,\alpha_2+\delta_i)$ on which $a_{i1}(x),\cdots,a_{in}(x)$ do not all have a common zero.

We now prove

LEMMA 1. *Let the* $a_{ij}(x)$ *of*

$$y_i' = \sum_{j=1}^{n} a_{ij}(x)y_j, \qquad i = 1,\cdots,n \tag{3}$$

satisfy (A), (B), (C) *for some* $\alpha_1, \alpha_2, \alpha_3$ *of* $[a, b]$ *and let* $(y_{h1}(x),\cdots,y_{hn}(x))$ $h=1$ or n, *be the solutions of* (3) *satisfying the conditions*

$$y_{hj}(\alpha_2) = \delta_{hj} = \begin{cases}1 & j = h \\ 0 & j\neq h,\end{cases} \qquad j = 1,\cdots,n. \tag{4}$$

Then $x=\alpha_2$ *is an isolated zero of* $y_{hm}(x)$, $m=1,\cdots,n$; $m\neq h$.

Proof. We give the proof for $h=1$ and n even; the proof for the other cases follows in a similar manner. From (4), $y_{11}(\alpha_2)=1$; hence $y_{11}(x)>0$ on $(\alpha_2-\delta,\alpha_2+\delta)$ for some $\delta>0$. Thus, from (3),

$$y_{1n}'(x) = a_{n1}(x)y_{11}(x) > 0 \text{ on } (\alpha_2-\epsilon_n,\alpha_2),\ (\alpha_2,\alpha_2+\epsilon_n),$$

where $\epsilon_n=\min(\delta,\delta_n)$. This, with (4) gives

$$y_{1n}(x) < 0 \text{ on } (\alpha_2-\epsilon_n,\alpha_2), \qquad y_{1n}(x) > 0 \text{ on } (\alpha_2,\alpha_2+\epsilon_n).$$

Now

$$y_{1\,n-1}'(x) \equiv a_{n-1\,n}(x)\,y_{1n}(x) < 0 \text{ on } (\alpha_2-\epsilon_{n-1},\alpha_2),$$
$$y_{1\,n-1}'(x) > 0 \text{ on } (\alpha_2,\alpha_2+\epsilon_{n-1}),$$

where $\epsilon_{n-1}=\min(\delta_{n-1},\epsilon_n)$. Hence, by (4),

$$y_{1\,n-1}(x) > 0 \text{ on } (\alpha_2-\epsilon_{n-1},\alpha_2),\ (\alpha_2,\alpha_2+\epsilon_{n-1}).$$

By continuing in this manner we find that

$$y_{1j}(x) > 0 \text{ on } (\alpha_2-\epsilon_j,\alpha_2),\ (\alpha_2,\alpha_2+\epsilon_j),\ \ j \text{ odd}$$

and

$$y_{1j}(x) < 0 \text{ on } (\alpha_2-\epsilon_j,\alpha_2), \qquad y_{1j}(x) > 0 \text{ on } (\alpha_2,\alpha_2+\epsilon_j),\ \ j \text{ even}$$

for $j=1,\cdots,n$, where $\epsilon_j=\min(\delta_j,\epsilon_{j+1})$.

We now have each $y_{1m}(x)$, $m=2,\cdots,n$, either positive or negative to the

immediate left and right of $x=\alpha_2$. Thus, $x=\alpha_2$ is an isolated zero of these functions.

The main theorem is now given.

THEOREM 1. *Let the* $a_{ij}(x)$ *of* (1) *satisfy* (A), (B), (C) *for some* $\alpha_1, \alpha_2, \alpha_3$ *of* $[a, b]$. *Then there exists a unique solution of* (1) *satisfying* (2).

Proof. Let $(y_{i1}(x), \cdots, y_{in}(x))$, $i=1, \cdots, n$, be the solutions of (3) satisfying

$$y_{ij}(\alpha_2) = \delta_{ij}, \qquad i, j = 1, \cdots, n. \tag{5}$$

Then the general solution $(y_1(x), \cdots, y_n(x))$ of (1) is given by

$$y_i(x) = c_1 y_{1i}(x) + \cdots + c_n y_{ni}(x) + y_{pi}(x), \qquad i = 1, \cdots, n,$$

where $(y_{p1}(x), \cdots, y_{pn}(x))$ is a particular solution of (1). Imposing (2) and (5) we obtain

$$\begin{aligned} c_h &= \beta_h - y_{ph}(\alpha_2), \qquad h = 2, \cdots, n-1 \\ c_1 y_{11}(\alpha_1) + \cdots + c_n y_{n1}(\alpha_1) &= \beta_1 - y_{p1}(\alpha_1) \\ c_1 y_{1n}(\alpha_3) + \cdots + c_n y_{nn}(\alpha_3) &= \beta_n - y_{pn}(\alpha_3). \end{aligned}$$

This system has a solution, and (2) can be satisfied uniquely, if

$$y_{11}(\alpha_1) y_{nn}(\alpha_3) - y_{n1}(\alpha_1) y_{1n}(\alpha_3) \neq 0.$$

We show this by first establishing $y_{11}(x) > 0$ on $[\alpha_1, \alpha_2)$.

Assume that at least one $y_{11}(x), \cdots y_{1n}(x)$ has a zero on $[\alpha_1, \alpha_2)$. Let c be the largest of all zeros of these functions on this interval; by virtue of Lemma 1 this is possible. Then the sign of $y_{1j}(x)$, as found in the proof of the lemma, holds over the segment (c, α_2). If $y_{11}(c) = 0$ then $y_{11}'(x) > 0$ to the immediate right of $x=c$. But, from the proof of Lemma 1 with the fact that $a_{1n}(x) \geqq 0$, it is found that $y_{11}'(x) \leqq 0$ on (c, α_2); hence $y_{11}(c) \neq 0$. The function $y_{1m}(x)$, $m=2, \cdots, n$, does not vanish at $x=c$ since it vanishes at $x=\alpha_2$ and since further, from the proof of Lemma 1, its derivative does not change sign on (c, α_2). Hence $y_{11}(x), \cdots, y_{1n}(x)$ do not vanish on $[\alpha_1, \alpha_2)$ and $y_{11}(x) > 0$ on $[\alpha_1, \alpha_2)$.

In a similar manner we find $y_{n1}(x) < 0$ on $[\alpha_1, \alpha_2)$, $y_{1n}(x), y_{nn}(x) > 0$ on $(\alpha_2, \alpha_3]$. These inequalities imply a unique solution for the c_i and, thus, a unique solution of (1), (2).

COROLLARY. *Let the* $a_{ij}(x)$ *satisfy* (A), (B), (C) *over* $[a, \alpha_2)$, $(\alpha_2, b]$ *for some* $\alpha_2 \in [a, b]$. *Then Theorem* 1 *holds without restricting* α_1, α_3 *further than requiring them to belong to* $[a, \alpha_2]$, $[\alpha_2, b]$, *respectively.*

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FINITE HOMOGENEOUS BOLYAI-LOBACHEVSKY PLANES

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L. M. Graves [1] and T. G. Ostrom [2] have recently published articles in this MONTHLY on the subject of finite Bolyai-Lobachevsky Planes (defined in [2]). Graves asks for additional constructions of such planes, and Ostrom has constructed an infinite class. In this note, another infinite class is defined and is seen to have some interesting properties.

Let π be a finite projective plane of order $n \geqq 7$ (i.e., there are $n+1$ points on each line of π and n^2+n+1 points in π). Let π_0 be the set of points obtained by removing from π all of the points contained in three lines L_1, L_2, L_3 of π, subject only to the restriction that the L_i do not contain a common point. The lines of π_0 are to be the lines of $\pi(\neq L_1, L_2, L_3)$ with the points contained in L_1, L_2, L_3 removed. Note that the construction is quite similar to the classical construction of affine planes from projective planes.

The following facts are immediate:

(1) Every point of π_0 is contained in $n+1$ lines of π_0.

(2) Every line of π_0 contains either $n-1$ or $n-2$ points of π_0 (depending on whether a line of π_0, considered as a line of π, contains an intersection $L_i \cap L_j$, $i, j=1, 2, 3$).

(3) Two distinct points of π_0 determine a unique line of π_0.

(4) Through each point $P \in \pi_0$ not on a line $L \in \pi_0$, there pass at least two lines of π_0 which do not intersect L.

The last property is obtained by considering the lines (of π) through P which intersect L in the points $L \cap L_i$, $i=1, 2, 3$.

Properties 3 and 4 are the first two axioms (see [2]) in the definition of B-L planes. The third axiom states that if a subset of the points of π_0 contains three noncollinear points, and all of the points on the line through any two points of the subset, then the subset contains all the points of the plane π_0. To see that this last condition is satisfied, let P_1, P_2, P_3 be three noncollinear points of π_0. The line $L=P_1 \cdot P_2$ contains at least $n-2$ points of π_0, so there must be at least $n-2$ lines through P_3. Let L' be a line of π through P_3 which has not yet been generated. Then $L' \neq M=P_1 \cdot P_3$, $L' \neq N=P_2 \cdot P_3$. For each $Q \in L'$, with $Q \in \pi_0$ and $Q \neq P_3$, consider the point $(Q \cdot P_1) \cap N$. At most three such points can be excluded from π_0, so if there are at least 8 points on L', there must be at least four $Q_j \in L'$, $Q_j \in \pi_0$, $Q_j \neq P_3$. For one of these four, say Q_1, $R=(Q_1 \cdot P_1) \cap N \in \pi_0$. But then the line $R \cdot P_1 \in \pi_0$ contains Q_1, hence the points of the line L' are all in the subset. In this way the two or three lines not immediately defined by the points P_i are easily picked up and all of the lines of π_0 passing through P_3 are generated. But these lines contain all of the points of π_0, and the axiom is verified.

A B-L plane π_0 is called *homogeneous* if its collineation group is transitive on the points of π_0. Clearly any collineation of π which maps the lines L_i, $i=1, 2, 3$, among themselves induces a collineation of π_0. In particular, then, if π is

Desarguesian, it is known that the collineation group of π is transitive on quadrilaterals. Thus the subgroup of the collineation group of π which leaves the points $L_i \cap L_j$ fixed $(i, j = 1, 2, 3)$, will induce a group of collineations of π_0 which is transitive on the points of π_0.

The construction given here can be extended to the case where more than three lines are removed from the plane π, and also to the case where π is infinite to obtain new examples of B-L planes satisfying the axioms of [2].

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NOTE ON AN EXAMPLE OF HEATH

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Let $\langle S, \mathfrak{J}\rangle$ be a topological space, and let ρ be a nonnegative, real-valued function having domain $S \times S$ and satisfying for all $y, y \in S \times S$ the following conditions: (i) $\rho(x, y) = 0$ if and only if $x = y$; (ii) $\rho(x, y) = \rho(y, x)$. If, for each $p \in S$, the collection $\{S_\rho(p; r) \mid r > 0\}$ of ρ-spheres about p is a local base at p for $\mathfrak{J}$, we say that the semi-metric ρ is admissible on $\langle S, \mathfrak{J}\rangle$ and refer to $\langle S, \rho\rangle$ as a semi-metric space. If $\langle S, \rho\rangle$ is a semi-metric space and if ρ satisfies the additional condition (iii) for all $x, y, z \in S$, $\rho(x, z) \leqq \rho(x, y) + \rho(y, z)$, then $\langle S, \rho\rangle$ is called a metric space.

In [1], R. W. Heath gives an example of a regular semi-metric space for which there is no semi-metric under which all spheres are open. This example provides a negative answer to the question about openness of semi-metric spheres, which was raised by Brown in [2]. Brown also stated in [2] that it is not known whether or not every collectionwise normal semi-metric space has a semi-metric which is continuous in each variable separately. It is the purpose of this note to point out that Heath's example also furnishes a negative answer to this question.

THEOREM. *If $\langle S, \rho\rangle$ is a semi-metric space and if ρ is continuous in each variable, then $S_\rho(p; r)$ is open for each $p \in S$ and each $r > 0$.*

Proof. Let q be a limit point of $S - S_\rho(p; r)$. For each $n \in I^+$, there exists $q_n \in S - S_\rho(p; r)$ such that $0 < \rho(q, q_n) < 1/n$, and thus $\lim_{n\to\infty} q_n = q$. For each $n \in I^+$, we have $\rho(p, q_n) \geqq r$, and thus $\lim_{n\to\infty} \rho(p, q_n) = \rho(p, q) \geqq r$ since ρ is continuous in its second variable. Hence, $q \in S - S_\rho(p; r)$, which implies $S - S_\rho(p; r)$ is closed and $S_\rho(p; r)$ is open.

COROLLARY. *There exists a collectionwise normal semi-metric space for which there exists no admissible semi-metric which is continuous in each variable.*

Proof. The example given by Heath is easily seen to be collectionwise normal since it is paracompact. Suppose this space had an admissible semi-metric ρ

which was continuous in each variable. By our Theorem, $S_\rho(p; r)$ would be open for each $p \in S$ and each $r>0$. This would contradict Heath's result in [1].

The contents of this paper are contained in the author's Ph.D. dissertation (1962), written at Iowa State University under the supervision of Dr. D. E. Sanderson.

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ON SETS OF INTEGERS WITH g.c.d. 1 HAVING NO PROPER SUBSET WITH g.c.d. 1

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Throughout this paper the symbol S denotes generically a set of $n (n>1)$ positive integers with greatest common divisor (g.c.d.) 1 having no proper subset with g.c.d. 1. Such sets have certain interesting properties.

Suppose that $S=\{a_1, \cdots, a_n\}$ and let $k=1, \cdots, n$. Define $A_k=S-\{a_k\}$ and let p_k be a prime common divisor of the elements of A_k. The primes p_k are distinct because $(a_1, \cdots, a_n)=1$. On defining

$$P_k = \prod_{j=1}^{n} p_j/p_k$$

we see that $a_k=P_kN_k$, where $(N_1, \cdots, N_n)=1$ and, moreover, p_k does not divide N_k. We have established the necessity of the condition given in the following theorem.

THEOREM 1. *A set $\{a_1, \cdots, a_n\}$ is a set S if and only if there exist integers N_k with $(N_1, \cdots, N_n)=1$ and distinct primes p_k such that $a_k=N_k\pi_{j=1}^{n} p_j/p_k$ and p_k does not divide N_k.*

To establish the sufficiency of the stated condition, we observe that p_k is a common divisor of the elements of A_k, whence no proper subset of $\{a_1, \cdots, a_n\}$ has g.c.d. 1. None of the primes p_k is a common divisor of all the integers a_k, for p_k does not divide N_k, and since $(N_1, \cdots, N_n)=1$ we see that $(a_1, \cdots, a_n)=1$.

THEOREM 2. *In a set S with $n>2$, no two elements, upon division by a third element, have equal remainders or one remainder divisible by the other.*

Let us consider first the following two lemmas.

LEMMA 1. *A necessary condition that the general solution of the linear Diophantine equation $a_1x_1+a_2x_2+\cdots+a_nx_n=c$, $(a_1, a_2, \cdots, a_n)=1$ contain the unknowns $x_1, \cdots, x_s (s<n)$ as parameters is that $(a_{s+1}, \cdots, a_n)=1$.*

Let $x_1, \cdots, x_s$ be parameters in the general solution of the equation, and suppose that $(a_{s+1}, \cdots, a_n) = d$. We may write the equation as

$$a_{s+1}x_{s+1} + \cdots + a_nx_n = c - a_1x_1 - \cdots - a_sx_s$$

and consider the following sets of particular values for the parameters $x_1, \cdots, x_s$. If $x_1 = x_2 = x_3 = \cdots = x_s = 0$, we conclude that $d \mid c$. By taking the sets of parameter values $x_k = 1$, $x_i = 0$, $i \neq k$, $1 \leqq i \leqq s$ for k: $1, \cdots, s$, we conclude $d \mid c - a_k$, k: $1, \cdots, s$ and hence $d \mid a_k$, k: $1, \cdots, s$. So we have $d \mid a_i$, i: $1, \cdots, n$, and thus $d = 1$.

LEMMA 2. *If in a set of $n > 2$ integers $a_1, \cdots, a_n$ with g.c.d. 1 the remainders of two integers a_r, a_s upon division by a third a_t are equal, or one is divisible by the other, then the subset of integers $a_1, \cdots, a_n$ excluding either one of a_r and a_s if the remainders are equal, or that with largest numerical remainder, has g.c.d.* 1.

The Diophantine equation

$$a_1x_1 + \cdots + a_nx_n = 1 \tag{1}$$

is solvable since $(a_1, \cdots, a_n) = 1$. Let $a_r = a_n$, $a_s = a_{n-1}$, $a_t = a_1$, and r_n, r_{n-1} the remainders of a_n, a_{n-1} upon division by a_1. By dividing all coefficients of (1) by a_1 we obtain the equation

$$a_1X_1 + r_2x_2 + \cdots + r_{n-1}x_{n-1} + r_nx_n = 1, \tag{2}$$

where

$$X_1 = x_1 + q_2x_2 + \cdots + q_nx_n. \tag{3}$$

Let

$$Y = x_{n-1} + x_n \text{ if } r_{n-1} = r_n \tag{4}$$

or

$$Y = x_{n-1} + r_n'x_n \text{ if } r_n = r_n'r_{n-1}$$

in (2) obtaining

$$a_1X_1 + r_2x_2 + \cdots + r_{n-2}x_{n-2} + r_{n-1}Y = 1. \tag{5}$$

From the general solution of equation (5) and the corresponding expression (4) we obtain x_{n-1} taking x_n as parameter, and x_1 is obtained from (3). Since then a form of the general solution of (1) contains x_n as a parameter, it follows from Lemma 1 that $(a_1, \cdots, a_{n-1}) = 1$. If in particular $r_{n-1} = r_n$, then x_{n-1} may be chosen as a parameter instead of x_n and thus we conclude that $(a_1, \cdots, a_{n-2}, a_n) = 1$.

Theorem 2 follows from Lemma 2. From Theorem 2 we have the following

COROLLARY. *If in a set of integers S any integer a is chosen as modulus, then the remaining $n-1$ integers distribute into different residue classes modulo a.*

CLASSROOM NOTES

EDITED BY A. L. SHIELDS, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, Mathematics Department, University of Michigan, Ann Arbor, Michigan.

AN ALTERNATIVE PROOF OF THE CHARACTERIZATION OF THE DENSITY Ax^B

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Characterizations of the normal, Poisson and gamma distributions are well known and are sometimes obtained by nontrivial methods (see [1]). The proof below, for a different density, is posed as one utilizing several basic principles at a level appropriate to a first course in mathematical statistics. The same theorem has been proved by Fisz in a different way (see [2]).

THEOREM. (i) *Let X be a real valued random variable with absolutely continuous distribution function $F(x)$ on the interval 0 to $b>0$.* (ii) *Let $X_1 \leqq X_2 \leqq \cdots \leqq X_n$ be the order statistics based on a random sample of size n from this distribution.* (iii) *Let $Z=X_m/X_{m+1}$ for fixed m, $1 \leqq m \leqq n-1$. Then a necessary and sufficient condition (NASC) that Z and X_{m+1} be stochastically independent (stochind) is that $F(x)=(x/b)^c$, where b is finite and $c>0$.*

Proof. For convenience let $X_m=Y$, $X_{m+1}=S$. If Z and S are stochind, the ordinary and conditional expectation (given $S=s$) of Z are equal; symbolically $E[Z]=E[Z|S=s]$. Note that $0<E[Z]\leqq 1$. Since $E[Z]$ is free of s, so is $E[Z|S=s]$ so that, with probability one,

$$\frac{d}{ds}E[Y/S \mid S=s] = \frac{d}{ds}\int_0^s (ymF(y)^{m-1}f(y)/sF(s)^m)dy = 0,$$

where $f(y)=F'(y)$. Performing the differentiation and simplifying, one obtains

$$mf(s)/F(s) - (mf(s)/F(s) + 1/s)E[Y/S] = 0.$$

If $mf(s)/F(s)+1/s$ is zero then $mf(s)/F(s)=0$, and this condition has probability zero. Therefore, assume these two quantities are not zero. Then $E[Z]=E[Y/S]=[mf(s)/F(s)]/(mf(s)/F(s)+1/s)$. Since the left hand side is a constant, so is the right hand side. Let $msf(s)/F(s)=D$. This differential equation has solution $F(s)=Es^c$ for $E>0$ and c both constant. Since $f(s)=Ecs^{c-1}\geqq 0$, $c>0$. Since $F(b)=1$, $b<\infty$ and $E=1/b^c$. Thus $F(x)=(x/b)^c$ and the proof of necessity is complete.

To show sufficiency, assume the form $F(x)=(x/b)^c$ and calculate the joint density and marginal densities of Z and S; the nonzero portions are:

$$g(z, s) = n!s^{cm+c-1}z^{cm-1}c^2(b^c - s^c)^{n-m-1}/(m-1)!(n-m-1)!b^{cn},$$
$$0 \leqq s \leqq b,\ 0 \leqq z \leqq 1,$$
$$h(s) = n!s^{cm+c-1}c(b^c - s^c)^{n-m-1}/m!(n-m-1)!b^{cn}, \qquad 0 \leqq s \leqq b,$$
$$g(z) = mcz^{cm-1}, \qquad 0 \leqq z \leqq 1.$$

Since $g(z, s) = g(z)h(s)$ and the "range of (Z, S) is free of the variables," Z and S are stochind.

COROLLARY 1. (i) *Let Y be a real valued random variable with absolutely continuous distribution function $G(y)$, on the interval $-\infty$ to $a<0$.* (ii) *Let $Y_1 \leqq \cdots \leqq Y_n$ be the order statistics based on a random sample of size n from this distribution.* (iii) *Let $Z = Y_{m+1}/Y_m$ for fixed m, $1 \leqq m \leqq n-1$. Then a NASC that Z and Y_{m+1} be stochind is that $G(y) = (y/a)^d$, where $d<0$.*

Proof. Let $Y = -1/X$. Then $-\infty < Y_m \leqq Y_{m+1} \leqq a < 0$ implies

$$0 < X_m \leqq X_{m+1} \leqq -\frac{1}{a} = b < \infty \quad \text{and} \quad Z = (-1/X_{m+1})/(-1/X_m) = X_m/X_{m+1}.$$

By the theorem, $F(x) = (x/b)^c$ with $c>0$. Hence $G(y) = F(-1/y) = (-1/by)^c = (a/y)^c = (y/a)^d$ *with* $d<0$.

COROLLARY 2. *With density prescribed by the theorem [Corollary 1], Z is stochind of $X_k[Y_k]$ for $k \geqq m+1$.*

Either of these results may be proved directly by a method analogous to the case of sufficiency above or by application of the following.

THEOREM. *Under hypotheses* (i) *and* (ii), *if*

$$Z = Z(X_1, \cdots, X_j) \ [Z = Z(Y_1, \cdots, Y_j)]$$

is stochind of $X_j[Y_j]$, then Z is stochind of $X_k[Y_k]$, $j \leqq k \leqq n$ and conversely.

Proof. See [3].

A remark by the referee leads to the following observation. Since Z and S are stochind if and only if $1/Z$ and S are stochind, the theorem [Corollary 1] can be stated in terms of X_{m+1}/X_m and $X_{m+1}[Y_m/Y_{m+1}$ and $Y_{m+1}]$. The reader can easily supply further variations by using other one-to-one transformations.

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ON THE COMPOSITION OF FINITE ROTATIONS

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Suppose a rigid body is free to rotate about a fixed point O, and designate by $\mathbf{r}$ the radius vector $\mathbf{OP}$ between the point O and a point P fixed in the body. If the body rotates through an angle θ, about an axis through O coincident with the unit vector $\mathbf{u}$, it is a relatively easy geometric problem to show (see [1]) that the point P moves to P', where the vector $\mathbf{PP}'=\mathbf{r}'-\mathbf{r}$ is given by

$$\mathbf{r}' - \mathbf{r} = [(\mathbf{r}\cdot\mathbf{u})\mathbf{u} - \mathbf{r}](1 - \cos\theta) + (\mathbf{u}\times\mathbf{r})\sin\theta. \tag{1}$$

For brevity we shall refer to such a rotation as $\mathrm{Rot}(\mathbf{u}, \theta)$, where it is understood that θ and $\mathbf{u}$ are related as in a right-handed screw. If the body undergoes $\mathrm{Rot}(\mathbf{u}_1, \theta_1)$ followed by $\mathrm{Rot}(\mathbf{u}_2, \theta_2)$ there is, according to Euler's theorem, a single rotation, $\mathrm{Rot}(\mathbf{u}, \theta)$, which carries the body into the same position. We can find u and θ as functions of $(\mathbf{u}_1, \mathbf{u}_2, \theta_1, \theta_2)$ by strictly scalar methods [2], strictly geometric methods [3], quaternion methods [4], and matrix methods [5]. We can use vectors by applying (1) twice in succession and manipulating the vector dyadic equations which result. A second approach is to reduce the problem to

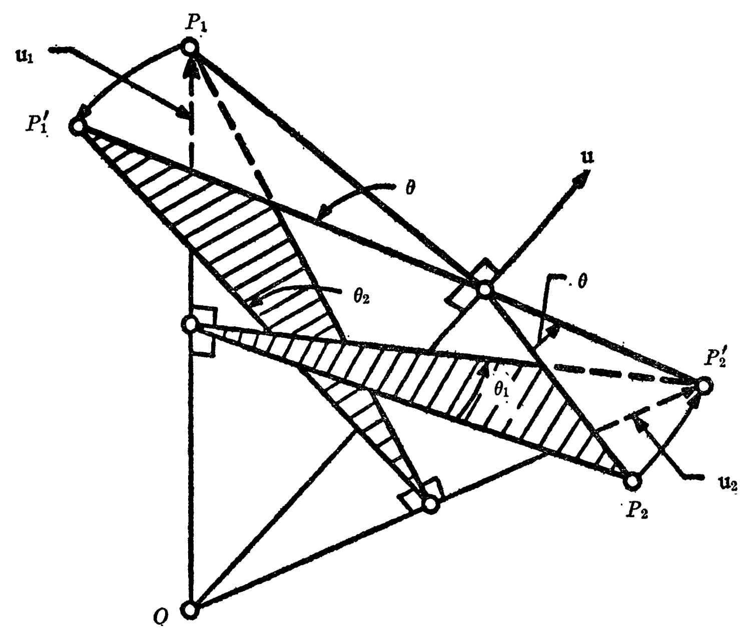

FIG. 1

the solution of a spherical triangle by means of the Rodrigues-Hamilton-Sylvester construction (see Lamb, [3]) and then use a combination of vector methods and spherical trigonometry as do Ames and Murnaghan [1] and Room [6].

In this note we present a new vector derivation for the composition of finite rotations, which does not use the foregoing methods.

Figure 1 shows a point P_1 initially at the tip of the unit vector $\mathbf{u}_1$. During the first rotation, Rot($\mathbf{u}_1$, θ_1), the point P_1 remains fixed but another point P_2 moves through an arc $P_2P_2' = \theta_1$, on the unit sphere, and occupies the position P_2' chosen to be at the tip of the given vector $\mathbf{u}_2$. During the second rotation, Rot($\mathbf{u}_2$, θ_2), the point P_2' remains fixed and point P_1 moves through arc $P_1P_1' = \theta_2$. There will be no loss of generality if both θ_1 and θ_2 are restricted to the range $0 \leqq \theta_i < \pi$ since $\theta_i > \pi$ corresponds to a rotation of $2\pi - \theta_i$ about the axis $-\mathbf{u}_i$. The special case $\theta_i = \pi$ may be treated as a limiting case and need not be treated separately.

The vector displacement $\mathbf{P}_1\mathbf{P}_1'$ may be found by use of (1) in either of two forms, depending upon whether we consider the displacement due to Rot($\mathbf{u}_2$, θ_2) or Rot($\mathbf{u}$, θ):

$$
\begin{aligned}
\mathbf{P}_1\mathbf{P}_1' &= [(\mathbf{u}_1 \cdot \mathbf{u}_2)\mathbf{u}_2 - \mathbf{u}_1](1 - \cos\theta_2) + (\mathbf{u}_2 \times \mathbf{u}_1)\sin\theta_2 \\
&= [(\mathbf{u}_1 \cdot \mathbf{u})\mathbf{u} - \mathbf{u}_1](1 - \cos\theta) + (\mathbf{u} \times \mathbf{u}_1)\sin\theta.
\end{aligned} \tag{2}
$$

Similarly, the displacement $\mathbf{P}_2'\mathbf{P}_2$ may be found from (1) by considering it a consequence of either Rot($\mathbf{u}_1$, $-\theta_1$) or Rot($\mathbf{u}$, $-\theta$); i.e.:

$$
\begin{aligned}
\mathbf{P}_2'\mathbf{P}_2 &= [(\mathbf{u}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 - \mathbf{u}_2](1 - \cos\theta_1) - (\mathbf{u}_1 \times \mathbf{u}_2)\sin\theta_1 \\
&= [(\mathbf{u}_2 \cdot \mathbf{u})\mathbf{u} - \mathbf{u}_2](1 - \cos\theta) - (\mathbf{u} \times \mathbf{u}_2)\sin\theta.
\end{aligned} \tag{3}
$$

Since the displacements of P_1 and P_2 could have been produced by a rotation about an axis parallel to $\mathbf{u}$, both $\mathbf{P}_1\mathbf{P}_1'$ and $\mathbf{P}_2'\mathbf{P}_2$ are perpendicular to the unit vector $\mathbf{u}$. In other words, $\mathbf{u}$ is parallel to the direction of the cross product $\mathbf{U}$, where

$$
\mathbf{U} = \mathbf{P}_1\mathbf{P}_1' \times \mathbf{P}_2'\mathbf{P}_2. \tag{4}
$$

Upon substitution of the first forms of (2) and (3) into (4) one finds that

$$
\begin{aligned}
&\mathbf{U}(1 - \cos\theta_1)^{-1}(1 - \cos\theta_2)^{-1}(1 - \mathbf{u}_1 \cdot \mathbf{u}_2)^{-1} \\
&= \frac{\sin\theta_2}{1 - \cos\theta_2}\mathbf{u}_1 + \frac{\sin\theta_2}{1 - \cos\theta_2}\mathbf{u}_2 - \mathbf{u}_1 \times \mathbf{u}_2 \\
&= \cot\left(\frac{\theta_2}{2}\right)\mathbf{u}_1 + \cot\left(\frac{\theta_1}{2}\right)\mathbf{u}_2 - \mathbf{u}_1 \times \mathbf{u}_2.
\end{aligned} \tag{5}
$$

From (5) we see that

$$
\mathbf{u} = \frac{\mathbf{U}}{|\mathbf{U}|} = \frac{1}{M}[\mathbf{u}_1 C_2 S_1 + \mathbf{u}_2 C_1 S_2 - S_1 S_2 \mathbf{u}_1 \times \mathbf{u}_2], \tag{6}
$$

where $C_i = \cos(\theta_i/2)$, $S_i = \sin(\theta_i/2)$, and M is the magnitude of the vector within brackets in (6), i.e.,

$$M^2 = C_2^2S_1^2 + C_1^2S_2^2 + 2C_1C_2S_1S_2(\mathbf{u}_1\cdot\mathbf{u}_2) + S_1^2S_2^2[1 - (\mathbf{u}_1\cdot\mathbf{u}_2)^2]. \tag{7}$$

Although it might appear that there is some ambiguity in the sense of $\mathbf{u}$ we note that $\mathbf{u}$ must reduce to $\mathbf{u}_1$ when S_2 vanishes and to $\mathbf{u}_2$ when S_1 vanishes; thus the sense of $\mathbf{u}$ specified by (6) is correct.

Having found $\mathbf{u}$ we are now in a position to find θ fairly quickly. If the scalar product of (3) by $\mathbf{u}_2$ is subtracted from the scalar product of (2) by $\mathbf{u}_1$, one finds:

$$1 - \cos\theta = \frac{[1 - (\mathbf{u}_1 - \mathbf{u}_2)^2](\cos\theta_2 - \cos\theta_1)}{(\mathbf{u}_1\cdot\mathbf{u})^2 - (\mathbf{u}_2\cdot\mathbf{u})^2}. \tag{8}$$

Upon substitution of (6) into (8) one finds that

$$\frac{1 - \cos\theta}{2} = M^2 \quad\text{or}\quad \sin\frac{\theta}{2} = M. \tag{9}$$

Equations (6) and (9) fully describe the combined rotation. They may be combined, however, in the compact form:

$$\mathbf{u}\sin\frac{\theta}{2} = \mathbf{u}_1S_1C_2 + \mathbf{u}_2S_2C_1 - \mathbf{u}_1S_1 \times \mathbf{u}_2S_2. \tag{10}$$

This vector equation, when resolved along orthogonal axes x, y, z, expresses the three Euler parameters (see [2]) ($u_x \sin\theta/2$, $u_y \sin\theta/2$, $u_z \sin\theta/2$) in terms of the Euler parameters of the component rotations ($u_{1x}S_1, \cdots, u_{1x}S_2, C_1, C_2$). The fourth Euler parameter, $\cos\theta/2$, may be found from the first three, or explicitly from (9) and (7) in the form

$$\cos\frac{\theta}{2} = \sqrt{(1 - M^2)} = C_1C_2 - S_1S_2(\mathbf{u}_1\cdot\mathbf{u}_2). \tag{11}$$

If (10) is "divided" by (11) one finds the known vector expression (see [6])

$$\mathbf{w} = \frac{\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_1 \times \mathbf{w}_2}{1 - \mathbf{w}_1\cdot\mathbf{w}_2}, \tag{12}$$

where

$$\mathbf{w} = \tan\left(\frac{\theta}{2}\right)\mathbf{u}, \qquad \mathbf{w}_i = \tan\left(\frac{\theta_i}{2}\right)\mathbf{u}_i.$$

Equation (12) illustrates the noncommutative character of finite rotations. It also shows that merely changing the sense of the "component" rotations does not merely change the sense of the combined rotation. Changing the sense *and* the order of the component rotations, however, will merely change the sense of the combined rotation, since

$$\frac{\mathbf{w}_2 - \mathbf{w}_1 - (-\mathbf{w}_2) \times (-\mathbf{w}_1)}{1 - (-\mathbf{w}_2\cdot -\mathbf{w}_1)} = -\frac{(\mathbf{w}_1 + \mathbf{w}_2 - \mathbf{w}_1 \times \mathbf{w}_2)}{1 - (\mathbf{w}_1\cdot\mathbf{w}_2)}. \tag{13}$$

Equation (13) is equivalent to the matrix identity $B^{-1}A^{-1}=(AB)^{-1}$, where A; B are matrix rotation operators, i.e., orthogonal 3×3 matrices.

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A NOTE ON AN INEQUALITY FOR THE INTEGRAL OF THE DEVIATION OF A FUNCTION FROM ITS MEAN

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For an integrable function ρ and a constant $\bar{\rho}$

$$\left|\int_a^x (\rho - \bar{\rho})dx\right| \leqq \int_a^b \left| \rho - \bar{\rho} \right| dx \qquad (a \leqq x \leqq b).$$

The object of this note is to show that in the case when $\bar{\rho}$ is the mean of ρ over (a, b) then it is possible to obtain a sharper result in that the $\bar{\rho}$ on the right-hand side can be replaced by any real number c. More precisely we have the following result:

Let ρ be $L(a, b)$ with mean value $\bar{\rho}=1/(b-a)\int_a^b \rho dx$. If c denotes a real number then

$$\text{(1)} \qquad \operatorname*{Max}_{a\leqq x\leqq b}\left|\int_a^x (\rho - \bar{\rho})dx\right| \leqq \operatorname*{Min}_{c} \int_a^b \left| \rho - c \right| dx;$$

and if equality holds then $\rho=\bar{\rho}$, a constant, almost everywhere.

The integral $\int_a^b|\rho-c|dx$ is a continuous function of the real number c. Let it take its minimum value for $c=c_m$. Write $p=\rho-c_m$. Then

$$\int_a^x (\rho - \bar{\rho})dx = \int_a^x (p - \bar{p})dx, \quad \text{where} \quad \bar{p} = \frac{1}{b-a}\int_a^b p dx,$$

and it is required to prove that

$$\text{(2)} \qquad \operatorname*{Max}_{a\leqq x\leqq b}\left|\int_a^x (p - \bar{p})dx\right| \leqq \int_a^b \left| p \right| dx.$$

Now $\left|\int_a^x(p-\bar{p})dx\right|$ is a continuous function of x. Let it take its maximum value

in (a, b) at $x=x_m$. Thus

$$
\begin{aligned}
\operatorname*{Max}_{a\leqq x\leqq b}\left|\int_a^x (p-\bar{p})dx\right| &= \left|\int_a^{x_m}(p-\bar{p})dx\right| \\
&= \left|\int_a^{x_m} pdx - \frac{x_m-a}{b-a}\int_a^b pdx\right| \\
&= \left|\frac{b-x_m}{b-a}\int_a^{x_m} pdx - \frac{x_m-a}{b-a}\int_{x_m}^b pdx\right| \\
&\leqq \frac{b-x_m}{b-a}\int_a^{x_m} |p|\,dx + \frac{x_m-a}{b-a}\int_{x_m}^b |p|\,dx \\
&\leqq \frac{b-x_m}{b-a}\int_a^b |p|\,dx + \frac{x_m-a}{b-a}\int_a^b |p|\,dx \\
&= \int_a^b |p|\,dx.
\end{aligned}
\tag{3}
$$

This proves (2) and hence also (1).

If equality holds in (1) then it also holds in (2) and throughout the steps involved in deriving (3). Thus

$$
\frac{b-x_m}{b-a}\int_a^{x_m}|p|\,dx + \frac{x_m-a}{b-a}\int_{x_m}^b |p|\,dx = \frac{b-x_m}{b-a}\int_a^b |p|\,dx + \frac{x_m-a}{b-a}\int_a^b |p|\,dx
$$

or

$$
(b-x_m)\int_{x_m}^b |p|\,dx + (x_m-a)\int_a^{x_m}|p|\,dx = 0.
$$

Since $a\leqq x_m\leqq b$, this is possible only if

$$
\int_a^{x_m}|p|\,dx = \int_{x_m}^b |p|\,dx = 0.
$$

Thus

$$
0\leqq \int_a^x |p|\,dx \leqq \int_a^b |p|\,dx = \int_a^{x_m}|p|\,dx + \int_{x_m}^b |p|\,dx = 0, \qquad (a\leqq x\leqq b),
$$

and differentiation with respect to x shows that p must be almost everywhere zero in (a, b). Hence $\rho=p+c_m$ is almost everywhere a constant which is plainly $\bar{\rho}$.

As an example consider the function

$$
\rho=0,\quad a\leqq x<b-\epsilon;\qquad \rho=1,\quad b-\epsilon\leqq x\leqq b,
$$

where ϵ is a small positive number. Then

$$\operatorname*{Max}_{a\leqq x\leqq b}\left|\int_a^x (\rho-\bar{\rho})dx\right| = \epsilon\left(1-\frac{\epsilon}{b-a}\right)$$

and

$$\operatorname*{Min}_{c}\int_a^b |\rho - c|\, dx = \epsilon, \qquad (c_m = 0).$$

Thus both sides of (1) have the same asymptotic value as $\epsilon\to 0$ and it follows that the inequality is a best possible one of its type in the sense that no positive number less than unity and independent of ρ can be inserted on the right-hand side.

Reference

1. E. C. Titchmarsh, Theory of Functions, Chapters X and XI, Oxford, 1939.

THE EXTREME VALUE THEOREM

GERALD JUNGCK, Bradley University

In this paper we offer a proof of the proposition that a real-valued function continuous on a closed interval $[a, b]$ attains a maximum value on $[a, b]$. We are aware of no proofs of the above theorem which employ methods identical to ours. We obtain the desired result, as have other writers (cf. [1]), without first proving that a function continuous on a closed interval is bounded. Beyond the assumed continuity, the proof relies solely on the l.u.b. property of the real numbers. Specifically employed is the fact that if $\bar{x}$ is the l.u.b. of a set S of reals, then for any $\epsilon>0$ there is a point x of S contained in $(\bar{x}-\epsilon, \bar{x}]$.

THEOREM. *If f is continuous on the closed interval $[a, b]$, then there is a point $\bar{x}\in[a, b]$ such that $f(\bar{x})\geqq f(x)$ for $x\in[a, b]$.*

Proof. If one of $f(a)$ or $f(b)$ is a maximum the theorem is proved. Hence we assume a point $y\in(a, b)$ such that $f(y)>f(a)$ and $f(y)>f(b)$. By continuity at a and b we can pick δ and δ', $0<\delta<y-a$ and $0<\delta'<b-y$, such that $f(y)>f(x)$ for $x\in[a, a+\delta)\cup(b-\delta', b]$. If ϵ is the smaller of δ and δ', we have (1) $f(y)>f(x)$ for $x\in[a, a+\epsilon)\cup(b-\epsilon, b]$, where $y\in[a+\epsilon, b-\epsilon]$.

Let $S=\{c\in(a, b)\mid \exists$ a point $d\in[c, b]$ such that $f(d)>f(x)$ for all $x\in[a, c)\}$.

Now $S\neq\varnothing$, since (1) assures us that $a+\epsilon\in S$. Moreover, b is an upper bound of S so that S has a l.u.b. $\bar{x}\in[a+\epsilon, b]$. But (1) yields a point $y\in[a, b-\epsilon]$ such that $f(y)>f(x)$ for $x\in(b-\epsilon, b]$. Thus for no $c\in(b-\epsilon, b]$ does there exist a point $d\in[c, b]$ such that $f(d)>f(x)$ for all $x\in[a, c)$. Hence $S\cap(b-\epsilon, b]$ is empty, and $\bar{x}\in[a+\epsilon, b-\epsilon]$.

We now assert that (2) $f(\bar{x})$ is maximum on $[a, b]$. For if $f(z)>f(\bar{x})$ for some $z\in[a, b]$, continuity permits us to choose a positive $\delta<\epsilon$ such that $(a) f(z)>f(x)$

for $x \in (\bar{x}-\delta, \bar{x}+\delta)$. By the definition of $\bar{x}$, there is a point c of S in $(\bar{x}-\delta, \bar{x}]$ and hence a point $d \in [c, b]$ such that $(b) f(d) > f(x)$ for $x \in [a, c) \supset [a, \bar{x}-\delta]$. Let $f(g)$ be the greater of $f(z)$ and $f(d)$. Then (a) yields $f(g) \geqq f(z) > f(x)$ for $x \in (\bar{x}-\delta, \bar{x}+\delta)$, and (b) yields $f(g) \geqq f(d) > f(x)$ for $x \in [a, \bar{x}-\delta]$; hence $f(g) > f(x)$ for $x \in [a, \bar{x}+\delta)$. Obviously $g \in [\bar{x}+\delta, b]$ so that $\bar{x}+\delta \in S$, which refutes $\bar{x}$ as an upper bound of S. Thus $f(z) > f(\bar{x})$ is absurd and (2) must be true.

One aspect of the above proof should be stressed. The point $\bar{x}$ is unique in that it is the minimum value of x on $[a, b]$ yielding the maximum value of f.

In closing we thank the referee for his helpful suggestion.

Reference

1. W. B. Pennington, Existence of a maximum of a continuous function, this MONTHLY, 67 (1960) 892–893.

A VARIANT OF TAYLOR'S THEOREM

W. R. BALLARD, A. E. LIVINGSTON AND W. M. MYERS, JR., Montana State University

There are two noncomparable versions of Taylor's Theorem commonly presented in the classroom: *If f is a real-valued function of a real variable and $f^{(n-1)}(c)$ is a real number, then*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x - c)^k + R_n(x)$$

for each x in a (one- or two-sided) neighborhood N of c, where

$$R_n(x) = \frac{f^{(n)}(\xi)}{n!} (x - c)^n \tag{1}$$

for some ξ between c and x, if $f^{(n-1)}$ has a (finite or infinite) derivative at each t in $N - \{c\}$; or

$$R_n(x) = \frac{1}{(n-1)!} \int_c^x (x - 1)^{n-1} f^{(n)}(t) dt \tag{2}$$

if $f^{(n-1)}(t) = f^{(n-1)}(c) + \int_c^t f^{(n)}(y) dy$, for each t in N.

A third version easily deducible from either of the above and frequently used in the study of central-limit theorems in probability theory, requires that $f^{(n)}$ be continuous at c and has

$$R_n(x) = \frac{f^{(n)}(c)}{n!} (x - c)^n + \phi(x),$$

where $\phi(x)/(x-c)^n \to 0$ as $x \to c$ (i.e., $\phi(x) = o[(x-c)^n]$).

All three of these versions require a global condition on the nth derivative.

We propose a variant similar to but not comparable with the third of these, which requires only that the function have an nth derivative at the point in question, and is perhaps even more amenable to classroom presentation than the three versions mentioned.

If $f^{(n)}(c)$ is a real number, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x - c)^k + \phi(x),$$

where $\phi(x)/(x-c)^n \to 0$ as $x \to c$.

For the proof we note first that the case $n=1$ is just the definition of $f'(c)$. To deal with the case $n>1$, we observe that the function

$$\phi(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x - c)^k$$

has $\phi(c)=\phi'(c)= \cdots =\phi^{(n)}(c)=0$. By $n-1$ applications of the Mean-Value Theorem we obtain

$$\phi(x) = [\phi^{(n-1)}(\xi_{n-1}) - \phi^{(n-1)}(c)](x - c)(\xi_1 - c) \cdots (\xi_{n-2} - c),$$

where ξ_1 is between c and x and ξ_i is between c and ξ_{i-1} for $i=2, \cdots, n-1$. Then

$$\frac{\phi(x)}{(x-c)^n} = \frac{\phi^{(n-1)}(\xi_{n-1}) - \phi^{(n-1)}(c)}{(\xi_{n-1} - c)} \cdot \frac{(\xi_{n-1} - c)(x - c)(\xi_1 - c) \cdots (\xi_{n-2} - c)}{(x - c)^n}.$$

Since

$$0 < \frac{\xi_i - c}{x - c} < 1 \quad \text{for} \quad i = 1, 2, \cdots, n - 1$$

and

$$\lim_{x \to c} \frac{\phi^{(n-1)}(\xi_{n-1}) - \phi^{(n-1)}(c)}{\xi_{n-1} - c} = \phi^{(n)}(c) = 0,$$

clearly

$$\lim_{x \to c} \frac{\phi(x)}{(x - c)^n} = 0.$$

Remark 1. An alternative method of proof is of some interest. The cases $n=1$ and $n=2$ could be treated as above and the remaining cases proved by induction. Assume then that our result is true for $n=m \geqq 2$, and consider an f such that $f^{(m+1)}(c)$ is a real number. Then $g=f'$ is a function for which $g^{(m)}(c)$ is a real number. By our inductive assumption,

$$g(x) = f'(x) = \sum_{k=0}^{m} \frac{f^{(k+1)}(c)}{k!} (x - c)^k + \phi(x),$$

where $\phi(x)/(x-c)^m \to 0$ as $x \to c$. Since $m+1 \geqq 3$, f' is continuous and, hence, Riemann-integrable on a neighborhood of c. Integration from c to x in this neighborhood then shows that our result also holds for $n=m+1$.

This alternative proof would be enhanced if we could pass directly from the case $n=1$ to $n=m$ without the special argument for $n=2$. The existence of $f''(c)$ does not guarantee, however, that f may be recaptured from f' by Riemann integration. An example of a function f differentiable on the unit interval but with f' not Riemann integrable on any interval $[0, x]$ will be given as Remark 2. The special proof for $n=2$ is not needed if the integral is taken in the sense of Lebesgue. For, since $f'(x)$ exists in a neighborhood of c and is continuous at c, f' is bounded and f accordingly absolutely continuous in a neighborhood of c. (See, for example, E. C. Titchmarsh, Theory of Functions, 2nd ed., Oxford, 1939, p. 368.)

Remark 2. There follows an example of a function differentiable on $[0, 1]$, with $f''(0)$ defined, but such that f' fails to be Riemann-integrable on every interval $[0, x]$, was suggested to us by Professor William M. Myers, of Montana State University.

We denote the unit interval $[0, 1]$ by X. Complementary subsets of X, each of measure $1/2$, are constructed in a manner analogous to the definition of the Cantor discontinuum. We give details because we will need the notations used. Let I_{01} be an open interval of length $1/4$ concentric with X. Then $X-I_{01}$ is the union of disjoint closed intervals J_{11} and J_{12}, where each element of J_{11} is a lower bound for J_{12}. Let I_{1i}, $i=1, 2$, denote the open interval of length $1/16$ concentric with J_{1i}. Then $X-(I_{01} \cup I_{11} \cup I_{12}) = J_{21} \cup J_{22} \cup J_{23} \cup J_{24}$, where J_{2i} is a closed interval and $x \in J_{2i}$, $y \in J_{2k}$ with $i<k$ implies $x<y$. The open interval concentric with J_{2i} of length $1/4^3=1/64$ is denoted I_{2i}. We continue in this way, obtaining, for each positive integer n, disjoint closed intervals J_{nk}, $k=1, 2, \cdots, 2^n$ such that $x \in J_{ni}, y \in J_{nk}$, with $i<k$ implies $x<y$, and open intervals I_{nk}, $k=1, 2, \cdots, 2^n$, such that I_{nk} is concentric with J_{nk} and of length $1/4^{n+1}$. If $G=\bigcup_{n,k} I_{nk}$ and $H=X-G$, it is easily seen that G, H are both of measure $1/2$. The set H is perfect and G is dense in X. More to the point for our present purposes, if $x \in H$ and x is not the left-hand (right-hand) end point of any I_{nk}, then every half-open interval $[x, x+\epsilon)((x-\epsilon, x])$, where $\epsilon>0$, contains an I_{nk}. Further, if $x>0$ then $[0, x] \cap H$ has positive measure.

We now define a preliminary function g on $[0, 1]$. If $x \in H$, let $g(x)=0$. In each I_{nk}, let K_{nk} be an open interval, concentric with I_{nk}, of length $2/4^{2n+3}$:

$$K_{nk} = \left(\alpha_{nk} - \frac{1}{4^{2n+3}},\ \alpha_{nk} + \frac{1}{4^{2n+3}}\right),$$

where we have chosen the symbol α_{nk} to denote the midpoint of K_{nk} and I_{nk}. For $x \in K_{nk}$, let $g(x)=1/4^{2n+3}(1+\cos 4^{2n+3}\pi(x-\alpha_{nk}))$ and for $x \in I_{nk}-K_{nk}$, let $g(x)=0$. It is easy to see that $g'(x)$ exists for $x \in X-H$, and it will be shown below that $g'(x)=0$ for $x \in H$. In every K_{nk}, $g'(x)$ attains the values π and $-\pi$.

Since every neighborhood of an arbitrary point of H contains an interval K_{nk}, g' is discontinuous at every point of H.

We will now take up the differentiability of g at points of H. If $x \in H$ is a left-hand end point of some I_{nk}, then, clearly, the right-hand derivative of g exists and is zero at x. Suppose $x \in H$ is not a left-hand end-point of an I_{nk} and let ϵ be an arbitrary positive number. Take $\delta = \epsilon/32$ and suppose $x < y < x + \delta$. We wish to estimate $(g(y) - g(x))/(y - x)$. This quotient is zero unless y is in some K_{nk}, so we assume $y \in K_{nk}$. Then a portion of the associated I_{nk} of length at least

$$\frac{1}{2 \cdot 4^{n+1}} - \frac{1}{4^{2n+3}} = \frac{2 \cdot 4^{n+1} - 1}{4^{2n+3}}$$

must be contained in $[x, y)$, so $(2 \cdot 4^{n+1} - 1)/4^{2n+3} < \delta$. It follows that $4^{n+1}/4^{2n+3} < \delta$, so that $2/4^n < 2^5 \delta = \epsilon$. If $x < y < x + \delta$, noting that for $y \in K_{nk}$,

$$|g(y)| \leqq \frac{2}{4^{2n+3}} \quad \text{and} \quad |y - x| > \frac{1}{2 \cdot 4^{n+1}} - \frac{1}{4^{2n+3}},$$

we have

$$\frac{g(y) - g(x)}{y - x} \leqq \frac{2/4^{2n+3}}{(1/(2 \cdot 4^{n+1})) - (1/(4^{2n+3}))} < \frac{2/4^{2n+3}}{(1/(4^{n+2})) - (1/(4^{2n+3}))} = \frac{2}{4^{n+1} - 1}$$
$$< \frac{2}{4^n} < \epsilon.$$

Thus g has a right-hand derivative equal to zero at every point $x \in H$. It may be shown in the same manner that g has left-hand derivative zero at every $x \in H$.

To complete the example, let $f(x) = x^2 g(x)$ for $x \in X$. Like g, f is differentiable throughout X. The derivative $f'(x) = x^2 g'(x) + 2xg(x)$ is discontinuous at all points where g' is discontinuous, except at $x = 0$. Consequently, for every $h > 0$, $\int_0^h f'(x)$ fails to exist since f', like g', must be discontinuous on a subset of $[0, h]$ which has positive measure. On the other hand,

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \to 0} (xg'(x) - 2g(x)) = 0.$$

Remark 3. Our result is clearly true also for one-sided derivatives if $x \to c$ from that side, as well as for a real-valued function f of m real variables if the Chain Rule is applicable n times to

$$g(t) = f[(1 - t)c_1 + tx_1, (1 - t)c_2 + tx_2, \cdots, (1 - t)c_m + tx_m] \quad \text{at} \quad t = 0,$$

and $o[(x - c)^n]$ is of course replaced by

$$o[\{(x_1 - c_1)^2 + (x_2 - c_2)^2 + \cdots + (x_m - c_m)^2\}^{1/2}]^n.$$

REGULA FALSI AND THE FIBONACCI NUMBERS

DMITRI THORO, San Jose State College

The classical regula falsi method for the solution of $f(x)=0$ is given by the iteration

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

For the equation $x^2=0$ this reduces to

$$x_{n+1} = \frac{x_{n-1}x_n}{x_{n-1} + x_n}.$$

From the definition of the Fibonacci numbers ($F_1=F_2=1$, $F_{n+2}=F_{n+1}+F_n$) it immediately follows that with starting values $x_0=1$, $x_1=1/2$ this application of *regula falsi* yields the reciprocals of the Fibonacci numbers since

$$\frac{\dfrac{1}{F_{i+1}}\cdot\dfrac{1}{F_{i+2}}}{\dfrac{1}{F_{i+1}}+\dfrac{1}{F_{i+2}}} = \frac{1}{F_{i+1}+F_{i+2}} = \frac{1}{F_{i+3}}.$$

ON THE MODULUS OF AN INTEGRAL

DAVID S. GREENSTEIN, Northwestern University

In a recent note [1], Mott has deduced a condition for equality in the well-known inequality $\left|\int_a^b f(x)dx\right| \leqq \int_a^b |f(x)|\,dx$, $f(x)$ being a complex-valued Lebesgue integrable function on the real interval $[a, b]$. It is possible to give a much shorter and simpler proof of Mott's principal theorem which simultaneously establishes the inequality.

Multiplying $f(x)$ by a suitably chosen constant of modulus one shows that we need consider only the case in which $\int_a^b f(x)dx$ is real and nonnegative. In this case we have

$$\left|\int_a^b f(x)dx\right| = \int_a^b \operatorname{Re} f(x)dx \leqq \int_a^b |f(x)|\,dx,$$

since $|f(x)| - \operatorname{Re} f(x) \geqq 0$. Thus equality holds if and only if $\operatorname{Re} f(x) = |f(x)|$ a.e., hence if and only if $f(x) \geqq 0$ a.e. It necessarily follows that in the general case we have equality if and only if $\arg f(x) = \arg \int_a^b f(x)dx$ a.e. on $\{x \mid f(x) \neq 0\}$.

Reference

1. Thomas E. Mott, Equality in $\left|\int_a^b f(x)dx\right| \leqq \int_a^b |f(x)|\,dx$, this MONTHLY, 69 (1962) 996–998.

Joseph Gallant. The *Guide* covers not only the physical and biological sciences and mathematics but also places an increased emphasis on the behavioral sciences. Nine hundred books are in the annotated list, including books of interest from the elementary to the specialist level.

NCTM PLACEMENT SERVICE

The National Council of Teachers of Mathematics arranged, for the first time in its history, to have a professional job placement service available at the 41st annual meeting in Pittsburgh, April 3 to 6, 1963. The service was conducted by the Pennsylvania State Employment Service, in cooperation with the U. S. Employment Service of the Department of Labor. It is recognized that many very competent high school teachers of mathematics do not have opportunities for better positions because they do not know about them. Placement service on a national basis for secondary school teachers is being considered by quite a number of professional societies.

Editorial Note. An article "Entering Levels of College Courses in Freshman Mathematics," by Clarence B. Lindquist, appears on page 14 of *School Life*, the official journal of the Office of Education, April 1963. For further information on the article, write directly to Dr. Lindquist at the Office of Education.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
COLLABORATING EDITOR: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1621. *Proposed by Arthur Engel, Stuttgart, Germany*

What is the smallest value of a for which 82^n+a69^n is divisible by 1963 for all odd positive integers n?

E 1622. *Proposed by Michael Gemignani, University of Notre Dame*

Determine for what values of x the following series converge:

$$(1)\qquad \sum_{n=1}^{\infty}(\sin 1/n)^x, \qquad\qquad (2)\qquad \sum_{n=1}^{\infty}(1-\cos 1/n)^x.$$

E 1623. *Proposed by R. C. Thompson, University of British Columbia*

Let $f(x)$ be a monic polynomial of degree n with distinct zeros $x_1, x_2, \cdots, x_n$. Let $g(x)$ be any monic polynomial of degree $n-1$. Show that

$$\sum_{j=1}^{n} g(x_j)/f'(x_j) = 1.$$

E 1624. *Proposed by C. M. Frye, San Mateo, California*

Prove, for all integers $n>2$, that $(2n-1)^n+(2n)^n<(2n+1)^n$ and that $(2n)^n+(2n+1)^n>(2n+2)^n$.

E 1625. *Proposed by J. L. Brown, Jr., Pennsylvania State University*

Let n be a positive integer, $\sigma(n)$ the sum of the positive divisors of n, and $t(n)$ the number of these positive divisors. Show that

$$\sigma(n)/t(n) \geqq \sqrt{n}.$$

E 1626. *Proposed by Cornelius Mack, Bradford Institute of Technology, Bradford, England*

Given that X, Y are points on the sides BC, AC of a triangle ABC such that $\measuredangle XAB : \measuredangle CAB = \measuredangle YBA : \measuredangle CBA = \lambda : 1$, where $0<\lambda<1$, show that

(1) $AX>BY$ implies $AC>BC$, and conversely,

(2) $CY>CX$ implies $AC>BC$, and conversely,

(3) $AY>BX$ implies $AC>BC$, and conversely, provided that $0<\lambda\leqq 0.5$, but that there exist triangles for which this is not true if $0.5<\lambda<1$.

E 1627. *Proposed by Ralph Greenberg, University of Pennsylvania*

Prove that every positive integer except 1 is the sum of two square-free integers.

E 1628. *Proposed by Leonard Carlitz, Duke University*

Let t_a, t_b, t_c denote the angle bisectors of a triangle, r the inradius, R the circumradius, and s the semiperimeter. Show that

(1) $t_a^2+t_b^2+t_c^2\leqq s^2$,

(1) $t_b^2t_c^2+t_c^2t_a^2+t_a^2t_b^2\leqq rs^2(4R+r)$,

(3) $t_at_bt_c\leqq rs^2$.

In each case there is equality if and only if the triangle is equilateral.

E 1629. *Proposed by F. M. Sioson, University of Hawaii*

Show that any associative system S satisfying the identity $x^2y=y=yx^2$ is a commutative group.

E 1630. *Proposed by Reuben Hersh, Stanford University*

If the polynomial $P(x)$ has no purely imaginary zeros, and if the function

f satisfies $|f(x)| < 1$ for all real x, then the ordinary differential equation $P(D)u = f$ has exactly one solution $u(x)$ which is bounded for all x, and that bound can be chosen as the product of the reciprocals of the real parts of the zeros of P.

SOLUTIONS

A Partitioned Quadrilateral

E 1548 [1962, 1008]. *Proposed by Taiichi Kutamura, Ibaraki University, Japan*

In any convex quadrilateral $ABCD$, we trisect the sides AB, BC, CD, DA in E, F; G, H; I, J; K, L respectively. Show that the area of the quadrilateral formed by the lines EJ, KH, IF, GL is one-ninth the area of $ABCD$. Generalize the problem.

Solution by M. F. Wiener, Marple-Newton Senior High School, Newtown Square, Pennsylvania. Let m_1 and m_2 be the medians of $ABCD$. The endpoints of m_1 and m_2 are the vertices of a parallelogram with diagonals m_1 and m_2. The parallelogram has area $K_1 = 2m_1m_2 \sin\theta$, where θ is an angle between the segments m_1 and m_2. It follows that $ABCD$ has area $K_2 = 2K_1 = 4m_1m_2 \sin\theta$.

Denote the quadrilateral formed as $MNOP$. The medians of $MNOP$ are segments of m_1 and m_2 with lengths $m_1/3$ and $m_2/3$ respectively. Hence the area of $MNOP$ is

$$K_3 = 2[2(m_1/3)(m_2/3)\sin\theta] = (1/9)4m_1m_2\sin\theta = (1/9)K_2.$$

In general, if the sides of $ABCD$ are divided into n equal segments, and we take the rth division point from each vertex on each side of $ABCD$, naming them E', F'; G', H'; I', J'; K', L', respectively, and draw the lines $E'J'$, $F'I'$, $G'L'$, $H'K'$, then the area of the quadrilateral formed is

$$K = 2\left[2\left(\frac{n-2r}{n}\right)m_1\left(\frac{n-2r}{n}\right)m_2\sin\theta\right] = \left(\frac{n-2r}{n}\right)^2 4m_1m_2\sin\theta$$

$$= \left(\frac{n-2r}{n}\right)^2 K_2,$$

for integers r and n such that $0 < r < n/2$.

Also solved by A. N. Aheart, H. L. Chow, R. J. Cormier, Michael Goldberg, S. H. Greene, K. S. Kalman, W. H. Laubach, D. C. B. Marsh, D. L. Shell, Sister M. Stephanie, H. Thornton, and the proposer.

Most of the other proofs used Cartesian coordinates, a few employed vectors, and one used complex coordinates. Other generalizations were given. For example, Goldberg stated that if $DK = LA = k(DA)$, $AE = FB = k(AB)$, etc., then the area of the central quadrilateral is $(1-2k)^2$ times the area of the original quadrilateral for all values of k between 0 and 1/2.

Curves with Arc Length Equal to Area

E 1549 [1962, 1008]. *Proposed by C. R. MacCluer, Ohio State University*

Find all functions $y=f(x)$ with second derivatives such that on each interval $[0, x]$, arc length = area.

Solution by D. A. Moran, University of Chicago. The conditions of the problem give the differential equation $y^2=1+(y')^2$, which has the solutions

$$y = \cosh(x-c) \quad \text{and} \quad y = 1. \tag{1}$$

These two functions intersect at $x=c$, and their first and second derivatives agree at this point, so in addition to the solutions given in (1) we also have as solutions the functions

$$y = \begin{cases} \cosh(x-a), & 0 \leqq x \leqq a \\ 1, & a < x < b \\ \cosh(x-b), & x \geqq b \end{cases}$$

Also solved by A. N. Aheart, R. G. Albert, J. W. Baldwin, Nyles Barnet and Roy Hines (jointly), M. E. Beatty, Joseph Beer, J. B. Bohac, D. A. Breault, Brother T. C. Wesselkamper, R. L. Carmichael, D. I. A. Cohen, R. J. Cormier, D. S. Daniels, J. W. Duke, Ragnar Dybvik, E. S. Eby, Michael Goldberg, Ralph Greenberg, F. C. Hall, W. J. Halm, D. L. Hansen, G. A. Heuer, A. R. Hyde, R. A. Jacobson, J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), M. S. Klamkin, Joel Kugelmass, A. E. Livingstone, Thomas Maddock, D. C. B. Marsh, Gus Mavrigian, J. C. Nichols, C. S. Ogilvy, Walter Penny, Stanton Philipp, B. E. Rhoades, H. J. Ricardo, L. A. Ringenberg, Perry Scheinok, D. L. Silverman, Barry Simon, H. W. Vayo, Andy Vince, Robert Weinstock, David Zeitlin, and the proposer.

The problem was located as Ex. 9, p. 45, *Ordinary Differential Equations*, by R. E. Langer, as Ex. 8, p. 25, *Elementary Differential Equations*, by G. E. F. Sherwood and A. E. Taylor, and on pp. 149–50 of *Through the Mathescope*, by C. S. Ogilvy.

Evaluation of a Sum

E 1550 [1962, 1008]. *Proposed by W. L. Nicholson, General Electric Company, Richland, Washington*

Show that

$$\sum_{(p,q)=1} 1/(2^{p+q}-1) = 1,$$

where the sum extends over all positive integers p and q such that p and q are relatively prime.

Solution by the proposer. Let a be a real number such that $a>1$. Then

$$(a-1)^{-2} = \left(\sum_{k=1}^{\infty} a^{-k}\right)^2 = \sum_{k_1,k_2=1}^{\infty} a^{-(k_1+k_2)}$$

$$= \sum_{(p,q)=1} \sum_{m=1}^{\infty} a^{-m(p+q)} = \sum_{(p,q)=1} (a^{p+q}-1)^{-1}.$$

The problem is the special case $a=2$.

Also solved by R. G. Albert, Joseph Beer, W. J. Blundon, Leonard Carlitz, M. J. Cohen, N. J. Fine, Alan Gart, Ralph Greenberg, D. C. B. Marsh, M. G. Murdeshwar, and Stanton Philipp.

Enumeration of Some Partitions

E 1551 [1962, 1008]. *Proposed by Azriel Rosenfeld, Budd Electronics, Inc., N. Y.*

Let $N>K>0$ be integers. How many partitions of K are there such that the parts all divide N? What is the smallest number of parts in such a partition? (This problem is, of course, suggested by the well-known Egyptian restriction to fractions with unit numerators.)

Solution by D. C. B. Marsh, Colorado School of Mines. Both questions may be answered in terms of algorithms.

(1) Let d range over all divisors of N. Euler showed that the number of partitions of K such that all parts divide N is given by c_K, where

$$\prod_{d|N}(1-x^d)^{-1}=1+\sum_{j=1}^{\infty}c_jx^j.$$

(2) For $d_1>d_2>\cdots$, the divisors of N, and $K=\sum_{d_j}b_jd_j$, the number of parts is $\sum b_j$. Thus if we define $K_m=K_{m-1}-\delta_m$, with $K_0=K$ and $\delta_m=\max d_j \leqq K_{m-1}$, the minimum number of parts for such a partition of K is p, where $K_p=0\neq K_{p-1}$.

Also solved, partially, by J. W. Baldwin.

Concerning Sequences of Positive Terms

E 1552 [1962, 1008]. *Proposed by David Rearick, University of Colorado*

Let $\{a_k\}$ be a sequence of positive terms approaching zero. For any positive x, let $N(x)$ denote the number of terms a_k which are greater than x. Prove that if $\sum a_k$ converges, then $xN(x)\to 0$ as $x\to 0$. Is the latter condition also sufficient for the convergence of $\sum a_k$?

Solution by Ralph Greenberg, University of Pennsylvania. Suppose that $\sum a_k$ converges. Then $\{a_k\}$ can be rearranged so that $a_k\geqq a_{k+1}$ without affecting the convergence of the series. Then, if $a_k\geqq x>a_{k+1}$, $xN(x)=kx\leqq ka_k$, so that $xN(x)\to 0$. That $xN(x)\to 0$ is not sufficient for the convergence of $\sum a_k$ is demonstrated by the sequence $a_k=1/(k\log k)$ for $k\geqq 2$, $a_1=1$. Here, if $a_k\geqq x>a_{k+1}$, $xN(x)=kx\to 0$, but $\sum a_k$ diverges.

Also solved by R. G. Albert, R. O. Atkinson, J. W. Baldwin, Joseph Beer, J. L. Brown, Jr., N. J. Fine, Stephen Fisk, Watson Fulks, Michael Goldberg, G. A. Heuer, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), Frank Knight, A. E. Livingstone, Brockway McMillan, D. C. B. Marsh, Stanton Philipp, Denis Sjerve, W. C. Waterhouse, and the proposer.

Fibonacci Numbers as Determinants

E 1553 [1962, 1008]. *Proposed by F. D. Parker, University of Alaska*

Find the value of the determinant of order n which is formed in the following way: The elements of the diagonals running in the direction of the main diagonal and beginning with $a_{21}, a_{11}, a_{13}, a_{15}, \cdots, a_{1,2k-1}, \cdots$ are all unity, and all other elements are zero.

Solution by R. G. Albert, Cambridge, Mass. Let A_n denote the value of the determinant of order n. Then $A_1 = A_2 = 1$. For $n > 2$, we expand A_n with respect to its first column, yielding $A_n = A_{n-1} - B_{n-1}$, and similar expansion of B_{n-1} gives $B_{n-1} = -A_{n-2}$. Thus $A_n = A_{n-1} + A_{n-2}$, the nth Fibonacci number.

Also solved by Jack Abad, A. N. Aheart, Joseph Beer, Marjorie Bicknell, Lydia T. Bills, D. A. Blaeuer, Robert Bowen, Maurice Brisebois, Brother T. C. Wesselkamper, David Carlson, R. L. Carmichael, H. L. Chow, D. I. A. Cohen, C. W. Conatser, R. J. Cormier, H. G. Demmert, J. W. Duke, Carol Echols, Patrick Endicott, J. A. Erbacher, N. J. Fine, F. E. Fischer, Stephen Fisk, C. M. Frye, Michael Goldberg, R. N. Gordon, Ralph Greenberg, S. H. Greene, Cornelius Groenewoud, W. J. Hansen, A. E. Hoffman, J. M. Horner, A. R. Hyde, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), Roman Kaluzniacki, P. G. Kirmser, M. S. Klamkin, L. A. Lambert, T. J. Lee, W. R. McEwen, C. F. Marion, D. C. B. Marsh, Otto Mond, Stephen Montague, D. A. Moran, P. J. Nickolai, Stanton Philipp, Lester Rubenfeld, Perry Scheinok, Donna J. Seaman, H. Thornton, Gary Venter, D. A. Vivian, Leonard Yap, K. L. Yocum, F. H. Young, Leonard Zacks, David Zeitlin, and the proposer.

Attention was called to the closely related problem E 834; if in the given $n \times n$ determinant we subtract the first row from the second row, then expand by minors along the first column, we obtain the $(n-1)\times(n-1)$ determinant of E 834, which was shown to be the nth Fibonacci number. Nickolai pointed out that A_n is equal to the permanent of its matrix.

A Functional Equation

E 1554 [1962, 1009]. *Proposed by Krishna Savati, Ann Arbor, Michigan*

Find all solutions of $f(x) = -f(1/x)$.

I. *Solution by R. J. Bumcrot, Ohio State University.* We might as well assume $f(x)$ is defined on the set F^* of nonzero elements of the field F of variables. If F is the complex field we may certainly choose f to be any arbitrary function on $0 < |z| < 1$ and for $z = e^{i\theta}$, $0 < \theta < \pi$. But this determines $f(z)$ for all z except ± 1, where f must be zero. Thus all solutions have been obtained. The solutions for the real field are now apparent. If $F = J_p$, then $f(1) = f(p-1) = 0$, and f is determined by assigning arbitrary values to $f(2), \cdots, f((p-1)/2)$. There are exactly $p^{(p-3)/2}$ such f's on J_p.

II. *Solution by R. H. Hines, Jr., Concord, Mass.* Any function

$$f(x) = g(x) - g(1/x), \tag{1}$$

defined for nonzero x, satisfies the functional equation $f(x) = -f(1/x)$. Conversely, if $f(x)$ satisfies the functional equation, then

$$f(x) = [f(x) - f(1/x)]/2.$$

Therefore, any solution is of the form (1).

Also solved by Jack Abad, R. G. Albert, R. H. Anglin, J. W. Baldwin, B. W. Banks, Joseph Beer, D. A. Breault, R. L. Carmichael, D. I. A. Cohen, E. S. Eby, N. J. Fine, Michael Goldberg, Ralph Greenberg, F. C. Hall, W. J. Hansen, G. A. Heuer, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), P. G. Kirmser, M. S. Klamkin, Joel Kugelmass, D. C. B. Marsh, F. J. Papp, Jr., Stanton Philipp, Perry Scheinok,, D. L. Silverman, Richard Sinkhorn, G. C. Thompson, H. Thornton, John Waddell, W. C. Waterhouse, L. B. Winrich, K. L. Yocum, David Zeitlin, and the proposer.

Sinkhorn expressed the solution in the form

$$f(x) = \begin{cases} F_1(\log x), & x > 0 \\ F_2(\log(-x)), & x < 0 \end{cases}$$

where each F_i is an arbitrary odd function. The proposer considered the somewhat more general functional equation, $f(x)=kf(x^k)$. A functional equation which has interested a number of correspondents is $f[f(x)]=1/x$.

A System of Diophantine Equations

E 1555 [1962, 1009]. *Proposed by Alfred Brauer and Aubrey Kempner, University of North Carolina and University of Colorado*

In his paper "On the integer solutions of the equation $x^2+y^2+z^2+2xyz=n$," *Journal of the London Math. Soc.*, 28 (1953) 500–10, L. J. Mordell states: "I do not know anything about the integer solutions of

$$x^3 + y^3 + z^3 = 3 \tag{1}$$

beyond the existence of the four solutions

$$(1, 1, 1),\quad (-5, 4, 4),\quad (4, -5, 4),\quad (4, 4, -5), \tag{2}$$

and it must be very difficult indeed to find out anything about any other solutions."

Prove that the triples (2) are the only solutions of (1) which also satisfy the Diophantine equation

$$x^3 + y^3 + z^3 = x + y + z. \tag{3}$$

I. *Solution by W. C. Waterhouse, Harvard University.* Eliminating z between (1) and (3), we get

$$0 = (3 - x)y^2 - (3 - x)^2y - 3x(3 - x) + 8,$$

and therefore $3-x$ must divide 8. This leaves only a finite number of values to check, and it is easily seen that the triples (2) are the only solutions.

II. *Solution by Andrzej Makowski, Warsaw, Poland.* We prove the following theorem: *The system of equations*

$$x + y + z = a, \tag{a}$$

$$x^n + y^n + z^n = b, \tag{b}$$

where a and b are integers, n is an odd integer >1, *has only a finite number of integral solutions x, y, z, which can be found in a finite number of steps, unless* $a^n=b$.

Suppose x, y, z are integers satisfying (a) and (b). Then

$$(x + y + z)^n - (x^n + y^n + z^n) = a^n - b.$$

The polynomial of the left-hand side is not identically 0 and (considered as a polynomial of one variable y) vanishes for $y=-z$, and hence is divisible by $y+z$. Similarly it is divisible by $x+z$ and $x+y$. Thus

$$(y + z)(x + z)(x + y)f(x, y, z) = a^n - b,$$

where f is a polynomial. By (a) we get

$$(a - x)(a - y)(a - z)f(x, y, z) = a^n - b.$$

Hence $a-x$, $a-y$, $a-z$ divide a^n-b. If $a^n\neq b$, there exist only a finite number of values x, y, z satisfying this condition, and the theorem is proved.

Now consider the case $n=3$. Then $f(x, y, z)=3$ identically. Thus a necessary condition of solvability of (a) and (b) is that $3\mid(a^3-b)$. If $b=a^3$, then one of x, y, z is equal to a. From (a) it follows that the sum of two others is equal to 0. All such triplets also satisfy (b).

These results are generalizations of problem E 1555.

Also solved by R. G. Albert, J. W. Baldwin, Nyles Barnert, Joseph Beer, W. J. Blundon, W. J. Carpenter, D. I. A. Cohen, M. J. Cohen, R. J. Cormier, Stephen Fisk, Michael Goldberg, Ralph Greenberg, W. J. Hansen, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), Joel Kugelmass, Viktors Linis, D. C. B. Marsh, Stanton Philipp, D. L. Silverman, H. Thornton, K. L. Yocum, David Zeitlin, and the proposers.

ADVANCED PROBLEMS AND SOLUTIONS

PROBLEMS FOR SOLUTION

5131. *Proposed by J. L. Pietenpol, Columbia University*

Let an element A of a Boolean algebra be called a "point" if and only if $A\neq 0$ and $A=B\cup C$ implies $B=A$ or $C=A$. Show that there exists a Boolean algebra having no points.

5132. *Proposed by H. T. Croft, University of California, Berkeley*

Let $f(z)$ be a complex polynomial of the complex variable z and let $f(1)=f(-1)=0$. Prove or disprove: $f'(z)=0$ has a root in the strip $-1\leqq \mathrm{Re}(z)\leqq 1$.

5133. *Proposed by D. S. Mitrinovic, Belgrade, Yugoslavia*

Let k, n be positive integers and $E=\{1, 2, \cdots, kn\}$. For every $r\in\{0, 1, \cdots, k-1\}$ let E_r denote the ordered set of all the numbers in E having r as remainder upon dividing by k. The sets $E_0, E_1, \cdots, E_{k-1}$ yield a set P of $k!$ permutations. If k is even, one of them is (*) $E_0E_2 \cdots E_{k-2}E_1E_3 \cdots E_{k-1}$. Determine the number of inversions in the permutation (*). Answer the same question for other elements of P.

5134. *Proposed by Joseph R. Landau, University of California, Berkeley*

Let G be a finite group of order mn, and let K be a subset of G containing m elements. If $g\in G$, the set of all kg for $k\in K$ may be called a coset of K. Prove: if K is not empty, then if K has just n cosets, one of those cosets is a subgroup of G; in particular, if K contains the identity and has just n cosets, then K itself is a subgroup.

5135. *Proposed by Seth Warner, Duke University*

Give an example of a quadratic over a field of characteristic 2 which is not solvable by radicals.

5136. *Proposed by A. V. Boyd, University of Witwatersrand, Johannesburg, So. Africa*

Prove that, for $x>0$,

$$\begin{aligned}(1)\qquad &\int_0^\infty\left\{e^{-t}-1+\frac{t}{1!}-\frac{t^2}{2!}+\cdots+\frac{(-1)^nt^{n-1}}{(n-1)!}-\frac{(-t)^n}{n!(tx+1)}\right\}t^{-n-1}dt\\ &\qquad=\frac{(-1)^{n+1}}{n!}\left\{\gamma-\log x-\sum_{r=1}^n\frac{1}{r}\right\}.\end{aligned}$$

$$\begin{aligned}(2)\qquad &\int_0^\infty\left\{e^{-t}-1+\frac{t}{1!}-\frac{t^2}{2!}+\cdots+\frac{(-1)^nt^{n-1}}{(n-1)!}-\frac{(-t)^n}{n!(t^2x^2+1)}\right\}t^{-n-1}dt\\ &\qquad=\frac{(-1)^{n+1}}{n!}\left\{\gamma-\log x-\sum_{r=1}^n\frac{1}{r}\right\}.\end{aligned}$$

5137. *Proposed by R. A. Rankin, Glasgow University, Scotland*

Let S_n be a set consisting of n different points and let $T_0(n)$ be the number of different T_0-topologies that can be formed on S_n; let $T_C(n)$ be the number of different connected topologies on S_n; and let $T^*(n)$ be the number of different connected T_0-topologies on S_n. For example,

$$T_0(1)=T_C(1)=T^*(1)=1;\quad T_0(2)=T_C(2)=3,\quad T^*(2)=2;$$

$$T_0(3)=T_C(3)=19,\quad T^*(3)=12;\quad T_0(4)=219,\quad T_C(4)=233,\quad T^*(4)=146.$$

Prove that $T_0(n)$ and $T_C(n)$ are odd for all $n \geqq 1$, and that $T^*(n)$ is even for $n \geqq 2$.

5138. *Proposed by N. S. Mendelsohn, University of Manitoba*

Show that in a free group, if a word of length $2n$ lies in the commutator subgroup, then it can be expressed as a product of at most $n-1$ commutators.

5139. *Proposed by Dennis Travis, Columbia University*

Let V be a finite dimensional normed vector space over R. Let $x_1, \cdots, x_n$ be a basis for V. Prove that there exists a $k>0$ such that if $y_1, \cdots, y_n$ is a set of vectors with the property that $\|x_i - y_i\| < k$ for $i=1, \cdots, n$, then the y_i are independent.

5140. *Proposed by A. A. Mullin, University of Illinois*

Consider the algebraic system determined by the additive and multiplicative monoids of nonnegative integers N. Put $E_n = \{p_{j_1} \cdot p_{j_2} \cdots p_{j_{2n+1}} : p_j \in A\}$ where A is the set of all odd primes and $n \in N$. Put $S = \bigcup_{n=0}^{\infty} E_n$. Let $T \subseteq N$ and consider the following two conditions:

(i) $\qquad T + T \subseteq \overline{T}, \qquad$ (ii) $\qquad T \cdot T \subseteq \overline{T},$

where $\overline{T}$ is the set-theoretical complement of T relative to N. Prove that S satisfies conditions (i) and (ii). Prove that if S is not maximal over N relative to both conditions (i) and (ii) holding simultaneously, then Goldbach's conjecture is not a theorem. (Incidentally, if one considers whether or not every even integer greater than four can be represented as the sum of two elements of S, then one has a weakened version of Goldbach's conjecture. Possibly the weakened version will yield more readily to solution.)

SOLUTIONS

Properties of Special Determinants

3172 [1926, 104]. *Proposed by H. T. Davis Indiana University*

Making the abbreviation

$$D_n(a_1, a_2, \cdots, a_n) = \begin{vmatrix} a_1 & a_2 & a_3 \cdots a_n \\ 1 & a_1 & a_2 \cdots a_{n-1} \\ 0 & 1 & a_1 \cdots a_{n-2} \\ \cdot & \cdot & \cdot \cdots \cdot \\ 0 & 0 & 0 \cdots a_1 \end{vmatrix}$$

prove the following:

(a) If

$$\delta_n = D_n\left(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \cdots, \frac{1}{(2n)!}\right),$$

then

$$\lim_{n\to\infty}\left|\frac{\delta_{n-1}}{\delta_n}\right| = \frac{\pi^2}{4}\cdot$$

(b) The Bernoulli numbers, B_n, are given by the determinant

$$B_n = (2n)!D_{2n}\left(\frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \cdots, \frac{1}{(2n+1)!}\right).$$

(c) If D_i denotes $D_i(a_1, a_2, \cdots, a_i)$ $(i=1, 2, \cdots, n)$, then we have $D_n(-D_1, D_2, -D_3, \cdots, (-1)^nD_n)=(-1)^na_n$.

Solution by D. A. Moran, University of Chicago.

(a) We use the following theorem of R. P. Agnew, J. B. Rosser, and R. J. Walker: (see this MONTHLY, 49 (1942), 462–463).

THEOREM. *Let $f(z)$ be an analytic function, and r a complex number, such that* (i) $f(0)=-1$, (ii) *r is a simple root of $f(z)=0$,* (iii) *if s is a root of $f(z)=0$ distinct from r then $|s|>|r|$,* (iv) *$f(z)$ has no singular points, except possible poles, in or on the circle $|z|=|r|$. Let $-1+a_1z+a_2z^2+\cdots$ be the Maclaurin expansion for $f(z)$, and define $A_0, A_1, \cdots$ by $A_0=1$,*

$$A_n = a_1A_{n-1} + a_2A_{n-2} + \cdots + a_nA_0 \qquad n > 0.$$

Then

$$r = \lim_{n\to\infty}\frac{A_n}{A_{n+1}}\cdot$$

If we examine the Maclaurin expansion for $-\cos\sqrt{x}$, we obtain for the numbers A_n precisely the numbers δ_n. The result follows from the theorem, once we note that $\pi^2/4$ is the zero of $-\cos\sqrt{x}$ which is smallest in absolute value.

(b) This is a well-known result. Its proof can be found in Muir's *History of the Theory of Determinants*, §762.

(c) Denote $D_n(-D_1, D_2, -D_3, \cdots, (-1)^nD_n)$ by Y_n. Expanding, we obtain

$$(*)\qquad Y_n = -D_1Y_{n-1} - D_2Y_{n-2} - D_3Y_{n-3} - \cdots - D_n$$

$$(\dagger)\qquad \begin{aligned} &= -a_1(-D_1Y_{n-2} - D_2Y_{n-3} - D_3Y_{n-4} - \cdots) \\ &\quad - (a_1D_1 - a_2)Y_{n-2} \\ &\quad - (a_1D_2 - a_2D_1 + a_3)Y_{n-3} \\ &\quad - (a_1D_3 - a_2D_2 + a_3D_1 - a_4)Y_{n-4}, \text{ etc.,}\end{aligned}$$

where the ith row of (†) corresponds to the ith term of (*). Now the first row of (†) adds with the first term of each of the remaining rows to give zero. Then a similar expansion of Y_{n-2} in the second row adds with the second terms of all the remaining rows to give zero. Continuing this process, we see that all that is left is $(-1)^na_n$.

An Identity on Hermite Polynomials

5046 [1962, 812]. *Proposed by Ronald C. Read, University College of the West Indies*

Prove the formal identity

$$\prod_{m=1}^{\infty}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}\left[H_n\left(\frac{i}{2\sqrt{m}}\right)\right]^2(-\tfrac{1}{2}t^m)^n\right\}$$
$$= (1-t)^{-1/2}(1-t^2)^{-1}(1-t^3)^{-1/2}(1-t^4)^{-1}\cdots$$

where $H_n(x)$ is the Hermite polynomial of degree n.

Solution by L. Carlitz, Duke University. We employ Mehler's formula (see, e.g., Rainville, *Special Functions*, p. 198):

$$\sum_{n=0}^{\infty}(-1)^n H_n(x)H_n(y)\frac{t^n}{2^n n!} = (1-t^2)^{-1/2}\exp\frac{-2xyt-(x^2+y^2)t^2}{1-t^2}.$$

Then we have

$$\prod_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n H_n\left(\frac{x}{\sqrt{m}}\right)H_n\left(\frac{y}{\sqrt{m}}\right)\frac{t^{mn}}{2^n n!}$$
$$= \prod_{m=1}^{\infty}(1-t^{2m})^{-1/2}\cdot\exp\left\{-\sum_{m=1}^{\infty}\frac{2xyt^m+(x^2+y^2)t^{2m}}{m(1-t^{2m})}\right\}.$$

Now

$$\sum_{m=1}^{\infty}\frac{2xyt^m+(x^2+y^2)t^{2m}}{m(1-t^{2m})}$$
$$= \sum_{m=1}^{\infty}\frac{1}{m}\left\{2xy\sum_{r=1}^{\infty}t^{m(2r-1)}+(x^2+y^2)\sum_{r=1}^{\infty}t^{2mr}\right\}$$
$$= -2xy\sum_{r=1}^{\infty}\log(1-t^{2r-1})-(x^2+y^2)\sum_{r=1}^{\infty}\log(1-t^{2r}).$$

Therefore

$$(*)\quad \prod_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n H_n\left(\frac{x}{\sqrt{m}}\right)H_n\left(\frac{y}{\sqrt{m}}\right)\frac{t^{mn}}{2^n n!} = \prod_{m=1}^{\infty}(1-t^{2m})^{x^2+y^2-1/2}(1-t^{2m-1})^{2xy}.$$

For $x=y=i/2$, this reduces to

$$\prod_{m=1}^{\infty}(1-t^{2m})^{-1}(1-t^{2m-1})^{-1/2},$$

in agreement with the stated result. Other noteworthy formulas result from (*) when $x=y=\frac{1}{2}$, when $x=i/2$, $y=-i/2$, etc.

Also solved by Stephen Fisk, H. W. Gould, K. R. Rajagopalan, Arnold Singer, and the proposer.

Nonnegative Hermitian Matrix

5048 [1962, 812]. *Proposed by Hans Schwerdtfeger, McGill University*

Let $m \geqq n$ and let A be a complex $m \times n$ matrix of rank n. Let A^* denote the conjugate transpose of A. Then the Gram matrix $G = A^*A$ is a positive definite hermitian $n \times n$ matrix. Show that the hermitian matrix $B = I_m - AG^{-1}A^*$ is nonnegative if I_m denotes the $m \times m$ unit matrix.

I. *Solution by J. Ernest Wilkins, Jr., John Jay Hopkins Laboratory, San Diego, California.* Since $(AG^{-1}A^*)A = A$, each of the n linearly independent columns of A is an eigenvector of $AG^{-1}A^*$ corresponding to the eigenvalue one. Moreover, the equation $A^*\zeta = 0$ has $m-n$ linearly independent solutions, each one of which is an eigenvector of $AG^{-1}A^*$ corresponding to the eigenvalue zero. Since an $m \times m$ matrix has at most m linearly independent eigenvectors, zero and one are the only eigenvalues of $AG^{-1}A^*$. The nonnegativity of B (and also of $AG^{-1}A^*$) follows at once.

II. *Solution by C. A. Rohde, North Carolina State College at Raleigh.* We have at once, $B = B^* = B^2$. Hence any characteristic root λ of B is zero or one, and the result follows. In fact, since Trace $(B) = m - n$ it follows that exactly $m - n$ characteristic roots of B are one.

If A is real the matrix B has extensive applications in statistical theory, particularly in the theory of analysis of variance.

Also solved by F. P. Callahan, Jr., D. Ž. Djoković, Stuart Haywood, J. C. Hickman, A. S. Householder, James Issos, Immanuel Marx, F. T. Metcalf, Thomas Porsching, Samuel Schechter, and the proposer.

Triangle in a Compact, Convex Subset of R^3

5049 [1962, 812]. *Proposed by G. D. Chakerian, California Institute of Technology*

Let K be a compact, convex subset of R^3, and let F be the area of the surface of K. Then K contains a triangle with area greater than or equal to $(3\sqrt{3}/16\pi)F$.

Solution by C. M. Petty, Lockheed Missiles and Space, Co., Palo Alto, California. Let $K(u)$, with area $\sigma(u)$, be the projection of K on a plane perpendicular to the direction u, and let $\Delta(u)$ be the area of a maximal triangle contained in $K(u)$. Since K is convex every triangle in $K(u)$ is the projection of a triangle in K, and consequently $\Delta(u) \leqq \Delta$, where Δ is the area of a maximal triangle in K. By Steiner symmetrization (see Blaschke, *Vorlesungen über Differential-geometrie* II, p. 50) one shows that $4\pi\Delta(u) - (3\sqrt{3})\sigma(u) \geqq 0$ and by Cauchy's formula (Bonnesen and Fenchel, *Konvexe Körper*, p. 48)

$$F = \frac{1}{\pi}\int_\Omega \sigma(u)d\omega \leqq \frac{16\pi\Delta}{3\sqrt{3}}$$

which solves the problem. Equality holds for the sphere.

Also solved by H. Guggenheimer, and the proposer.

Surface Area and Total Mean Curvature

5050 [1962, 813]. *Proposed by G. D. Chakerian, California Institute of Technology*

Let K be a convex subset of R^3 with constant width. Let F be the area of the surface of K and let M be the total mean curvature. Then

$$1 \leqq \frac{M^2}{4\pi F} \leqq \frac{\pi}{2(\pi - \sqrt{3})} \approx 1.114.$$

Solution by H. Guggenheimer, University of Minnesota. $M^2/4\pi F \geqq 1$ is a classical inequality of Minkowski. The other inequality is obtained as follows.

(a) The plane projections of a body of constant width are plane curves of constant width $w = M/2\pi$. (This follows immediately from the formulas of Cauchy. See Hadwiger, *Altes und Neues über konvexe Körper*, Basel, 1955, p. 48.)

(b) Among all plane curves of constant width w, the Reuleaux triangle has minimal area $(w^2/2)(\pi - \sqrt{3})$ (Yaglom and Boltyanskii, *Convex Figures*, problem 7–12).

(c) By Cauchy's formula, the mean area of a projection of the convex body onto a plane is $F/4$. Hence

$$\frac{F}{4} \geqq \frac{1}{2}\left(\frac{M}{2\pi}\right)^2 (\pi - \sqrt{3}).$$

Equality holds for the solid of revolution of a Reuleaux triangle.

Also solved by C. M. Petty, and the proposer.

Centrosymmetric and Centroskew Matrices

5051 [1962, 925]. *Proposed by Olga Taussky, California Institute of Technology*

Let $A = (a_{ik})$ be an $n \times n$ matrix with $a_{ik} = a_{n+1-i,n+1-k}$. Show that corresponding to any characteristic root there exists a vector $x = (x_1, \cdots, x_n)$ either with $x_i = x_{n+1-i}$ or with $x_i = -x_{n+1-i}$, $i = 1, \cdots, n$.

Solution by Imanuel Marx, Purdue University. Let S be the matrix with entries $s_{ik} = \delta_{i,n-k+1}$ (the Kronecker delta with $i, k = 1, \cdots, n$). The problem may be formulated in terms of S: if A is a matrix such that $SA = AS$, prove that, corresponding to each of its characteristic roots, A has a characteristic vector x such that either $x = Sx$ or $x = -Sx$. For the proof, if $Ay = \lambda y$, then left multiplication by S gives $SAy = A(Sy) = S\lambda y = \lambda(Sy)$, so that Sy is again a character-

istic vector belonging to λ. If $Sy=-y$, then $x=y$ is a vector of the desired sort; if $Sy\neq -y$, then $x=y+Sy$ is a vector of the desired sort (since $S^2=I$).

Also solved by H. E. Bell, D. Ž. Djoković, E. W. Ewing, Wallace Givens, A. S. Householder, D. E. Knuth, James Issos, A. G. Konheim, Marvin Marcus, E. A. Sällin, Michael Skalsky, Neal Zierler, and the proposer.

Editorial Note. Householder points out that a recent paper by A. R. Collar, *On centrosymmetric and centroskew matrices* (Quart. J. Mech. Appl. Math. 15 (1962), 265–281) solves problem 5051 and discusses these matrices in some detail.

Multiplicative Seminorm

5052 [1962, 925]. *Proposed by Seth Warner, Duke University*

A multiplicative seminorm on the algebra E (over the complex numbers) of all entire functions is, by definition, a function p from E into the nonnegative real numbers with the following properties: $p(0)=0$, $p(\lambda f)=|\lambda|p(f)$, $p(f+g)\leqq p(f)+p(g)$, and $p(fg)\leqq p(f)p(g)$ for all entire functions f, g and all complex numbers λ. If E is furnished with the topology of uniform convergence on compact sets (the compact-open topology), is every multiplicative seminorm on E necessarily continuous?

Solution by I. N. Baker, Imperial College of Science and Technology, London, England. Take the entire function $f(z)=\sum_{n=k}^{\infty} a_n z^n$, $a_k\neq 0$, and set $p(z)=|a_k|k!$. This clearly satisfies the conditions for a multiplicative seminorm. Now the sequence of functions $f_n(z)=z^n/n!$ satisfies $p(f_n)=1$, while in E we have $f_n(z)\to 0$ as $n\to\infty$ (since for fixed r, $r^n/n!\to 0$ as $n\to\infty$). Thus p is not continuous.

A Class of Polynomials

5053 [1962, 926]. *Proposed by H. S. Shapiro, New York University*

Let $P(x, y, z)$ be a polynomial such that

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}=0, \quad \text{where} \quad r=(x^2+y^2+z^2)^{1/2}.$$

Prove that P is a multiple of $x^2+y^2+z^2$.

Solution by Dragomir Djoković, Belgrade, Yugoslavia. We prove first the following statement: If $Q(x, y)$ is a nonzero polynomial then

$$\frac{\partial}{\partial z} Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{1}{r}\neq 0. \tag{1}$$

If we suppose, on the contrary, that the left member of (1) is identically zero, we use the formula

$$\frac{1}{r}=\frac{1}{|z|}\sum_{\nu=0}^{\infty}\binom{-1/2}{\nu}\frac{(x^2+y^2)^{\nu}}{z^{2\nu}} \qquad (x^2+y^2<z^2),$$

and find that for every nonnegative integer ν,

$$Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)(x^2 + y^2)^\nu = 0.$$

Hence

$$Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)e^{x^2+y^2} = Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\sum_{\nu=0}^{\infty} \frac{1}{\nu!}(x^2 + y^2)^\nu = 0. \tag{2}$$

If the degree of Q is $m+n$ and $Q(x, y) = ax^my^n + \cdots$ $(a \neq 0)$, we have

$$Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)e^{x^2+y^2} = 2^{m+n}ax^my^ne^{x^2+y^2} + \cdots$$

and (2) is impossible. Thus (1) holds.

We note that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{1}{r} = 0. \tag{3}$$

Taking P in the form

$$P(x, y, z) = (x^2 + y^2 + z^2)Q(x, y, z) + zQ_1(x, y) + Q_2(x, y), \tag{4}$$

where Q, Q_1, Q_2 are polynomials, and making use of (3), we obtain

$$P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = \frac{\partial}{\partial z}\, Q_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{1}{r} + Q_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{1}{r} = 0. \tag{5}$$

By differentiation, we get

$$\frac{\partial}{\partial z}\, Q_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{1}{r} = zR_1(x, y, r), \tag{6}$$

$$Q_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{1}{r} = R_2(x, y, r), \tag{7}$$

where R_1 and R_2 denote rational functions of x, y, r. If $R_1 \not\equiv 0$, it follows from (5), (6), (7) that

$$\sqrt{(r^2 - x^2 - y^2)} = -R_2(x,y, r)/R_1(x, y, r)$$

which is impossible. So $R_1(x, y, r) \equiv R_2(x, y, r) \equiv 0$ whence, by (6) and (7),

$$\frac{\partial}{\partial z}\, Q_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{1}{r} \equiv Q_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{1}{r} \equiv 0.$$

Hence, by (1), $Q_1(x, y) \equiv Q_2(x, y) \equiv 0$, and the desired result follows from (4).

Also solved by Robert Breusch, L. Carlitz, Harley Flanders, and the proposer.

Editorial Note. The proposer derives the solution from Hobson, *Spherical and Ellipsoidal Harmonics*, formula 8, p. 127.

A Limit Due to Ramanujan

5054 [1962, 926]. *Proposed by Richard Sinkhorn, The Boeing Company, Wichita, Kansas*

Evaluate

$$\operatorname*{Lim}_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}\,.$$

Solution by A. G. Konheim, IBM Research, Yorktown Heights, N. Y. We start with the identity

$$\int_0^A e^{-x}x^n dx = n!\left\{1 - e^{-A}\sum_{k=0}^{n}\frac{A^k}{k!}\right\}$$

which is easily established by induction on n. Setting $A=n$ and using the fact that $n! = \int_0^\infty e^{-x}x^n dx$, we find

$$e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{n^{n+1}e^{-n}}{n!}\int_1^\infty e^{-n(x-\log x-1)}dx.$$

Setting $u=x-\log x-1$ we note that when x ranges over $[1, \infty)$ u ranges over $[0, \infty)$ and u is an increasing function of x. Moreover, if $x=g(u)$ then $g'(u) = (2u)^{-1/2}+O(1)$ as $u\to 0^+$. Thus

$$\int_1^\infty e^{-n(x-\log x-1)}dx \sim \Gamma(\tfrac{1}{2})/(2n)^{1/2} = (\pi/2n)^{1/2} \qquad (n\to\infty),$$

and Stirling's formula, $n!\sim e^{-n}n^n\sqrt{(2\pi n)}$, gives finally

$$\operatorname*{Lim}_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2}\,.$$

Also solved by I. N. Baker, P. T. Bateman, W. H. Bonney, Robert Breusch, L. Carlitz, Leopold Flatto, Walter Gantschi, B. R. Henry, G. E. Henry, Fritz Herzog, S. Keller, John B. Kelly, J. Koekoek, M. A. Malik, Roger Pinkham, Ronald Pyke, John Raleigh, D. Ramakotarah and D. Suryanarayana, Vincent Sardella, Z. Šidák, Arnold Singer, O. E. Stanaitis, H. G. Tucker and H. D. Brunk, J. H. van Lint, David Zeitlin, and the proposer. Late solutions by J. H. Foster, Joseph Gayda, and A. E. Livingston.

Editorial Note. Many of the solutions received were interesting and ingenious. The result, however, is fairly well known. It was proved by S. Ramanujan (Journ. Indian Math. Soc., III, 128; See *Collected Papers of Srinivasa Ramanujan*, 323–324).

The stated result is an application of the Central Limit Theorem in Probability. It appears, in particular, as Problem 3 of Chapter 8 (p. 302) of *The Theory of Probability* by B. V. Gnedenko, Chelsea, 1962.

After the first step (as in the above proof) Bateman finds that the rest follows directly from Problems II, 210 and II, 211 of Polya and Szegö, *Aufgaben und Lehrsätze aus der Analysis I.*

In the Journal of the Indian Math. Soc., 1960, Karamata proves

$$\frac{1}{2}e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\theta_n, \qquad \frac{1}{3} < \theta_n < \frac{1}{2}.$$

Other references: J. V. Uspensky, *Introduction to Mathematical Probability*, problem 4, p. 304, Copson, *Theory of Functions of a Complex Variable*, prob. 18, p. 230; G. N. Watson, Proc. London Math. Soc. (2) 29 (1929), 293–308.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

COLLABORATING EDITORS: K. O. MAY, Carleton College and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Analogue Computation Techniques and Components. By R. W. Williams. Academic Press, New York, 1961. 271 pp. $9.50.

A typical reader of this magazine is likely to be interested primarily in the means by which the individual mathematical operations are combined to solve problems. Yet a knowledge of the capabilities and limitations of the individual electronic elements which perform these operations can enable one to utilize the computer more effectively. In this book the author writes about all of the major components, save one, in a clear, concise manner. The reader requires only a rudimentary background in electric circuits to understand how these devices work. In particular, the author emphasizes the shortcomings of the elements and the nature of their errors.

I would caution the reader that errors in individual elements are not easily related to errors in a problem solution. The nature of the differential equation determines the sensitivity of the solution to particular errors. Because the author is interested in the separate elements rather than their combination, he does not discuss this important point.

Unaccountably, the book says nothing about the relay. This device greatly enhances the versatility of the computer, because with the relay the computer can choose alternatives, i.e., make decisions.

Even though two pet subjects of mine are omitted, I would recommend this book to anyone interested in an elementary, clearly written discussion of analogue computer components.

LEON LEVINE, Hughes Aircraft Company

An Introduction to the Theory of Linear Spaces. By Georgi E. Shilov. Translated by Richard A. Silverman. Prentice-Hall, Englewood Cliffs, N. J., 1961. ix+310 pp. $10.00.

The table of contents suggests an approach to this subject which is now considered to be out of date. The exposition begins with a discussion of determinants, Cramer's rule, and the definition of the rank of a matrix as the order of the largest nonvanishing subdeterminant. Systems of linear equations are solved by a cumbersome application of Cramer's rule which depends upon finding a basis minor, a largest nonvanishing subdeterminant.

But except for Chapters 1 and 3 the book is very much up to date. Linear vector spaces are introduced abstractly, and the usual introductory theorems are obtained by coordinate-free methods. Linear functions and linear transformations are discussed at once as linear functions of a vector argument. Although matrices are introduced in the opening section, matrix algebra is delayed until motivated by the corresponding algebra of linear transformations.

Coordinate transformations are delayed until Chapter 5. Then the effects of a change of coordinates on the representations of vectors, linear functionals, and linear transformations are discussed at once. In Chapter 6 bilinear and quadratic forms are introduced. The canonical form for a symmetric bilinear form is obtained by the method of completing the square. Jacobi's method is given as an alternative. Completing the square is probably the easiest method to explain and the easiest to understand, but it leaves much to be desired for practical computation, especially for symmetric bilinear forms of large order. Jacobi's method utilizes exactly the same sequence of arithmetic steps as the method of orthogonal complements, but the computation in the latter method is more conveniently organized.

In Chapter 7 an arbitrary positive definite quadratic form is used to define the scalar product, which is made the basis for the theory of Euclidean spaces and orthogonalization in Chapter 8. A discussion of invariant spaces and eigenvectors is included in Chapter 9. This is done primarily to prove the symmetric operator theorem, the assertion that a symmetric operator has n mutually orthogonal eigenvectors. A number of important theorems in the theory of similarity of matrices are included, but this entire area is greatly de-emphasized. The main thread of the discussion of Euclidean spaces then continues in Chapter 10 with the diagonalization theorems for quadratic forms represented over orthogonal bases.

The first ten chapters contain the material that would normally be included in a course in linear algebra. The last two chapters present an application of linear algebra and an extension of the theory. Brief remarks on several important applications are scattered throughout the book, but only the application to quadric surfaces is given the space of a chapter for its development. The extension of the theory to infinite dimensional vector spaces is well done and an important feature of the book. As a natural continuation of the discussion for finite dimensional spaces it provides an easy introduction to Hilbert spaces and L_2 theory.

A number of unusual features of the book are: the theory of ideals in the algebra of linear operators, the Hamilton-Cayley Theorem as a problem for the reader, an introduction to tensor notation, Hadamard's inequality, least squares, Sturm-Liouville theory, an introduction to the theory of linear integral equations, and a large number of very interesting problems.

Surprisingly, the efficient method of solving linear equations by elimination is not recommended. Equally surprisingly, there is no reference to normal matrices and the problems and advantages brought by considering vector spaces over the field of complex numbers.

The translation is in smooth colloquial English and shows no sign of being a translation. The only real evidence within the text is the frequent reference to Russian mathematical works. The translator has added references to works in English.

Most of the problems are extensions and generalizations of the theory. For example, the algebraic part of the theory of Fourier series is contained in a set of problems. The few numerical problems are excellent, but an instructor desiring to use this book for a text will want many more.

On page 106 reference is made to the "adjoint" matrix, but at the reference point this matrix is called the "adjugate" matrix. Since the former term is used for several unrelated concepts, the later term might well be preferred.

In general, the Theory of Linear Spaces is a valuable addition to the available literature in linear algebra. It is particularly recommended for self study. It is adaptable as a text if some additional numerical exercises are provided. Updating the material on systems of linear equations is a little more difficult.

The rest of the exposition is arranged in such a way that extensions of the discussion to vector spaces over the complex numbers can be made very easily. Since the book contains much more material than can be covered in a one semester course, a number of cuts will have to be made. The translator has assisted in this selection by denoting by asterisks a number of sections that can be skipped without loss of continuity.

EVAR D. NERING, Arizona State University

Fundamentals of Galois Theory. By M. M. Postnikov. Translated from the Russian by Leo F. Boron. Noordhoff Ltd., Groningen, The Netherlands, 1962. 142 pp. $2.50.

Foundations of Galois Theory. By M. M. Postnikov. Translated by Ann Swinfen. Pergamon Press, New York, 1962. 109 pp. $6.50.

Both these translations of Postnikov's book seem honest and similar. They differ mainly in that the first is a paperback with an index, a bibliography, and a picture of the boy hero who has always stood for me as the symbol of the difference between the artist and the scholar, while the second is a beautifully bound edition without the extras. The book itself however, is not similar to other Galois theory treatments. The clean, effortless but sophisticated, lines of ab-

stract algebra have been sacrificed to make the subject available to senior undergraduates, and to make it appeal to the type of person who prefers matrices to linear transformations. The elegance of the book, and there is much of it, comes from the cleverness of the proofs and from techniques rather than from getting a skeleton of just the basic concepts in such a way that the results finally (and in abstract algebra it can take a while) seem obvious. For instance, the clear but sophisticated fact that isomorphisms of fields can be lifted has been replaced at a crucial point by the main theorem of symmetric polynomials. Also, that a finite extension is always simple is proved early by computations and then used in the later proofs, rather than made an obvious result of the theory. Finally, when an algebraically closed field is needed to contain $Q[x_1, \cdots, x_n]$ (Q the rationals) it is artificially but ingeniously constructed as the field of fractional formal power series with complex coefficients.

I strongly recommend this book to those who are interested in handling algebraic equations but who do not care for the basic philosophy of abstract algebra, and also for undergraduates who will later learn the theory with the usual less ingenious and less extraneous concepts.

D. K. HARRISON, University of Pennsylvania and New Mexico State University

An Introduction To Modern Calculus. By Wilhelm Maak. Holt, Rinehart and Winston, New York, 1963. x+390 pp. $7.00.

This book is a translation (apparently by the author himself) of the German text, *Differential- und Integralrechnung*, 2nd ed., Vandenhoeck and Ruprecht, Göttingen, 1960.

The first part of the book (six chapters, 154 pages) is devoted to calculus for real functions of one real variable. There are no exercises or problems. The intent is to present the essential theory and structure of elementary calculus, giving it (the author says) the form of an axiomatic theory. Apart from the brevity achieved by elimination of material extraneous to the author's purpose, this presentation of the calculus of functions of one real variable is basically conventional and classical: real numbers, functions, limits of sequences, infinite series, continuous functions, differential calculus, integrals, the fundamental theorem of calculus.

One may ask: How can this first part of Professor Maak's book be used in the United States? A mathematically inclined student might well read it independently after completing one year of a typical good American university calculus course. Or, it could be assigned as collateral reading for juniors in an advanced calculus course. It could serve as part of the required reading in a beginning university course for selected "honors" students. I would doubt the wisdom of making it the sole text for such a course. I hope that the evolution of honors courses for freshmen and sophomores does not lead, either to the elimination of due attention to the applications of calculus, or to over-concentration on pure theory at the expense of the development of skill in technique and in

problem-solving. I foresee the time, not too far off, when the standard university courses in calculus in the first two years will assume complete responsibility for the sound theoretical treatment of all aspects of the calculus for functions of one variable. This can be done without abdicating responsibility for applications, for technique, and for the stimulation of problem-solving. The normal "advanced calculus" course will then not need to redo this part of calculus.

I feel obliged to point out some minor drawbacks in the first part of the book here under review. The author's translation of his text into English is quite well done, on the whole; but there are many instances in which the English constructions sound strange. For example, on page 32, the phrase "grow smaller beyond bound" is used to mean "approach zero." In a few places the author's informal discussion contains unintentional mis-statements which are inaccurate: "A uniformly continuous function is always bounded, though" (on p. 64, with no reference to a domain of definition).

The second part of the book deals with functions of several real variables and with mappings from R^p into R^n. Here everything is organized so as to lead to the general Stokes formula

$$\int_{F^p} d\omega = \int_{\partial F^p} \omega,$$

where ω is an alternating differential form of degree $p-1$ and F^p is what is called a p-cell in R^n. This formula is presented as the multidimensional analogue of the fundamental theorem $\int_a^b f'(x)dx = f(b) - f(a)$, from elementary calculus, and hence as the culmination of the second part of the book.

Included in this second part of the book are the implicit function theorem for systems of functions of several variables (treated by vector methods, using a familiar iteration procedure to get the theorem as a particular instance of a fixed-point theorem), and a theory of volume and multiple integrals. The underlying concept of a measurable set is that of Jordan. The integral is characterized uniquely by certain axioms, and this uniqueness leads to theorems on the evaluation of integrals in R^n by successive integrations in R^{n-1} and R.

The second part of Professor Maak's book deserves to be read by all teachers of advanced calculus. The point of view of linear algebra and finite-dimensional vector spaces is going to be strongly felt in all our instruction in analysis. I do not regard the exposition in this book as completely successful from a student's point of view. The subject is not easy, and there is danger that a student will be discouraged by the rather massive machinery and terminology which must be developed in tackling some of the work in the general n-dimensional situation at the very outset. Nevertheless, I feel that the aim of this part of the book is praiseworthy, and that it is useful to have Professor Maak's conception of multivariable calculus made available to us in English.

ANGUS E. TAYLOR, University of California, Los Angeles

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor L. M. Blumenthal of the University of Missouri has been elected Foreign Corresponding Member of the Spanish Royal Academy of Sciences. He spent the year 1962–63 as Fulbright Professor at the University of Madrid.

Professor Don Kirkham of Iowa State University received Belgium's highest academic honor, "Doctor Honoris Causa," at a special convocation in Ghent, Belgium, May 7, 1963 sponsored by the King, in recognition of his eminent work on mathematical soil physics.

Dr. Solomon Lefschetz, Head of the Mathematics Center at the Research Institute for Advanced Studies of the Martin Company, Baltimore, Maryland, has received the honorary degree of Doctor of Science from Princeton University.

Professor Raoul Hailpern, SUNY at Buffalo, represented the Association at the inauguration of Dr. W. Allen Wallis as President of the University of Rochester on May 17. At the inaugural convocation, the degree of Doctor of Humane Letters was conferred on Warren Weaver, Vice President of the Sloan Foundation.

Professor C. T. Salkind, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Dr. Dumont F. Kenny as the second President of Queensborough Community College on October 29.

Long Beach State College: Associate Professor Anthony Mardellis has been appointed Chairman of the Mathematics Department; Associate Professor A. H. Smith will be on sabbatical leave for the academic year 1963–64.

Polytechnic Institute of Brooklyn: Professor Harry Hochstadt has been named Head of the Department of Mathematics; Professor R. M. Foster has retired with the title of Professor Emeritus; Assistant Professor Clifford Marshall has been promoted to Associate Professor.

Rensselaer Polytechnic Institute: Associate Professors W. E. Boyce and C. E. Lemke have been promoted to Professors; Assistant Professor G. F. Guilford, Jr. has been promoted to Associate Professor.

Wabash College: Assistant Professor W. C. Swift, Rutgers, The State University, has been appointed Associate Professor; Professor J. C. Polley has retired as Chairman of the Department of Mathematics but will continue teaching; Associate Professor P. T. Mielke has been promoted to Professor and appointed Chairman of the Department of Mathematics.

Dr. Eugene Albert, Dartmouth College, has been appointed Assistant Professor at the University of California, Davis.

Dr. D. W. Bailey, Yale University, has been appointed Assistant Professor at Amherst College.

Dr. R. E. Barlow, General Telephone Laboratories and San Jose State College, has been appointed Assistant Professor in the Department of Industrial Engineering at the University of California, Berkeley.

Assistant Professor Homer Bechtell, Lebanon Valley College, has been appointed Associate Professor at Bucknell University.

Assistant Professor H. E. Bell, Union College, has been appointed Assistant Professor at Harpur College.

Mr. T. B. Boone, Northwestern State College of Louisiana, has been promoted to Assistant Professor.

Mr. Alphonse Buccino, University of Chicago, has been appointed Assistant Professor and Acting Chairman of the Department of Mathematics at DePaul University.

Associate Professor R. J. Buehler, Iowa State University, has been appointed Professor at the University of Minnesota.

Associate Professor R. G. Buschman, Oregon State University, has been appointed Professor at SUNY at Buffalo.

Associate Professor John Christopher, University of Nebraska, has been appointed Associate Professor at Sacramento State College.

Dr. Joyce C. Cundiff, University of Florida, has been appointed Associate Professor at the State University of New York at Fredonia.

Dr. S. E. Dickson, New Mexico State University, has been appointed Assistant Professor at the University of Nebraska.

Associate Professor W. H. Fleming, on leave of absence at the Mathematics Research Center of the University of Wisconsin, has returned to Brown University and been promoted to Professor.

Dr. Arthur Grad is on a year's leave of absence from the National Science Foundation to serve as Associate Dean of the Graduate Division of Stanford University.

Mr. C. V. Heuer, University of Nebraska, has been appointed Assistant Professor at the University of Missouri.

Professor P. G. Hodge, Jr., Illinois Institute of Technology, has received an NSF Senior Postdoctoral Fellowship and will spend the year 1963–64 at Stanford University.

Mr. J. E. Homer, Jr., St. John's University, Minnesota, has been promoted to Assistant Professor.

Assistant Professor Edgar Karst, on sabbatical leave from Evangel College, will spend the summer and the academic year 1963–64 at the University of Oklahoma.

Mr. R. M. Kennedy, American International College, has been promoted to Assistant Professor.

Assistant Professor John Kenelly, University of Southwestern Louisiana, has been appointed Associate Professor at Clemson College.

Dr. A. M. Krall, University of Virginia, has been appointed Assistant Professor at Pennsylvania State University.

Associate Professor Max Kramer, San Jose State College, has been promoted to Professor.

Associate Professor O. C. Kreider, Iowa State University, has been promoted to Professor.

Mr. W. M. Lambert, Jr., University of California, Los Angeles, has been appointed Assistant Professor at Loyola University, Los Angeles.

Associate Professor F. C. Leone, Case Institute of Technology, has been appointed Acting Head of the Department of Mathematics.

Dr. Jiang Luh, University of Michigan, has been appointed Associate Professor at Indiana State College.

Mr. R. G. McDermot, University of Pittsburgh, has been appointed Assistant Professor at Duquesne University.

Assistant Professor D. V. Meyer, State University of Iowa, has been appointed Assistant Professor at Central College.

Professor Henryk Minc, University of Florida, has been appointed Professor at the University of California, Santa Barbara.

Assistant Professor Amin Muwafi, American University of Beirut, Lebanon, has been promoted to Associate Professor.

Assistant Professor Hajimu Ogawa, University of California, Riverside, will be on leave during the academic year 1963–64 as Visiting Assistant Professor at the University of California, Berkeley.

Mr. J. D. Patterson, Idaho State College, has been appointed Associate Professor of Physics at the South Dakota School of Mines and Technology.

Professor Everett Pitcher, Lehigh University, has been designated Distinguished Professor for outstanding teaching and service.

Dr. R. C. Prim, Bell Telephone Laboratories, Murray Hill, New Jersey, has accepted a position as Vice-President for Research with the Sandia Corporation, Albuquerque, New Mexico.

Associate Professor R. E. Roth, St. Bonaventure University, on leave during the past year at the University of Oklahoma, has been promoted to Professor.

Assistant Professor W. M. Sanders, University of Illinois, has been appointed Associate Professor at the University of Southern Mississippi.

Professor R. D. Sheffield, University of Mississippi, has been appointed Professor and Head of the Department of Mathematics at Mississippi State University.

Dr. D. P. Squier has returned to the California Research Corporation after spending a year at the University of Arizona on leave of absence.

Assistant Professor Louis Sucheston, University of Wisconsin, has been appointed Associate Professor at Ohio State University.

Professor H. C. Trimble, Iowa State Teachers College, has been appointed Professor of Education at Ohio State University.

Associate Professor M. E. Turner, Medical College of Virginia, has been appointed Professor and Chairman of the Department of Biometry at Emory University.

Assistant Professor George Van Zwalenberg, Calvin College, has been appointed Assistant Professor at Fresno State College.

Associate Professor Alexander Wittenberg, Laval University, has been appointed Professor at York University.

Professor R. L. Beinert, Hobart and William Smith College, died on June 4, 1963. He was a member of the Association for twenty-one years.

Professor Emeritus C. E. Melville, Clark University, died on April 8, 1963. He was a member of the Association for forty-three years.

Dr. B. D. Roberts, Arizona State University, died on April 25, 1963. He was a member of the Association for thirty-five years.

Professor Emeritus R. C. Staley, University of North Dakota, died on May 4, 1963. He was a member of the Association for thirty-eight years.

Professor Emeritus Roscoe Woods, State University of Iowa, died on June 19, 1963. He was a member of the Association for forty years.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

The American Society for Engineering Education wishes to invite into its membership all mathematicians who teach engineering students, or who are employed by industrial organizations having an interest in engineering education. Mathematics is one of the twenty-three divisions of ASEE which represent the various professional branches of science, engineering and related fields. Membership in ASEE offers a unique opportunity for mathematicians to associate and consult with those who are anxious to make the best possible use of mathematics in their courses and in their work. In turn the ASEE needs a broad representation of the best mathematicians to fulfill its purposes.

Annual dues are $10 for those 36 years of age and over, and $7 for those under 36. Further information and application forms may be obtained from Professor W. Leighton Collins, Executive Secretary, ASEE, University of Illinois, Urbana, Illinois.

INFORMATION FROM SMSG FOR PROSPECTIVE AUTHORS

A Panel on Supplementary Publications (POSP) of the School Mathematics Study Group has been established. This notice is to tell prospective authors of the opportunity for publishing through this panel.

POSP is concerned with the writing, evaluating and editing of manuscripts for pamphlets to be used in grades 7 through 12. When published, the pamphlets will range in length from 16 to 64 pages. They are intended for use as supplementary units to be studied by an entire class in addition to a regular text, and as enrichment units for students with special interests. Some pamphlets may serve both purposes and may be useful for a teacher or a group of teachers to study alone. The primary purpose, however, is for class use.

Although POSP is preparing for immediate publication a few manuscripts which have been already submitted to SMSG, and is re-editing some material already published, there is a continuing need for new manuscripts on a variety of topics. Authors are encouraged to submit manuscripts on subject matter which they consider suitable for use as mentioned above, or on topics suggested by members of the Panel. Pictures, diagrams, graphs and charts may be used to enliven the presentation. It is quite appropriate to include answers to problems, typical solutions, or comments to the teacher in an appendix or in a separate Teacher's Commentary.

Further information may be secured from the chairman of the panel, Dr. H. W. Syer, Kent School, Kent, Connecticut.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE IOWA SECTION

The 50th regular meeting of the Iowa Section of the Mathematical Association of America was held at Iowa State University, Ames, on April 19, 1963. Chairman Lyle E. Pursell presided. Total attendance was 96, including 53 members of the Association. Routine business was considered during the afternoon meeting.

A report of the Iowa 1963 high school mathematics contest was given by Lewis Workman of the Des Moines Actuaries Club, who sponsors it. A treasurer's report was given and a balance of $222.18 was indicated.

The following officers were elected: Chairman, C. H. Lindahl, Iowa State University; Vice-Chairman, R. S. Jacobsen, Luther College; Secretary-Treasurer, E. L. Canfield, Drake University.

Conflicting dates between the Iowa Section spring meeting date and the regional meeting of the American Mathematical Society were again discussed. No specific action to remedy the conflict was taken.

The following papers completed the program:

1. *Propagation and growth of waves in hypo-elastic media*, by Professor G. A. Nariboli, Iowa State University, introduced by the Chairman.

By the use of the theory of singular surfaces, wave phenomena in a hypo-elastic medium are discussed. In a medium, arbitrarily stressed initially, the waves, when all three fronts exit, do not

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FREDERICK A. FICKEN, *Editor*

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ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, N. Y.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

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AN EXAMPLE IN MATHEMATICAL LOGIC

HARTLEY ROGERS, JR., Massachusetts Institute of Technology

The following example serves to illustrate some of the chief concerns, both traditional and current, of *mathematical logic*. We assume no previous knowledge of logic and we adopt, initially, a rather naive point of view. The example is fairly well known to logicians.

1. The subject. We begin by briefly describing, in broadest outlines, our subject matter. It is as follows.

First of all, we shall have a *language*, a precisely specified collection of symbolic expressions: what a logician sometimes refers to as a *formal calculus*, although we, of course, are not concerned with derivatives and integrals. Then, secondly, and quite separate from this first, we shall have the *mathematical objects* about which the language talks; and we will think of these as entities existing separately and apart from the language. This may strike the reader as a little dangerous. He may say, "Well, the only way that we *know* mathematical objects is through the language that we use to talk about them. As a matter of fact, maybe the only sense in which mathematical objects exist is in our talking about them in some particular language." These are subtleties that we shall not be concerned with here. We remark that they are subtleties with which the modern logician is, on the whole, well equipped to deal. For our purposes, for the sake of efficient communication below, we suggest that the reader adopt a *platonistic* attitude, simply viewing the situation, for the present anyway, as being concerned with our *language* on the one hand, and these independently existing mathematical *objects* on the other.

2. The language. So much for a broad outline. Now let us be more specific and look at the language, or formal calculus, with which we shall be concerned.

First of all, it will involve the following symbols:

$$x \quad y \quad z \quad w.$$

These letters will eventually be used to stand for certain basic mathematical objects about which the language will speak. They will play a role similar to that of pronouns in ordinary English. We call these symbols *variables*. Variables affected by quantifiers (see below) function as indefinite pronouns, while variables elsewhere function as relative pronouns. If we need more variables, we make new ones by adding numerical subscripts; e.g.,

$$x_4 \quad y_{25} \quad z_7.$$

We will have the symbol "$=$", which we read as "equals".

We will also have the symbol "$<$". We read this symbol as "less than," just to have a name for it, but we do not let this phrase prejudice our decision as to what the symbol will stand for when, eventually, we introduce the mathematical objects about which the language will speak. At the moment, of course, we

are simply describing symbolic expressions, quite apart from any meaning that they may later be given.

If we take variables and put them on either side of "$=$" or of "$<$", we obtain what we shall call *atomic formulas.* E.g.,

$$x = w, \qquad y < y, \qquad x_4 < z_2,$$

are atomic formulas. These are the basic formulas (i.e., expressions) from which we shall build the larger formulas of our language.

We put formulas together to make larger formulas using the following symbols:

"$\leftrightarrow$",	which we read as	"if and only if";
"$\rightarrow$",	which we read as	"only if";
"&",	which we read as	"and";
"$\vee$",	which we read as	"or";
"$\sim$",	which we read as	"not".

"$\leftrightarrow$", "$\rightarrow$", "&", and "$\vee$" are used between two formulas; "$\sim$" is used in front of a formula. We use brackets to show grouping, and any "$\sim$" is assumed to apply to the smallest following string of symbols that the brackets permit. Thus,

$$x = y \rightarrow [\sim y < z \,\&\, z = x]$$

could be read: "x equals y only if both y not less than z and z equals x." We shall use the words "if . . . then . . . " as an alternative reading for "only if". Thus, the above could be read: "if x equals y, then both y not less than x and z equals x."

We use two further symbols for building larger formulas from smaller,

$$\forall \qquad \exists.$$

When either of these symbols is used, it is joined to a variable and enclosed in parentheses. The result is called a *quantifier.* A quantifier is used in front of a formula. Like "$\sim$", a quantifier is assumed to apply to the smallest following formula that brackets permit. Quantifiers are read as follows. "$(\forall y)$" is read as "for every y." "$(\exists x)$" is read as "there exists an x such that."

Rather than give more precise rules for constructing larger formulas from smaller, we content ourselves with several further examples of formulas

(1) $$(\forall x)(\exists y)[y < x],$$

("for every x there is a y such that y less than x").

(2) $$(\forall x)(\exists y)[x < y \,\&\, (\forall z)[x < z \rightarrow [y = z \vee y < z]]],$$

("for every x there exists a y such that x less than y and for every z, if x less than z then either y equals z or y less than z").

Most readers will, at some time or other, have had a secret curiosity about logic, and these symbols or minor variations on them will be mediumly familiar.

The above examples should give a fairly clear idea of what we shall take as the formulas in our language. Incidentally, we shall for the most part be dealing with formulas in which every variable is controlled or acted on by some quantifier. Such formulas we call *closed formulas.*

Several further technical terms will prove useful in discussing formulas. If we take any formula, call it A, and put a "$\sim$" in front of that formula, we then call this larger formula the *negation* of A. Similarly, the *conjunction* of two formulas A and B is the larger formula obtained by putting "&" between formula A and formula B; and the *disjunction* of A and B is obtained by putting "$\vee$" between A and B. In talking *about* formulas, we may sometimes indicate the conjunction of A and B by the marks "A & B". Of course, the marks "A" and "B" are not formulas; they are used, from time to time in our discussion, as names for formulas. We agree to use the marks "A & B" as a name for the corresponding conjunction. We do similarly with other symbols of our language.

3. Realizations. So much for our language. Let us now turn and consider the mathematical objects to which this language might refer. To put it another way, let us consider ways of giving *meaning* to formulas of the language. We proceed here, as we did in setting up the language, on a common sense level. We avoid precise definitions; and attempt to suggest the ideas through instructive examples.

First of all, we agree that henceforth "$\leftrightarrow$", "$\rightarrow$", "&", "$\vee$", "$\sim$", "$\exists$", "$\forall$" and "$=$" shall have the meanings suggested by the English words used for them. (In particular, "$x=y$" will mean "x is one and the same object as y". "$\vee$" will mean "or" in the inclusive sense of "and/or".) What remains then is for us to choose: (a) the basic objects to which the variables shall refer; and (b) the meaning that "$<$" shall have for those basic objects.

For a first illustration, let us take the *real numbers* as the basic objects and let us interpret "$<$" as meaning "less than" in the usual ordering of the reals. (To put it another way, we will take the relation over the reals given by this usual ordering, and associate this relation with "$<$".) Now, at our common sense level, we can see how any closed formula will be either true or false of this particular mathematical system: the real numbers together with their usual ordering. For instance, formula (1) will be true. Formula (2), which asserts, in effect, the existence of an *immediate successor* to any element, will be false.

For a second illustration, take the positive integers together with the "less than" relation of their natural ordering. Here, in this *new* interpretation of the formulas of our language, we find that the formula (1) is false, whereas formula (2) becomes true. Here is a third formula:

$$(\forall x)(\exists y)[x < y]. \tag{3}$$

Formula (3) is true in both the above interpretations.

Let us introduce a somewhat more technical word. We call an interpretation, such as either of the two interpretations mentioned above, a *realization.* More

generally, for our given language, a *realization* is obtained when we take some *set* s together with some *relation* r, where r is any subset of $s \times s$, the cartesian product of s with itself. If we take s and r together and call them $\mathcal{R}$, then $\mathcal{R}$ is a realization. Given any closed formula A and any realization $\mathcal{R}$, we can see what it means to say that A is *true* of $\mathcal{R}$, —or, to use an equivalent technical word, what it means to say that the closed formula A is *satisfied* by the realization $\mathcal{R}$.

Now any given realization is a precisely determined mathematical object. Similarly, the formulas of our language, as strings of symbols, are precisely definable, finite, combinatorial mathematical entities. Therefore, this relationship of a closed formula being *satisfied* by a realization is something which can be given a precise mathematical definition. We do not take space to do this, other than to observe that it can be done. Rather, we content ourselves with the common sense indication of this relationship given above. Let us make several brief digressive comments before going on.

First comment. The mathematical reader will observe that we now have a situation that could be described as follows. We have a space $\mathcal{X}$ of all closed formulas of our language, and a space $\mathcal{Y}$ of all possible realizations. Furthermore, we have a general relation (a many-many relation) between these two spaces, given by the relation of *satisfaction.* This relation has a structure which is itself related in a natural way to the structure of formulas as they are built up out of smaller formulas. The mathematical reader will recognize this as a situation of a kind that is fraught with mathematical possibilities. We can take a particular realization and ask for the collection of all formulas which it satisfies. We can take a particular formula and ask for the collection of all realizations satisfying it. We can proceed to ask a variety of interesting structural questions. As a matter of fact, such investigations turn out to be fruitful and rewarding. Below, however, we shall be treating a more specific and limited objective.

A second brief comment. The language we are considering is limited in what it can express about any given realization. For instance, it cannot talk about sets of basic objects. If we are using the real number realization, it cannot say "for all *sets* of real numbers, . . . ". Such a language, where we have a single domain of objects to which the quantifiers can refer, and where the quantifiers can refer to nothing else, is called an *elementary* or *first-order* language, or *lower predicate calculus* language.

A third brief comment on terminology. When a realization satisfies a particular formula, we sometimes say that the realization is a *model* for the particular formula.

4. The laws of logic. At this point we have our language, and we have discussed in rather general terms possible mathematical objects, i.e., realizations, to which it might refer. Next, before going on to our specific problem, we consider what we shall call the *laws of logic.* Logicians and mathematicians, in dealing with formulas in a language like ours from, if you like, a common sense point

of view, soon find that there are certain cases where two formulas appear to be completely interchangeable. They appear to say exactly the same thing. They are interchangeable, not only one for the other, but they are interchangeable as parts of larger formulas. Such pairings of interchangeable formulas we call *equivalences*, and we include them among what we shall call the *laws of logic.* Here are a few such equivalences.

For any formulas A, B, and C,

(E_1)	$A \leftrightarrow B$	is equivalent to	$[A \rightarrow B]\ \&\ [B \rightarrow A]$
(E_2)	$A \rightarrow B$	is equivalent to	$\sim A \vee B$
(E_3)	$\sim[A\ \&\ B]$	is equivalent to	$\sim A \vee \sim B$
(E_4)	$\sim[A \vee B]$	is equivalent to	$\sim A\ \&\ \sim B$
(E_5)	$\sim\sim A$	is equivalent to	A
(E_6)	$A\ \&\ B$	is equivalent to	$B\ \&\ A$
(E_7)	$A\ \&\ [B \vee C]$	is equivalent to	$[A\ \&\ B] \vee [A\ \&\ C]$
(E_8)	$[A\ \&\ B]\ \&\ C$	is equivalent to	$A\ \&\ [B\ \&\ C]$
(E_9)	$[A \vee B] \vee C$	is equivalent to	$A \vee [B \vee C]$
(E_{10})	$(\forall x)A$	is equivalent to	$\sim(\exists x) \sim A$
(E_{11})	$(\exists x)[A \vee B]$	is equivalent to	$(\exists x)A \vee (\exists x)B.$

Let A be any formula and D be any formula in which the variable "x" does not occur, then

(E_{12})	$(\exists x)D$	is equivalent to	D
(E_{13})	$(\exists x)[A\ \&\ D]$	is equivalent to	$(\exists x)A\ \&\ D.$

Let $A(x)$ be any formula containing the variable "x" but no quantifiers on "x" or "y", then

(E_{14}) $\quad (\exists x)[x = y\ \&\ A(x)]$ is equivalent to $A(y)$.

In $E_{10}-E_{14}$ we have used the variables "x" and "y". Clearly, these equivalences hold with other variables.

Below, we shall be almost entirely concerned with equivalences; it is customary, however, to include other rules as laws of logic, rules that describe cases where mathematicians and logicians go *from* formulas which they have accepted as true in certain circumstances *to* a new formula which they then accept as true in those circumstances. We call these rules *implications*, and say that the former formulas *imply* the new one. Here are two examples of implications.

For any formulas A and B:

$$(\exists x)(\forall y)A \quad \text{implies} \quad (\forall y)(\exists x)A;$$

$$A \quad \text{and} \quad A \rightarrow B \quad \text{imply} \quad B.$$

At this point the reader may well complain, "You have been rather vague about what are to be laws of logic. Is there any way of more precisely specifying what equivalences and implications are to be included?" This can be answered in two quite different ways. One rather natural way that suggests itself—once we have defined our language and once we have defined the notion of satisfaction—is simply to say that two formulas are equivalent if they are satisfied by exactly the same realizations, and that one or more formulas imply another if the latter formula is satisfied by every realization that satisfies the former. (For these purposes, satisfaction can be made to apply to nonclosed as well as closed formulas. We omit details.) Another way of doing it, a quite different way and one much closer to the historical development of the subject, is to observe logicians and mathematicians at work, empirically to observe the equivalences and implications they use, to make a brief empirical catalogue of the simplest cases, and to take as laws of logic all those equivalences and implications which can be obtained by repeated use of equivalences and implications in the catalogue. It turns out that with a rather simple (in fact finite) catalogue, these two different ways lead us to exactly the same equivalences and implications. This is not too surprising, because, as a matter of historical fact, the precise definition of satisfaction was arranged as far as was possible to make things turn out this way. The fact that we get exactly the same laws of logic either way is sometimes called the *completeness of elementary logic*. Note that, while the second way refers only to the formal symbolism of the language, the first way refers to both language and realizations.

5. Our problem. So much for the laws of logic. Now, finally, we turn to our particular problem. We have our language. Let us choose a specific realization: the real numbers together with the usual "less than" ordering relation on the real numbers. Let us define $\mathfrak{T}$ to be the set of all those closed formulas of the language which are satisfied by this particular realization. (We might call $\mathfrak{T}$ the *elementary theory of the ordering of the reals.*) And let us ask this question: is there an effective procedure, i.e., an algorithm, which we can apply to the symbols of any formula and by which we can decide whether or not that formula is in $\mathfrak{T}$? This is a classic sort of question to ask. It takes us back to the 17th century German philosopher and mathematician, Leibnitz. Leibnitz was one of the first scientists to ask questions of this kind. It was Leibnitz's dream that man should discover: (a) a precisely definable universal symbolism for making statements of science (Leibnitz called this a *characteristica universalis*); and (b) an algorithm which, when applied to the symbols of any formula of the characteristica universalis, would determine whether or not that formula were true as a statement of science, Leibnitz called this algorithm a *calculus ratiocinator.*

In our little problem, we have a very limited language and are far from a characteristica universalis. However, we *are* going to find a calculus ratiocinator for this language under the chosen realization. To use a modern term, we will find an *effective decision procedure* for our language and realization.

6. The solution. We begin, in good mathematical fashion, by making the problem apparently a little bit harder. We add to our language two new symbols, "t" and "f", which we take as additional atomic formulas. As far as meaning (i.e., satisfaction) goes, we treat "t" as an atomic formula which is always true and "f" as an atomic formula which is always false. We can now list several further, rather natural, laws of logic:

$$(\mathrm{E}_{15}) \qquad \begin{cases} A \vee t & \text{is equivalent to} \quad t \\ A \vee f & \text{is equivalent to} \quad A \\ A \;\&\; t & \text{is equivalent to} \quad A \\ A \;\&\; f & \text{is equivalent to} \quad f \\ \sim t & \text{is equivalent to} \quad f \\ \sim f & \text{is equivalent to} \quad t. \end{cases}$$

Now we are ready for the algorithm. Assume we are given some closed formula. Our general procedure will be to operate on our formula in steps by taking a particular subformula (possibly the whole formula) and replacing it by some equivalent formula. This means that the successive large formulas that we get are equivalent; so that if the original formula is true or false of the real numbers, then any formula we subsequently get from it will be, respectively, true or false of the real numbers.

First step. Using the logical equivalences E_1 and E_2, we can eliminate all occurences of "$\leftrightarrow$" and "$\rightarrow$".

Second step. Using E_{10}, we can eliminate all occurrences of "$\forall$" by putting "$\sim(\exists x)\sim$" in place of "$(\forall x)$", (and similarly for other variables than "x").

Third step. Let us take an *inside* quantifier in our formula. An inside quantifier is a quantifier which acts on a subformula that itself contains no quantifier. Let us choose a particular inside quantifier and the formula that it acts on and let us call this

$$(\exists x)B,$$

if "x" happens to be the variable in the quantifier. Now B must be made up of atoms, conjunctions, negations, disjunctions. We can indicate $(\exists x)B$ as

$$(\exists x)[\text{atoms}, \&, \sim, \vee].$$

We look through B and locate any occurrence of an atomic formula that consists of "$=$" with the same variable on each side; e.g., "$x=x$" or "$y=y$". Any such occurrence we replace by "t". This is clearly a law of logic. Then we look for occurrences of "$<$" with the same variable on each side. Any such occurrence we replace by "f". Here for the first time we make a replacement which is not a law of logic but which is, rather, related to the particular realization we have in mind. There are going to be some more of these. We list these as *special*

equivalences.

(SE$_1$) $\qquad x < x \quad$ is equivalent to $\quad f.$

We now have a string of symbols involving atomic formulas each of which is either "t" or "f" or has distinct variables on the two sides of "$=$" or "$<$".

Fourth step. We distribute any "$\sim$" through the various conjunctions and disjunctions to which it may apply. This is possible by E_3 and E_4. Furthermore, if "$\sim$" symbols begin to pile up, we can eliminate them in pairs by E_5. Finally, when we get the "$\sim$" symbols distributed all the way through and pairs of them eliminated, we may be left with some single "$\sim$" symbols applied to atoms. Any part of the form

$$\sim[x = y]$$

we replace by

$$[x < y \vee y < x],$$

and any part of the form

$$\sim[x < y]$$

we replace by

$$[x = y \vee y < x],$$

(and similarly for other variables in place of "x" and "y"). These two substitutions are again special equivalences that depend for their validity on our chosen realization:

(SE$_2$) $\qquad \sim[x = y] \quad$ is equivalent to $\quad [x < y \vee y < x],$

(SE$_3$) $\qquad \sim[x < y] \quad$ is equivalent to $\quad [x = y \vee y < x].$

Any part of the form "$\sim t$" we replace by "f", and of the form "$\sim f$" we replace by "t", according to E_{15}. At this point, we have our inside quantifier applied to a formula which consists only of atoms, conjunctions and disjunctions. We indicate this as

$$(\exists x)[\text{atoms}, \&, \vee].$$

Fifth step. We operate further by distributing conjunctions through disjunctions, using equivalences E_6 and E_7. Doing this as far as we can, we reach, as the reader will see, a point where we have

$$(\exists x)[[\text{atoms}, \&] \vee [\text{atoms}, \&] \vee \cdots \vee [\text{atoms}, \&]].$$

(By E_8 and E_9, it is immaterial how we group the terms in a repeated conjunction or a repeated disjunction, so we do not indicate any grouping by brackets.) Now we take the inside quantifier, and using E_{11}, we distribute it through this disjunction. So we now have

$$(\exists x)[\text{atoms}, \&] \vee (\exists x)[\text{atoms}, \&] \vee \cdots \vee (\exists x)[\text{atoms}, \&].$$

Sixth step. We focus on a particular one of these parts of the form

$$(\exists x)[\text{atoms}, \&].$$

A first possibility is that "x" does not occur at all in any of the atoms in the conjunction. In this case, we can simply eliminate the quantifier by E_{12}. The other possibility is that "x" occurs in some of the atomic formulas in this conjunction. In this case, using E_6, we rearrange the atomic formulas so that all those atomic formulas involving "x" come last in the conjunction, and than using E_{13}, we distribute the quantifier through until it applies only to those atoms which do involve the variable "x".

Seventh step. We now work on this subformula of the form

$$(\exists x)[\text{atoms involving } x, \&].$$

(Incidentally, as the reader will begin to see, our objective here is to carry out a sequence of transformations by which we can totally eliminate all occurrences of our original inside quantifier. Then we will go back and successively eliminate the various other quantifiers, as they become inside quantifiers in due course.) There are several possible cases:

Case 1. We just have a formula of the form

$$(\exists x)[x = y].$$

This clearly gets replaced by "t", an obvious law of logic.

Case 2. We have,

$$(\exists x)[x = y \,\&\, A(x)].$$

By E_{14}, this gets replaced by $A(y)$. Again we have got rid of the quantifier.

Case 3. The only remaining possibility is that all the atoms are made up of "$<$". Here we consider three different subcases:

Subcase 3a. We have

$$(\exists x)[x < y_1 \,\&\, x < y_2 \,\&\, \cdots \,\&\, x < y_n].$$

(Here, as elsewhere, we have chosen specific variables to illustrate, but any other variables may be used, subject to the obvious restrictions. Here, the obvious restriction is that the variables in place of "y_1", $\cdots$, "y_n" must all be distinct from the variable in place of "x". This restriction will always hold as a result of the third step.) This formula can clearly be replaced by "t". This is again an equivalence that depends upon our particular realization. It simply says that given any collection of real numbers "y_1", $\cdots$, "y_n", there is a real smaller than all of them.

$$(\mathrm{SE}_4) \qquad (\exists x)[x < y_1 \,\&\, \cdots \,\&\, x < y_n] \quad \text{is equivalent to} \quad t.$$

Subcase 3b. This is similar. If we have

$$(\exists x)[z_1 < x \,\&\, \cdots \,\&\, z_m < x],$$

this clearly can be replaced by "t". We have

(SE$_5$) $(\exists x)[z_1 < x \,\&\, \cdots \,\&\, z_m < x]$ is equivalent to t.

Subcase 3c. The only remaining possibility is that we have a mixture of atoms, some with "x" before "$<$" and some with "x" after "$<$". Thus, using E$_6$, we have something of the form

$$(\exists x)[z_1 < x \,\&\, \cdots \,\&\, z_m < x \,\&\, x < y_1 \,\&\, \cdots \,\&\, x < y_n].$$

For this we need another equivalence depending on the chosen realization in order to get rid of our quantifier. With a little thought, the reader will see that the following suffices.

(SE$_6$) $(\exists x)[z_1 < x \,\&\, \cdots \,\&\, x < y_n]$ is equivalent to

$$[z_1 < y_1 \,\&\, z_1 < y_2 \,\&\, \cdots \,\&\, z_1 < y_n \,\&\, z_2 < y_1 \,\&\, \cdots \,\&\, z_2 < y_n \,\&\, \cdots \,\&\, z_m < y_1 \,\&\, \cdots \,\&\, z_m < y_n].$$

We could indicate this last formula more briefly as "$\underset{\substack{1 \leqq i \leqq m, \\ 1 \leqq j \leqq n}}{\&} [z_i < y_j]$".

This concludes the seventh step.

By repeating the sixth and seventh steps for each part of the disjunction that resulted from the fifth step, we can get rid of the original inside quantifier. Then by repeating the whole procedure, from the third step, for other remaining inside quantifiers, we can work our way through until we get rid of all quantifiers. The reader will notice that at each stage the formula remains a closed formula. This means that when we finally get rid of all the quantifiers, the only thing that can be left is some combination of "t" and "f" under "$\sim$", "&", and "$\vee$".

Final step. By laws E$_{15}$ on "t" and "f", what remains can be transformed until we reach either "t" or "f" alone. If we reach "t", the original formula must be true of the reals; if we reach "f", the formula must be false.

We thus have our decision procedure. We have found a calculus ratiocinator for our limited language and chosen realization.

7. Illustrations. So far we have imposed on the reader's attention and powers of concentration by describing this procedure without at the same time giving an example. We now look at several. First of all, let us take the formula

$$(\exists x)(\forall y)[y < x].$$

Applying the procedure, and noting the steps used on the left, we have

(1) no change

(2) $(\exists x) \sim (\exists y) \sim [y < x]$

(3) $\cdots (\exists y) \sim [y < x]$ (no change)

(4)	$\cdots(\exists y)[y = x \vee x < y]$
(5)	$\cdots[(\exists y)[y = x] \vee (\exists y)[x < y]]$
(6)	no change
(7)	$\cdots[t \vee t]$
(3)	$(\exists x) \sim [t \vee t]$ (no change)
(4)	$(\exists x)[f \& f]$
(5)	no change
(6)	$f \& f$
(7)	no change
(8)	f.

Thus our original formula is false, and this checks with our immediate intuitive recognition of it as a false statement about the real numbers.

Next, let us take a formula that we would expect to be true of the reals, but let us be a little careless about it. We know that given any two distinct reals, there is a third one between them. So, let us take the formula

$$(\forall x)(\forall y)[\sim x = y \rightarrow (\exists z)[x < z \& z < y]].$$

Let us apply our algorithm and see what happens. We omit steps that cause no change.

(1)	$(\forall x)(\forall y)[\sim\sim x = y \vee (\exists z)[x < z \& z < y]]$
(2)	$\sim(\exists x)\sim\sim(\exists y)\sim[\sim\sim x = y \vee (\exists z)[x < z \& z < y]]$
(7)	$\sim(\exists x)\sim\sim(\exists y)\sim[\sim\sim x = y \vee x < y]$
(4)	$\sim(\exists x)\sim\sim(\exists y)[\sim x = y \& \sim x < y]$
(4 cont'd)	$\sim(\exists x)\sim\sim(\exists y)[[x < y \vee y < x] \& [x = y \vee y < x]]$
(5)	$\sim(\exists x)\sim\sim(\exists y)[[x = y \& x < y] \vee [x = y \& y < x]$ $\vee [y < x \& x < y] \vee [y < x \& y < x]]$
(5 cont'd)	$\sim(\exists x)\sim\sim[(\exists y)[x = y \& x < y] \vee (\exists y)[x = y \& y < x]$ $\vee (\exists y)[y < x \& x < y] \vee (\exists y)[y < x \& y < x]]$
(7)	$\sim(\exists x)\sim\sim[x < x \vee x < x \vee x < x \vee t]$
(3)	$\sim(\exists x)\sim\sim[f \vee f \vee f \vee t]$
(4)	$\sim(\exists x)[f \vee f \vee f \vee t]$
(5)	$\sim[(\exists x)f \vee (\exists x)f \vee (\exists x)f \vee (\exists x)t]$
(6)	$\sim[f \vee f \vee f \vee t]$
(8)	$\sim t$
(8 cont'd)	f.

So this formula turns out to be false. This really should not be too surprising to us, because we can go back and see that we were in fact careless. The formula that we should have used was the formula

$$(\forall x)(\forall y)[x < y \rightarrow (\exists z)[x < z \,\&\, z < y]].$$

The reader can check that the algorithm carries this latter formula to "t".

Incidentally, the reader will probably note from both the above examples that the algorithm could be made slightly more efficient by allowing more freedom in the order in which equivalences E_1–E_{16} and SE_1–SE_6 are used.

Here is a third illustration that we leave as an exercise for the reader. Consider any closed formula of the following kind,

$$[(\exists x)A \,\&\, (\exists y)B \,\&\, (\forall x)(\forall y)[[A \,\&\, B] \rightarrow x < y]] \\ \rightarrow (\exists z)[(\forall x)[A \rightarrow \sim z < x] \,\&\, (\forall y)[B \rightarrow \sim y < z]],$$

where A contains "x" but not "y", and B contains "y" but not "x". A and B may contain other variables, but they must be acted on by quantifiers. Clearly we cannot apply the algorithm to this formula directly unless we know what the formulas A and B are. However, by a general argument, which we leave to the reader (it is not difficult), one can show that whatever A and B happen to be, the algorithm must always lead to "t". Hence such a formula must always be true of the reals. This is as it should be; because we see, after a moment's thought, that such a formula expresses an instance of the least upper bound principle for the real numbers, and we know, as a mathematical fact, that any such instance must be true.

8. Applications. What are some of the other conclusions that we can draw from the above? In a sense, much of the modern history of logic can be read from this example. There are many ideas here, and we will not take space even to mention some of them. We look at a few.

First of all, we consider the particular equivalences $SE_1, \cdots, SE_6$. These were not laws of logic, but were, instead, related to our chosen realization. Let us ask the question: for what realizations does this set of equivalences hold? It is not difficult to show (another exercise for the reader) that the realizations can be characterized as those which satisfy all of the following six formulas:

(A_1) $\quad (\forall x) \sim [x < x]$

(A_2) $\quad (\forall x)(\forall y)(\quad)z[[x < y \,\&\, y < z] \rightarrow x < z]$

(A_3) $\quad (\forall x)(\forall y)[x < y \vee x = y \vee y < x]$

(A_4) $\quad (\forall x)(\exists y)[y < x]$

(A_5) $\quad (\forall x)(\exists y)[x < y]$

(A_6) $\quad (\forall x)(\forall y)[x < y \rightarrow (\exists z)[x < z \,\&\, z < y]].$

This means that if we had chosen any other realization satisfying $A_1, \cdots, A_6$,

exactly the same decision procedure would work; exactly the same formulas would come out to be true, and exactly the same formulas would come out to be false. But there are other realizations satisfying $A_1, \cdots, A_6$. The rational numbers, with their usual ordering, are one such. Thus, exactly the same formulas come out true for the rationals as for the reals. In fact any *dense linear ordering without endpoints* will have this same decision procedure. This may seem a little paradoxical, since we saw that our decision procedure made the least upper bound principle come out to be true. The answer to this apparent paradox is as follows: in our formulation of the least upper bound principle, we of course restrict A and B to be formulas of our rather limited language; it then happens that the resulting class of instances of the least upper bound principle holds for the rationals as well as for the reals. To put this another way, $A_1, \cdots, A_6$ characterize those realizations which are indistinguishable from the reals by the formulas of our limited language.

It is occasionally convenient to express equivalences as single formulas. This is possible since, as can be easily shown, a necessary and sufficient condition for A and B to be equivalent with respect to a realization is that the formula $A \leftrightarrow B$ be true for that realization. We would then say that the formula $A \leftrightarrow B$ *expressed* the equivalence. (We continue to omit details concerning satisfaction and truth when formulas are not closed.) Thus the set of equivalences $SE_1, \cdots, SE_6$ can be replaced by a corresponding set of single formulas, (and "t" and "f" can be eliminated from these formulas in a way that we do not describe). It is not difficult to show that each of these single formulas is *logically derivable* from $A_1, \cdots, A_6$; where *logically derivable* means: obtainable by iterated use of implications and equivalences occurring in the simple catalogue mentioned in Section 4. Furthermore, by considering, in reverse order, the substitutions made in any application of the decision procedure, it is not difficult to show that a formula carried to "t" by the decision procedure is logically derivable from the single formulas expressing $SE_1, \cdots, SE_6$. It follows that any formula carried to "t" is logically derivable from $A_1, \cdots, A_6$. Conversely, each of $A_1, \cdots, A_6$ is satisfied by our chosen realization and so, as described in Section 4, must be any formula logically derivable from $A_1, \cdots, A_6$. Hence any formula derivable from $A_1, \cdots, A_6$ is carried to "t" by our decision procedure. Therefore, the set of formulas true of the usual ordering of the reals coincides with the set of formulas derivable from $A_1, \cdots, A_6$. We might refer to formulas $A_1, \cdots, A_6$ as *axioms*. We hence can draw the conclusion that the elementary theory of the ordering of the reals is *finitely axiomatizable*; its true formulas are the formulas derivable from a finite list of axioms.

Sometimes (usually, in fact), a theory is initially given in terms of axioms rather than in terms of realizations. Thus we might have begun our investigation by defining $\mathfrak{J}^*$ to be the set of closed formulas logically derivable from $A_1, \cdots, A_6$. We would then have faced a number of interesting and nontrivial problems. One problem would be to ascertain whether or not $\mathfrak{J}^*$ is *formally complete*; where $\mathfrak{J}^*$ *formally complete* means that for every closed formula A,

either A or $\sim A$ is in $\mathfrak{I}^*$. We can solve this as follows. From the preceding paragraph, we know that $\mathfrak{I}^*$ coincides with $\mathfrak{I}$. $\mathfrak{I}$ is formally complete, since the decision procedure, for any closed formula A, must take either A or $\sim A$ into "t". Hence $\mathfrak{I}^*$ is formally complete and our decision procedure has led us to the proof.

When a theory is given in terms of axioms, another natural problem is that of *consistency*. A set of axioms is said to be *consistent* if there is no closed formula A such that both A and $\sim A$ are derivable. The mathematician's usual way of showing the consistency of a set of axioms is to give a satisfying realization and to observe that all derivable formulas must also be satisfied by that realization. Since a closed formula is satisfied if and only if its negation is not, there cannot be any closed formula A such that both A and $\sim A$ are derivable. We have consistency for $A_1, \cdots, A_6$ since the real number ordering gives a satisfying realization.

An important traditional question in logic, however, is to try to show that axioms are consistent by very elementary combinatorial arguments, arguments that do not involve us with infinite and rather abstract set theoretical objects like the set of real numbers or the set of rational numbers. Our decision procedure can be used to give us such a basic, or to use the traditional logician's word, *finitistic* consistency proof. We first observe that when the procedure is applied to any one of our axioms, that axiom is carried into "t". Then specifying the simple catalogue of Section 4 more carefully than we have done, we show, as a combinatorial fact, that the property of being carried into "t" by our decision procedure is preserved under application of the equivalences and implications of the catalogue. In this way we can conclude that the only formulas in $\mathfrak{I}^*$ are formulas carried into "t" by our decision procedure. Since for any closed A it is not possible for both A and $\sim A$ to be carried into "t", we have our finitistic consistency proof. It is a proof that has made no reference to realizations.

Returning to the proof of formal completeness for $\mathfrak{I}^*$, the reader will note, incidentally, that the argument just used in the finitistic consistency proof can also be employed to make the proof of formal completeness entirely finitistic; for it shows, without reference to realizations, that formulas derivable from $A_1, \cdots, A_6$ are carried to "t" by the decision procedure. As a corollary, we have, of course, a finitistic proof that our decision procedure is a decision procedure for $\mathfrak{I}^*$.

9. Another solution. In the last section, we saw that our decision procedure for $\mathfrak{I}$ is also a decision procedure for $\mathfrak{I}^*$, and we noted that it gave us a finitistic proof, indeed, of the existence of a decision procedure for $\mathfrak{I}^*$. In this section we shall give a shorter, nonfinitistic proof of this latter fact. The proof will not depend on the decision procedure previously given, but will lead us to a different decision procedure that is considerably less efficient than that of Section 6. It will illustrate both a strength (in brevity) and a weakness (in efficiency) of nonfinitistic proofs in logic. The decision procedure for $\mathfrak{I}^*$ will also be a decision procedure for $\mathfrak{I}$ and hence an alternative solution to our original problem. This

section uses a few simple notions from set theory. The reader can omit it and proceed directly to Section 10 if he wishes.

We use two facts, one from set theory and one from logic, which we state without proof.

(a) Any two denumerably infinite realizations which satisfy $A_1, \cdots, A_6$ are isomorphic as linear orderings. (This is a theorem due to Cantor. Its proof can be found in most elementary texts on set theory. A realization $\mathfrak{R}=(s, r)$ is *infinite*, if the set s is infinite.)

(b) For any formula A, if A is satisfied by an infinite realization, it is satisfied by a denumerably infinite realization. (This is a standard logical result for first-order languages. It is called the Löwenheim Theorem.)

As a first step, we show that $\mathfrak{J}^*$ is formally complete. (We do not use our previous proof of formal completeness, since that depended on the previous decision procedure, of which we wish to remain independent.) Let B be the conjunction of $A_1, \cdots, A_6$. Then B is satisfied by the rationals under the usual ordering. Take any closed formula A. We wish to show that either A or $\sim A$ is in $\mathfrak{J}^*$. Either A or $\sim A$ is satisfied by the rationals, for if one is false of the rationals, the other must be true. Without loss of generality, we assume A is satisfied by the rationals. Then the formula $A \,\&\, B$ is satisfied by the rationals. If two realizations are isomorphic, they must satisfy the same formulas; hence, by (a), $A \,\&\, B$ is satisfied by all denumerably infinite realizations which satisfy B. Hence, by (b), $A \,\&\, B$ is satisfied by all infinite realizations which satisfy B. It is easily shown that B is satisfied by no finite realization. Hence $A \,\&\, B$ is satisfied by all realizations which satisfy B. A fortiori, A is satisfied by all realizations which satisfy B. By the completeness of elementary logic, described in Section 4, A must be logically derivable from B. In other words, A must be in $\mathfrak{J}^*$. Therefore, $\mathfrak{J}^*$ is formally complete.

As a second step, we show that $\mathfrak{J}^*$ is consistent. The first proof of consistency in Section 8 achieves this without reference to the decision procedure of Section 6.

As a third and final step, we observe (again omitting details) that we can effectively list, in some order, all permissible logical derivations from $A_1, \cdots, A_6$. We can do this by moving, in a process of exhaustion, through all possible derivations of each finite length n, as n increases. The length of a derivation may be taken as the total number of basic symbols occurring in that derivation.

To test whether or not A is in $\mathfrak{J}^*$, for any closed A, we start generating the list of all derivations and search for a derivation having either A or $\sim A$ as its conclusion. Eventually, by completeness of $\mathfrak{J}^*$, we know that one or the other must occur; and by consistency, we know that only one will occur. This is our decision procedure. It is obviously much less efficient than the preceding one.

That it is also a decision procedure for $\mathfrak{J}$ follows as a corollary by observing, from (a) and (b), that if a formula A is satisfied by the reals, it is satisfied by the rationals, and, similarly, if $\sim A$ is satisfied by the reals, $\sim A$ is satisfied by the rationals. Hence A is satisfied by the reals if and only if A is satisfied by the

rationals. From the preceding argument, $\mathfrak{I}^*$ consists of exactly those A which are satisfied by the rationals. Thus $\mathfrak{I}$ and $\mathfrak{I}^*$ must coincide (a fact that we proved in Section 8, but in whose previous proof we used the decision procedure of Section 6).

The approach just described is sometimes useful if we seek only to learn whether or not a first-order theory has a decision procedure. (Some do not, as we shall see in Section 10.) In many cases (all the cases in Section 10), an appropriate counterpart to (a) cannot be found and the approach cannot be used. The approach is due to Vaught, a student of Tarski, and to Łoś, a Polish mathematician.

10. Other problems. The decision procedure that we have described in Sections 4–8 is an example of what is generally called, for obvious reasons, an *elimination of quantifiers* decision procedure. It is worth noting that it has a family resemblance to decision algorithms that occur in other parts of mathematics. It has a similar flavor, for instance, to the well-known procedure in topology by which one can determine, given triangulations for two different 2-manifolds, whether or not the manifolds are homeomorphic.

What other logical decision procedures can be found? Can we carry Leibnitz's dream further and find precise languages and appropriate decision procedures for more general parts of mathematics? By 1910, it had been made clear, in works of Frege, Russell, Whitehead and others, that rather natural and fully precise languages could be obtained for virtually all of mathematics; the first part of Leibnitz's dream, as it applied to mathematics, was accomplished. The problem of decision procedures remained. In the 1920's, Hilbert and his associates, including John von Neumann, hoped to find a decision procedure for very general parts of mathematics. They also hoped to find a finitistic consistency proof for these more general parts of mathematics. We shall comment further on Hilbert's goals in a moment.

Here are several further problems as exercises. They are not too difficult, and the solutions are similar, in general outline, to ours in Section 6. *First*, find a decision procedure for the ordering of the integers. This is fairly easy, although the reader will find that in addition to "t" and "f", he will have to add basic expressions of the form "$x+7$", "$y-2$", etc. This means that he will have to add the numerals to the language and also "$+$" and "$-$", and then take any equation with on variable and one numeral on each side, e.g., "$x+7=y-14$", as an atomic formula. *Second*, find a decision procedure for the ordering of the ordinal numbers. This is a slightly harder problem. For a *third* problem, change the language and in place of "$<$" use the symbol "$+$", taking as atomic formulas any equations whose two sides are combinations of variables under "$+$", e.g., "$(x+y)+z=y+w$". Take the real numbers under ordinary addition as realization, and find a decision procedure. For a *fourth* problem, take this latter language and find a decision procedure with ordinary addition over the integers as realization. Here the reader will have to augment his language by introducing

expressions for congruences; but the problem is still not too hard. For a *fifth* problem, consider a language with both "$+$" and "$\times$" symbols, and take the real numbers under ordinary addition and multiplication as realization. A decision procedure for this was discovered by Tarski some years ago. The procedure uses Sturm's Theorem—a result that has to do with finding the number of real roots of a given polynomial lying in a given interval.

Still a further problem would be to take this last language and to try to find a decision procedure when the realization is the *integers* under ordinary addition and multiplication. Here we run into the famous theorem of Gödel which shows us that *no decision procedure can exist.* A variation on this problem would be to start with Peano's axioms in this language and ask for a decision procedure for the set of formulas logically derivable from them. Gödel also shows us that *such a procedure cannot exist if the axioms are consistent.* It is easy to show (nonfinitistically) that the axioms are consistent; it follows (nonfinitistically) that *such a procedure cannot exist.* Gödel further shows that the set of derivable formulas is formally incomplete (Gödel's "incompleteness theorem"), and he shows that no finitistic consistency proof for these axioms can exist. This remarkable constellation of results dashed the hopes of Hilbert and his associates, by showing their major goals to be impossible of achievement. Gödel published his results in 1931.

It is, of course, also possible in very simple cases to start with axioms that lead to an incomplete theory. For instance, if we just take the axioms A_1, A_2, and A_3, —the axioms for a linear ordering, —then the derivable formulas form an incomplete theory; e.g., neither (2) in Section 2 nor its negation is in the theory. We can ask for a decision procedure for this latter theory. The problem of finding one is difficult, and it is only recently that Ehrenfeucht, a Polish mathematician, made the first discovery of such a procedure.

In conclusion, several men should be mentioned. We have mentioned Leibnitz and Hilbert. We should also mention Boole; some of the more algebraic steps in our decision procedure are closely related to Boole's pioneering work. Herbrand should be mentioned; he explored some of the general structural features of certain elimination of quantifiers procedures. Tarski should be mentioned. He is a major founder of the modern approach in which one carefully distinguishes between the formulas of a language and the possible realizations about which the language may speak. In the last fifteen years, Tarski and his students have pursued deeper problems concerning the satisfaction relationship between formulas and realizations. Finally we mention C. H. Langford. The particular decision procedure that we have given is a slightly modified version of a procedure published by Langford in the *Annals of Mathematics* in 1927.

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POLAR ZONOHEDRA

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1. Introduction. Convex polyhedra whose faces are centrally symmetrical were named *zonohedra* by Fedorov ([5], pp. 256–266). In particular, those whose faces consist of $n(n-1)$ congruent rhombi were named *isozonohedra.* When $n=3$, such a polyhedron is a rhombohedron, including the cube as a special case. When $n=4$, the obvious instance is Kepler's rhombic dodecahedron (or rhombdodecahedron). When $n=5$, we have the isohedral rhombic icosahedron, which is derived from Kepler's triacontahedron ($n=6$) by making any one of the 6 zones collapse ([1], p. 143; [3], p. 29; [4], p. 147). Bilinski [2] noticed that, by making any one of the 5 zones of the rhombic icosahedron collapse, we can obtain a *second* rhombic dodecahedron, whose faces are the same shape as those of the triacontahedron.

Three of the five isozonohedra, namely the rhombohedron ($n=3$), first rhombic dodecahedron ($n=4$), and rhombic icosahedron ($n=5$), have an axis of n-gonal symmetry and thus belong to the family of *polar* zonohedra, which were first systematically discussed by Franklin ([6], p. 363).

The present paper answers the following question: What shape does the polar zonohedron approach when n tends to infinity?

2. Zonohedra. A zonohedron is a convex polyhedron whose faces are centrally symmetrical polygons, e.g., parallelograms. Every edge determines a *zone* of faces in which every two adjacent faces meet in an edge parallel to the given edge. Let n denote the number of zones, i.e., the number of sets of parallel edges. Let $\mathbf{e}_k$ $(k=1, 2, \cdots, n)$ denote either of the two opposite vectors that can be used to indicate the length and direction of such a set of edges. Clearly, there is a set of vectors $\pm\mathbf{e}_k$ for each zonohedron. Conversely, any n vectors $\mathbf{e}_k$, not all coplanar and no two collinear, determine a zonohedron consisting of the points whose position vectors are

$$\theta_1\mathbf{e}_1 + \theta_2\mathbf{e}_2 + \cdots + \theta_n\mathbf{e}_n,$$

where each θ_k varies continuously from 0 to 1. Since the range of θ_k could equally well have been chosen from $-\frac{1}{2}$ to $\frac{1}{2}$, the sign of $\mathbf{e}_k$ can be reversed without altering the shape of the zonohedron. (There is a misprint, "θ_k and $-\theta_k$" for "$\mathbf{e}_k$ and $-\mathbf{e}_k$," near the end of Section 3 in [4], p. 141.)

If no three of the n vectors $\mathbf{e}_k$ are coplanar, all the faces are parallelograms, each zone consists of $n-1$ pairs of opposite faces, there are altogether $\binom{n}{2}$ pairs of opposite faces, $n(n-1)$ pairs of opposite edges, and therefore (by Euler's formula) $\binom{n}{2}+1$ pairs of opposite vertices.

If the n vectors are disposed like the edges of an ordinary solid angle (with all its dihedral angles less than π), we may name them $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ in cyclic order, and then the position vectors of the $1+(n-1)n+1$ vertices are just the

sums of sets of *consecutive* e's, namely

$$0;\ \mathrm{e}_1,\ \mathrm{e}_2,\ \cdots,\ \mathrm{e}_n;\ \mathrm{e}_{12},\ \mathrm{e}_{23},\ \cdots,\ \mathrm{e}_{n1};\ \mathrm{e}_{123},\ \mathrm{e}_{234},\ \cdots,\ \mathrm{e}_{n12};\ \cdots;\ \mathrm{e}_{12\cdots n},$$

where $\mathrm{e}_{j\cdots k}$ is a convenient abbreviation for $\mathrm{e}_j+\cdots+\mathrm{e}_k$. In this case every zone passes through the two special vertices 0 and $\mathrm{e}_{12\cdots n}$; for instance, the zone determined by the edge $0\mathrm{e}_n$ consists of the $2(n-1)$ faces

$$0\mathrm{e}_n\mathrm{e}_{n1}\mathrm{e}_1,\ \mathrm{e}_1\mathrm{e}_{n1}\mathrm{e}_{n12}\mathrm{e}_{12},\ \mathrm{e}_{12}\mathrm{e}_{n12}\mathrm{e}_{n123}\mathrm{e}_{123},\ \cdots,\ \mathrm{e}_{12\cdots n-2}\mathrm{e}_{n12\cdots n-2}\mathrm{e}_{n12\cdots n-1}\mathrm{e}_{12\cdots n-1},$$

$$\mathrm{e}_{12\cdots n-1}\mathrm{e}_{12\cdots n}\mathrm{e}_{2\cdots n}\mathrm{e}_{2\cdots n-1},\ \mathrm{e}_{23\cdots n-1}\mathrm{e}_{23\cdots n}\mathrm{e}_{3\cdots n}\mathrm{e}_{3\cdots n-1},\ \cdots,\ \mathrm{e}_{n-1}\mathrm{e}_{n-1n}\mathrm{e}_n0.$$

In particular, any three independent vectors e_1, e_2, e_3 form a parallelepiped $0\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3\mathrm{e}_{23}\mathrm{e}_{31}\mathrm{e}_{12}\mathrm{e}_{123}$ (Figure 1), in which the edge $0\mathrm{e}_3$ determines the zone

$$0\mathrm{e}_3\mathrm{e}_{31}\mathrm{e}_1,\quad \mathrm{e}_1\mathrm{e}_{31}\mathrm{e}_{312}\mathrm{e}_{12},\quad \mathrm{e}_{12}\mathrm{e}_{123}\mathrm{e}_{23}\mathrm{e}_2,\quad \mathrm{e}_2\mathrm{e}_{23}\mathrm{e}_30.$$

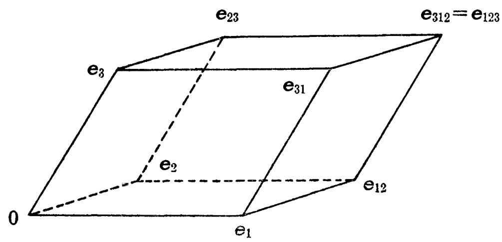

FIG. 1

3. Polar zonohedra. A *polar* zonohedron is the special case in which the n edges at the special vertex 0 are of equal length l and have the directions of n evenly spaced generators of a cone of revolution, so that the faces round 0 consist of n equal rhombi. The free edges of these faces belong to another cycle of n equal rhombi, and this construction continues till we reach the opposite vertex $\mathrm{e}_{12\cdots n}$, which again belongs to n faces.

The n rhombi at 0 may have any angle less than $2\pi/n$. In the extreme case when we make the angle exactly $2\pi/n$, we have a plane figure consisting of cycles of rhombi of angle $2k\pi/n$ $(1\leq k<n/2)$ fitting together to fill a regular polygon $\{n\}$ (n even) or $\{2n\}$ (n odd). The case when $n=7$ is illustrated in Figure 2, which may be interpreted *either* as a $\{14\}$ filled with 21 rhombi of various shapes *or* as the view of a solid zonohedron looked at from far away in the direction of the axis of heptagonal symmetry (which joins the special vertices 0 and $\mathrm{e}_{1234567}$).

The $n(n-1)$ faces of the solid zonohedron fall into $n-1$ cycles of n, those in the kth cycle being rhombi of angle

$$2\alpha_k = \angle\ \mathrm{e}_10\mathrm{e}_{k+1} = \angle\ \mathrm{e}_n0\mathrm{e}_k \quad (k = 1, 2, \cdots, n-1).$$

This angle can be computed, in terms of

$$\gamma = \angle\ \mathrm{e}_{1\cdots n}0\mathrm{e}_1,$$

by considering the right-angled spherical triangle ABC (on the unit sphere round $\mathbf{0}$) whose vertices lie in the directions of the vectors $\mathbf{e}_{1\cdots n}$, $\mathbf{e}_n$, $\mathbf{e}_{nk}$. Clearly, the angle at A is $k\pi/n$, the opposite side is α_k, and the hypotenuse is γ. By one of the classical formulae of spherical trigonometry, $\sin \alpha_k = \sin \gamma \sin (k\pi/n)$. The "extreme case" (when $\alpha_1 = \pi/n$) is derived by setting $\gamma = \pi/2$.

When $\sin \gamma = \sqrt{\frac{2}{3}}$, so that $\cos 2\gamma = -\frac{1}{3}$, the zonohedron is the "shadow" of an n-dimensional cube, projected orthogonally onto a suitable 3-space ([3], p. 256; [6]).

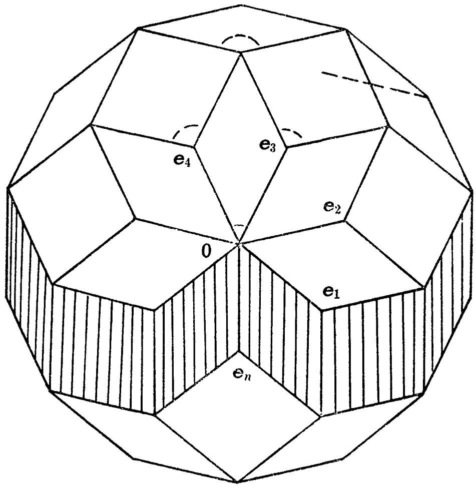

FIG. 2

4. The helices. In Figure 2, the vector $\mathbf{e}_n$ $(n=7)$ goes vertically down from the middle. The front half of the corresponding vertical zone has been shaded. The dark edges at the top of this zone are in the directions of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ (and $\mathbf{e}_5$, marked as a broken line because it is on the far side of the solid). When the figure is regarded as lying in a plane, these dark edges, being sides of a regular n-gon, are a sequence of equal chords of a circle. In the solid, the same edges, being all equally inclined to the "line of sight" $\mathbf{0e}_{1\cdots n}$, are a sequence of equal chords of a circular *helix*.

The entire zone is bounded by two congruent skew $2(n-1)$-gons

$$\mathbf{0e}_1\mathbf{e}_{12}\mathbf{e}_{123} \cdots \mathbf{e}_{12\cdots n-1}\mathbf{e}_{2\cdots n-1}\mathbf{e}_{3\cdots n-1} \cdots \mathbf{e}_{n-1}\mathbf{0}$$

and

$$\mathbf{e}_n\mathbf{e}_{n1}\mathbf{e}_{n12} \cdots \mathbf{e}_{n12\cdots n-2}\mathbf{e}_{12\cdots n}\mathbf{e}_{23\cdots n}\mathbf{e}_{3\cdots n} \cdots \mathbf{e}_n,$$

each inscribed in a pair of oppositely congruent helices on cylinders having the

line $0e_{1\cdots n}$ as a common generator. Rotations through multiples of $2\pi/n$ exhibit the helix

$$0e_1e_{12}e_{123}\cdots e_{1\cdots n-1}e_{1\cdots n}$$

as one of a family of n directly congruent helices which together contain all the $1+n(n-1)+1$ vertices. In other words, the polar zonohedron is inscribed in the surface obtained by revolving any one of the helices about the line $0e_{1\cdots n}$ (which is a generator of the cylinder on which the helix lies).

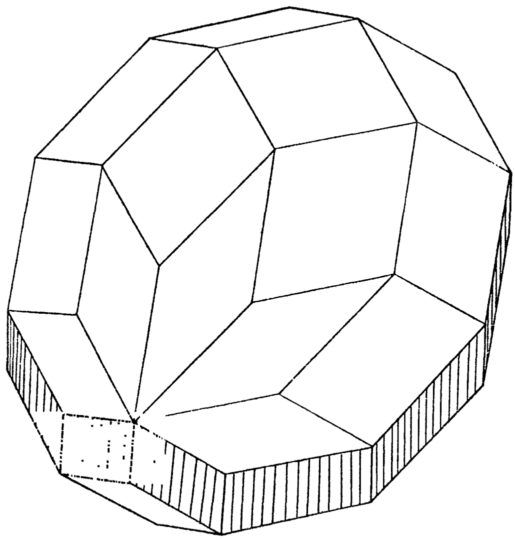

FIG. 3

5. The surface of revolution. Figure 3 was copied from Franklin's drawing of 1938; it shows, more clearly than Figure 2, the appearance of a polar zonohedron with $n=7$. The surface described above is evidently nothing like a spheroid, but rather spindle-shaped. We naturally ask what is its meridian: the section by a plane through the axis of revolution.

Figure 4, like Figure 2, is a view along the axis of revolution, which we take to be the x-axis and indicate in the figure by the single point O. Let a variable point on the helix project into a point P on the circle and into a point Q on the axis, so that $OQ=x$. The nature of the helix is such that the line OP makes with the tangent at O an angle proportional to x, say x/c. The meridian of the surface of revolution is the plane curve whose equation expresses the "radius" $y=OP$ in terms of the abscissa x. Simple trigonometry shows that this equation is

5.1 $$\frac{y}{a}=\sin\frac{x}{c},$$

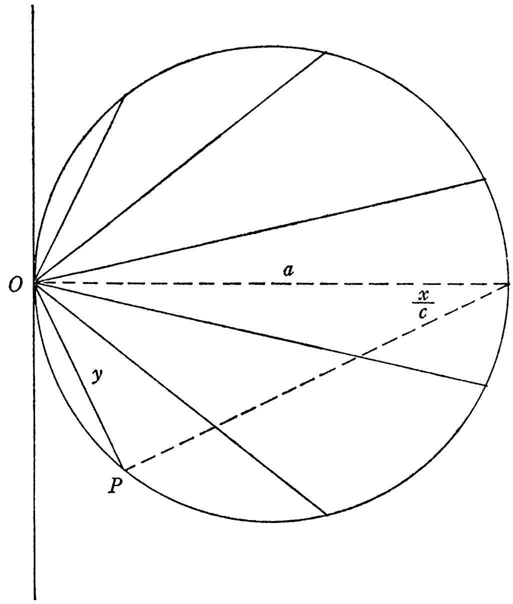

FIG. 4

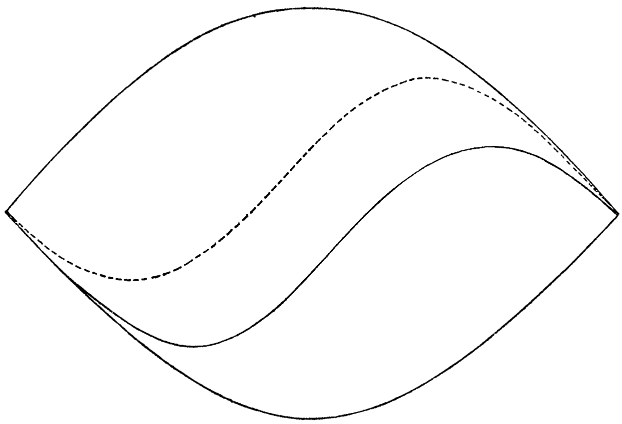

FIG. 5

where a is the diameter of the cylinder. Thus the meridian is a *sine curve.* In other words, the spindle-shaped surface is obtained by revolving an arc of a sine-curve, from one inflection to the next, about the line joining these inflections.

Figure 5 depicts this surface as seen from a great distance along a line perpendicular to the axis of revolution. The two curves drawn on the surface (one supposed to be on the far side) are the "pair of oppositely congruent

helices" mentioned in Section 4. Since all the vectors $\mathbf{e}_k$ are of length l and make the same angle γ with the axis $0\mathbf{e}_{1\ldots n}$, the distance between the two "poles" is

$$\pi c = |\mathbf{e}_1 + \cdots + \mathbf{e}_n| = nl \cos \gamma.$$

Taking the meridian to be the section of the surface by the plane $0\mathbf{e}_{1\ldots n}\mathbf{e}_1$, we see that its equation 5.1 must be satisfied by the coordinates of the vertex $\mathbf{e}_1$, namely

$$x = l \cos \gamma = \pi c/n, \qquad y = l \sin \gamma.$$

Hence

$$\frac{l \sin \gamma}{a} = \sin \frac{\pi}{n},$$

and the "equatorial radius" is $a = l \sin \gamma \csc \pi/n$.

The same surface

$$y^2 + z^2 = a^2 \sin^2 \frac{x}{c} \qquad (0 \leqq x \leqq c\pi)$$

contains the vertices of a polar zonohedron for every value of n, if we take

$$\tan \gamma = \frac{na}{\pi c} \sin \frac{\pi}{n}, \qquad l = \frac{\pi c}{n} \sec \gamma.$$

When n increases, so that γ tends to arctan (a/c) and l to zero, the polyhedron approximates the surface more and more closely.

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THE GEOMETRIC THEORY OF QUATERNIONS

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In a previous paper [1] we made reference in passing to a connection with quaternions. It seems worthwhile to enlarge on this point. The resulting coordinate-free theory of (real) quaternions is much closer to the original geometric ideas of Hamilton [2] than the $(1, i, j, k)$ coordinate formalism of the algebraists. We assume familiarity with the terminology and results of [1] and shall call the elements of the spin model $\mathfrak{S}_3$ *3-vectors.*

Let A and B be 3-vectors. Then

$$AB = \frac{1}{2}(AB + BA) + i\left[\frac{1}{2i}(AB - BA)\right] = (A\cdot B)I + i(A \times B).$$

DEFINITION. *A quaternion is an element of $B(H_2)$ of the form $Q=a_0I+iA$, where a_0 is real and A is a 3-vector.*

(Since $a_0=\frac{1}{2}$ trace Q, such a representation is unique.) Thus the product of two 3-vectors is a quaternion. Conversely,

THEOREM. *Every quaternion is the product of two 3-vectors.*

Proof. Let $Q=a_0I+iA$ and write $A=a_3e_3$, where e_3 is a unit 3-vector. Add orthogonal unit 3-vectors e_1, e_2 to obtain a right-handed coordinate system (e_1, e_2, e_3) for $\mathfrak{S}_3$. Set $B=e_1$, $C=a_0e_1+a_3e_2$. Then B, C are 3-vectors and $BC=a_0e_1^2+a_3e_1e_2=a_0I+ia_3e_3=Q$.

The quaternions clearly form a real linear space, which is closed under multiplication since the product of two quaternions

$$(a_0I + iA)(b_0I + iB) = a_0b_0I + i(b_0A + a_0B) - AB$$

is the difference of two quaternions.

Denote the norm of a linear transformation T in H_2 by $|T|$.

THEOREM. *Let $Q=a_0I+i(a_1e_1+a_2e_2+a_3e_3)$, where (e_1, e_2, e_3) is a coordinate system for $\mathfrak{S}_3$. Then $|Q|=(a_0^2+a_1^2+a_2^2+a_3^2)^{\frac{1}{2}}$.*

Proof. The operator adjoint $Q^*=a_0I-i(a_1e_1+a_2e_2+a_3e_3)$ is also a quaternion, and

$$Q^*Q = QQ^* = a_0^2I + (a_1e_1 + a_2e_2 + a_3e_3)^2 = (a_0^2 + a_1^2 + a_2^2 + a_3^2)I.$$

Then $|Q|^2=|Q^*Q|=a_0^2+a_1^2+a_2^2+a_3^2$.

Note that a nonzero quaternion Q has the quaternion inverse $Q^{-1}=Q^*/|Q|^2$. We have thus established:

THEOREM. *The quaternions form a division algebra over the real numbers.*

The multiplicative structure of the quaternions is implicit in the following

THEOREM. *The quaternions are just the elements of $B(H_2)$ of the form kU, where k is real and $U \in \mathfrak{SU}(2)$.*

For example, the following results are immediate:

COROLLARIES. 1) $|Q_1Q_2| = |Q_1| \cdot |Q_2|$.
2) $\det Q = |Q|^2$.

Proof of theorem. It is sufficient to show that the quaternions of unit norm coincide with the elements of $\mathfrak{SU}(2)$. If $Q = a_0I + iA$ is any quaternion, pick an orthonormal basis (ϕ_1, ϕ_2) yielding a diagonal representation for A. Then

$$Q \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} + i\begin{pmatrix} a_3 & 0 \\ 0 & -a_3 \end{pmatrix} = \begin{pmatrix} a_0 + ia_3 & 0 \\ 0 & a_0 - ia_3 \end{pmatrix}.$$

But

$$Q^*Q \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} a_0^2 + a_3^2 & 0 \\ 0 & a_0^2 + a_3^2 \end{pmatrix} = (a_0^2 + a_3^2)I.$$

Then $|Q| = |Q^*Q|^{\frac{1}{2}} = 1$ implies $a_0^2 + a_3^2 = 1$, and Q is represented by a unitary unimodular matrix. Conversely, given an element U in $\mathfrak{SU}(2)$ pick an orthonormal basis (ϕ_1, ϕ_2) in H_2 yielding a diagonal representation

$$U \underset{(\phi)}{\longleftrightarrow} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + i\begin{pmatrix} \sin\theta & 0 \\ 0 & -\sin\theta \end{pmatrix},$$

where θ is real. Since $|U| = 1$, the matrix represents a quaternion of unit norm.

Introducing an orthonormal basis (ϕ_1, ϕ_2) in H_2 and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ as a basis for the resulting matrix representation of $\mathfrak{E}_3$, we note that the quaternions of unit norm are the elements of $B(H_2)$ with the representation

$$Q \underset{(\phi)}{\longleftrightarrow} a_0I + i(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3) = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix},$$

where the a's are real and $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. It follows that such matrices are just the unimodular unitary 2×2 matrices, a familiar fact employed without proof in [1] but easily established directly.

To make contact with the algebraic theory of quaternions introduce a right-handed coordinate system (e_1, e_2, e_3) in $\mathfrak{E}_3$ and set

$$\mathbf{1} = I$$

$$(\mathbf{k}, \mathbf{j}, \mathbf{i}) = i(e_1, e_2, e_3).$$

Then $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ form a basis for the quaternions, while the identities 1) and 3) of

[1] imply

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ ij &= k = -ji \\ jk &= i = -kj \\ ki &= j = -ik. \end{aligned}$$

For example, $ij=(ie_3)(ie_2)=-e_3e_2=e_2e_3=ie_1=k$.

The complex quaternions may be defined as operators of the form $a_0I+i(a_1e_1+a_2e_2+a_3e_3)$ with complex a's. But then one obtains all of $B(H_2)$ and divisors of 0 enter.

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ON THE ASYMPTOTIC BEHAVIOR OF BESSEL FUNCTIONS

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1. The behavior of the Bessel function $J_n(x)$ for large real values of x is well known. In fact, it is possible to obtain an asymptotic series for $J_n(x)$. This requires, however, a long and involved argument. It is far easier, and for many applications just as useful, to express $J_n(x)$ in terms of trigonometric functions ([3], p. 334). This expression contains an amplitude and a phase angle, whose calculation requires writing $J_n(x)$ in terms of certain contour integrals in the complex plane and evaluating these integrals ([3], p. 524).

Our goal in this note is to obtain the same expression in terms of trigonometric functions by a simpler method based on [2]. This method will show that the resulting formula can be differentiated. Then the well-known recursion relations for Bessel functions ([1], p. 93) can be used to calculate the amplitude and phase angle. The treatment is elementary; using only the Bessel differential equation and the recursion relations we can establish the asymptotic formula, and express the amplitude and phase angle for $J_n(x)$ in terms of the amplitude and phase angle for $J_0(x)$. As the amplitude and phase angle for $J_0(x)$ can be found by evaluating one real definite integral ([4], p. 172), the problem is then completely solved.

2. We begin with the Bessel equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{1}$$

The usual change of variable $v=y\sqrt{x}$ transforms (1) to

$$v'' + [1 + (\tfrac{1}{4} - n^2)/x^2]v = 0. \tag{2}$$

We write this as a system of two first order equations by setting $v_1=v$, $v_2=v'$. This gives $v_1' = v_2$, $v_2' = [-1+(n^2-\frac{1}{4})/x^2]v_1$. If we define the matrices $A(x)$ and $B(x)$ by

$$A(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B(x) = \begin{pmatrix} 0 & 0 \\ (n^2 - \frac{1}{4})/x^2 & 0 \end{pmatrix},$$

then we can write this system as

$$v' = [A(x) + B(x)]v. \tag{3}$$

We compare (3) with the system

$$u' = A(x)u, \tag{4}$$

which has a fundamental matrix solution

$$\Phi(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}.$$

Any solution $u(x)$ of (4) has the form $u(x)=\Phi(x)c$ for some constant vector c. We try to find a solution $v(x)$ of (3) in the form $v(x)=\Phi(x)z(x)$. This requires

$$z' = \Phi^{-1}(x)B(x)\Phi(x)z. \tag{5}$$

It is easy to calculate

$$\Phi^{-1}(x) = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}.$$

For our estimates, we will use the sum of the absolute values of the components of a vector for the norm of the vector, and the sum of the absolute values of the elements of a matrix for the norm of the matrix. This is a matter of convenience; a different norm would serve just as well.

The matrices $\Phi(x)$ and $\Phi^{-1}(x)$ are bounded, $|\Phi(x)| \leqq 4$, $|\Phi^{-1}(x)| \leqq 4$, and $|\Phi^{-1}(x)B(x)\Phi(x)| \leqq M_0/x^2$, with $M_0=4(4n^2-1)$. Let $r(x)$ be the solution of $r'=M_0r/x^2$ with $r(x_0)=|z(x_0)|$ for some $x_0>0$. Then, because $\int_{x_0}^\infty dx/x^2$ converges, $z(x)$ tends to a limit vector $z(\infty)$ as $x\to\infty$, and there exists a constant K, possibly depending on n, such that

$$|z(x) - z(\infty)| \leqq KM_0/x$$

for large x (see [2] Lemmas 1 and 3). Now let $u(x)$ be the solution of (4) such that $u(x)=\Phi(x)z(\infty)$. Corresponding to the solution $z(x)$ of (5), there is a solution $v(x)$ of (3), and

$$|u(x) - v(x)| = |\Phi(x)z(x) - \Phi(x)z(\infty)| \leqq |\Phi(x)| \cdot |z(x) - z(\infty)| \leqq M/x,$$

where $M = 4KM_0$.

Now, for the Bessel function $J_n(x)$, we can write

$$x^{1/2}J_n(x) = v_1(x) = u_1(x) + p(x),$$

where $|p(x)| \leqq M/x$. In addition,

$$v_2(x) = v_1'(x) = x^{1/2}J_n'(x) + \tfrac{1}{2}x^{-1/2}J_n(x) = u_2(x) + q(x),$$

where $|q(x)| \leqq M/x$.

We can write $u_1(x) = A_n \cos(x - \delta_n)$, $u_2(x) = u_1'(x) = -A_n \sin(x - \delta_n)$, and it follows that

$$\begin{aligned} J_n(x) &= A_n x^{-1/2} \cos(x - \delta_n) + O(x^{-3/2}) \\ J_n'(x) &= -A_n x^{-1/2} \sin(x - \delta_n) + O(x^{-3/2}). \end{aligned} \tag{6}$$

From the differential equation,

$$J_n''(x) = J_n'(x)/x - J_n(x) + n^2 J_n(x)/x^2 = -A_n x^{-1/2} \cos(x - \delta_n) + O(x^{-3/2}).$$

Repeated differentiation of the differential equation and use of (6) gives

$$\begin{aligned} J_n^{(k)}(x) &= x^{-1/2} \frac{d^k}{dx^k}[A_n \cos(x - \delta_n)] + O(x^{-3/2}) \\ &= \frac{d^k}{dx^k}[x^{-1/2}A_n \cos(x - \delta_n)] + O(x^{-3/2}), \quad k = 0, 1, 2, \cdots. \end{aligned}$$

Now let us try to compute the amplitude A_n and the phase angle δ_n. For this, we need the recurrence relations

$$\begin{aligned} J_n'(x) &= J_{n-1}(x) - nJ_n(x)/x \\ J_n'(x) &= -J_{n+1}(x) + nJ_n(x)/x, \end{aligned} \tag{7}$$

(see, for example, [1], p. 93). We replace n by $(n+1)$ in the first formula of (7), substitute (6), and neglect terms which are $O(x^{-3/2})$. This gives

$$\begin{aligned} -A_{n+1} \sin(x - \delta_{n+1}) &= A_n \cos(x - \delta_n) \\ A_{n+1} \cos(x - \delta_{n+1}) &= A_n \sin(x - \delta_n). \end{aligned} \tag{8}$$

Squaring and adding these two equations, we obtain $A_{n+1}^2 = A_n^2$, or $A_{n+1} = \pm A_n$.

If $A_{n+1} = A_n$, (8) implies $\cos(x - \delta_n) = -\sin(x - \delta_{n+1}) = \sin(\delta_{n+1} - x)$, whence $\delta_{n+1} - x = 2k\pi + \pi/2 - (x - \delta_n)$ for some integer k, or $\delta_{n+1} - \delta_n = 2k\pi + \pi/2$. Since δ_n appears in (6) only as an argument of a trigonometric function, we can take $k = 0$, so that $\delta_{n+1} = \delta_n + \pi/2$.

If $A_{n+1} = -A_n$, (8) implies $\cos(x - \delta_{n+1}) = -\sin(x - \delta_n) = \sin(\delta_n - x)$, whence $x - \delta_{n+1} = 2k\pi + \pi/2 - (\delta_n - x)$ for some integer k, or $\delta_n - \delta_{n+1} = 2k\pi + \pi/2$. Since δ_n appears in (6) only as an argument of a trigonometric function, we can take $k = 0$, so that $\delta_{n+1} = \delta_n - \pi/2$.

We now have two possible expressions for $A_{n+1} \cos(x - \delta_{n+1})$, either $A_n \cos(x - \delta_n - \pi/2)$ or $-A_n \cos(x - \delta_n + \pi/2)$. But $\cos(x - \delta_n - \pi/2) = -\cos(x - \delta_n + \pi/2)$, and the two expressions are the same. Thus we can take $A_{n+1} = A_n$, $\delta_{n+1} = \delta_n + \pi/2$ for all n. Now we can express A_n and δ_n in terms of A_0 and δ_0, obtaining

$$A_n = A_0, \qquad \delta_n = \delta_0 + n\pi/2, \qquad n = 0, 1, 2, \cdots. \tag{9}$$

We are left with the task of calculating A_0 and δ_0. This can be done using the real integral

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \cos \theta) d\theta, \tag{10}$$

which can be derived easily from the power series for $J_0(x)$ ([1], p. 57). Using (10) it is easy to calculate that

$$x^{1/2} J_0(x) \sim (2/\pi)^{1/2} \cos(x - \pi/4), \qquad x \to \infty, \tag{11}$$

(see [4], pp. 172–173). Comparison of (11) and (6) shows that $A_0 = (2/\pi)^{1/2}$, $\delta_0 = \pi/4$. It follows from (9) that

$$A_n = (2/\pi)^{1/2}, \qquad \delta_n = (2n + 1)\pi/4, \qquad n = 0, 1, 2, \cdots. \tag{12}$$

Now, substitution of (12) in (6) gives the final result, which we state formally as a theorem.

THEOREM. *The asymptotic behavior of the Bessel function $J_n(x)$ as $x \to \infty$ is given by $J_n(x) = (2/\pi x)^{1/2} \cos[x - (2n+1)\pi/4] + O(x^{-3/2})$, $n = 0, 1, 2, \cdots$, and this relation can be differentiated any number of times with the same error term $O(x^{-3/2})$.*

As we have said, this result is well known, but the proof seems to be more elementary and shorter than the usual proofs (see, for example, [3], pp. 333–334, 524–526). The only properties of Bessel functions required are easily established, using only the power series expansions for these functions.

It is clear that a result analogous to Theorem 1 can be obtained for Bessel functions of the second kind. As the purpose of this note is to outline a method rather than to reestablish a well-known formula, we will not state this result explicitly.

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SOME ELEMENTARY CHARACTERIZATIONS OF THE POISSON DISTRIBUTION

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1. It is well known (see [1] p. 223, Ex. 6) that if X and Y are independent Poisson random variables (r.v.) then the conditional distribution of X given $X+Y$ is binomial. More specifically, if X and Y are independent r.v.'s and

$$\Pr(X = i) = e^{-\lambda}\frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \cdots$$

$$\Pr(Y = i) = e^{-\mu}\frac{\mu^i}{i!} \qquad \lambda > 0,\ \mu > 0$$

then

$$\Pr(X = k \mid X + Y = t) = \binom{t}{k} p^k(1-p)^{t-k}, \qquad 0 \leqq k \leqq t$$

$$= 0, \qquad k > t$$

where $p = \lambda/(\lambda+\mu)$. The purpose of this note is to show that quite weak forms of the above-mentioned property characterize the Poisson distribution. (We use t to denote a nonnegative integer.) For example, one has,

THEOREM 1. *Let X and Y be independent r.v.'s such that*

$$\Pr(X = i) = f(i)$$

$$\Pr(Y = i) = g(i) \qquad i = 0, 1, 2, \cdots,$$

where $f(i) > 0$, $g(i) > 0$, $\sum_{i=0}^{\infty} f(i) = \sum_{i=0}^{\infty} g(i) = 1$. Let

$$\Pr(X = k \mid X + Y = t) = \binom{t}{k} p_t^k(1-p_t)^{t-k} \qquad 0 \leqq k \leqq t$$

$$= 0 \qquad k > t.$$

Then $p_t \equiv p$, $t = 0, 1, 2, \cdots$, and

$$f(i) = e^{-\theta\alpha}\cdot\frac{(\theta\alpha)^i}{i!}$$

$$g(i) = e^{-\theta}\cdot\frac{\theta^i}{i!},$$

where $\alpha = p/(1-p)$ and $\theta > 0$ is arbitrary.

Proof. It is given that for $t \geqq 0$ and $0 \leqq k \leqq t$

$$\Pr(X = k \mid X + Y = t) = \binom{t}{k} p_t^k(1-p_t)^{t-k}.$$

Using the independence of X and Y one has

$$\Pr(X = k \mid X + Y = t) = \frac{\Pr(X = k) \cdot \Pr(Y = t - k)}{\Pr(X + Y = t)}$$
$$= f(k)g(t-k) \Big/ \sum_{i=0}^{t} f(i)g(t-i).$$

Hence one is given that for $0 \leqq k \leqq t$

$$\frac{f(k)g(t-k)}{\sum_{i=0}^{t} f(i)g(t-i)} = \binom{t}{k} p_t^k (1 - p_t)^{t-k}. \tag{1}$$

Equation (1) yields for $t \geqq 1$ and $1 \leqq k \leqq t$

$$\frac{f(k)g(t-k)}{f(k-1)g(t-k+1)} = \left(\frac{p_t}{1-p_t}\right) \cdot \frac{(t-k+1)}{k}. \tag{2}$$

This in turn shows that for $0 \leqq k \leqq t$, $t \geqq 0$

$$f(k)g(t-k) = \left(\frac{p_t}{1-p_t}\right)^k \cdot \binom{t}{k} g(t)f(0). \tag{3}$$

It is easy to show that equations (1) and (3) are equivalent. Let $\alpha_t = p_t/(1-p_t)$, $t \geqq 1$. Setting $k = t$ in (2) one has for $t \geqq 1$

$$f(t) = \alpha_t \cdot \left(\frac{g(1)}{g(0)}\right) \cdot \frac{1}{t} \cdot f(t-1). \tag{4}$$

Let $\theta = g(1)/g(0)$. Then (4) yields the formula

$$f(t) = \prod_{i=1}^{t} \alpha_i \cdot \theta^t \cdot f(0)/t! \qquad t \geqq 1. \tag{5}$$

Similarly setting $k = 1$ in (2) one has

$$g(t) = \frac{1}{t \cdot \alpha_t} \cdot \left(\frac{f(1)}{f(0)}\right) \cdot g(t-1),$$

giving

$$g(t) = (\theta\alpha_1)^t \cdot g(0) \Big/ \prod_{i=1}^{t} \alpha_i \cdot t! \qquad t \geqq 1. \tag{6}$$

On the other hand, from (3) one has for $k+1 \leqq t$

$$\frac{g(t-k)}{g(t-k-1)} = \frac{1}{\alpha_t} \cdot \frac{(k+1)}{(t-k)} \cdot \frac{f(k+1)}{f(k)}.$$

This combined with (4) gives for $k+1 \leqq t$, $t \geqq 1$

$$g(t-k) = \frac{\alpha_{(k+1)}}{\alpha_t} \cdot \frac{\theta}{(t-k)} \cdot g(t-k-1). \tag{7}$$

From (7) one also has

$$g(t-k) = g(t+1-(k+1)) = \frac{\alpha_{(k+2)}}{\alpha_{(t+1)}} \cdot \frac{\theta}{(t-k)} \cdot g(t-k-1). \tag{8}$$

Comparing (7) and (8) one has

$$\frac{\alpha_{(k+1)}}{\alpha_t} = \frac{\alpha_{(k+2)}}{\alpha_{(t+1)}} \qquad k \geqq 0, t \geqq (k+1).$$

In particular $\alpha_{(t+1)} = (\alpha_2/\alpha_1) \cdot \alpha_t$, $t \geqq 1$, giving

$$\alpha_t = \left(\frac{\alpha_2}{\alpha_1}\right)^{t-1} \cdot \alpha_1 \qquad t \geqq 1. \tag{9}$$

Substituting (9) in (5) and (6) one has for $t \geqq 1$

$$f(t) = \frac{(\theta\alpha_1)^t}{t!} \cdot \left(\frac{\alpha_2}{\alpha_1}\right)^{t(t-1)/2} \cdot f(0) \tag{10}$$

and

$$g(t) = \frac{\theta^t}{t!} \cdot \left(\frac{\alpha_1}{\alpha_2}\right)^{t(t-1)/2} \cdot g(0). \tag{11}$$

But if $f(t)$ is given by (10), $\sum_{t=0}^{\infty} f(t)$ is convergent only if $\alpha_2/\alpha_1 \leqq 1$ and if $g(t)$ is given by (11), $\sum_{t=0}^{\infty} g(t)$ is convergent only if $\alpha_1/\alpha_2 \leqq 1$. Hence $\alpha_1 = \alpha_2$. This means from (9) that $\alpha_t \equiv \alpha$, $t \geqq 1$ and

$$f(t) = \frac{(\theta\alpha)^t}{t!} \cdot f(0), \qquad g(t) = \frac{\theta^t}{t!} \cdot g(0).$$

The hypothesis that $\sum_{t=0}^{\infty} f(t) = \sum_{t=0}^{\infty} g(t) = 1$ yields $f(0) = e^{-\theta \cdot \alpha}$ and $g(0) = e^{-\theta}$. This completes the proof of Theorem 1.

It is interesting to observe that there are r.v.'s X and Y having distributions other than Poisson such that

$$\Pr(X = k \mid X + Y = t) = \binom{t}{k} p_t^k (1 - p_t)^{t-k}$$

holds for $0 \leqq t \leqq L$ and p_t's are different from each other for $0 \leqq t \leqq L$. A precise statement of this is as follows:

THEOREM 2. *Let X and Y be independent nonnegative integer-valued r.v.'s such that* $\Pr.(X=i)=f(i)>0$, $\Pr.(Y=i)=g(i)>0$, $i=0, 1, 2, \cdots, L$. *Then for* $0 \leqq t \leqq L$

$$\Pr(X = k \mid X + Y = t) = \binom{t}{k} p_t^k (1 - p_t)^{t-k} \qquad 0 \leqq k \leqq t$$

$$= 0 \qquad k > t$$

iff $\alpha_t = (\alpha_2/\alpha_1)^{t-1} \cdot \alpha_1$ *and*

$$(12) \qquad \begin{aligned} f(t) &= \frac{(\theta\alpha_1)^t}{t!}\left(\frac{\alpha_2}{\alpha_1}\right)^{t(t-1)/2} \cdot f(0) \qquad 1 \leqq t \leqq L \\ g(t) &= \frac{\theta^t}{t!}\left(\frac{\alpha_1}{\alpha_2}\right)^{t(t-1)/2} \cdot g(0), \end{aligned}$$

where $\theta>0$ is arbitrary, $\alpha_t = p_t/(1-p_t)$.

A proof of Theorem (2) can be obtained by verifying directly that $f(t)$, $g(t)$ satisfying (12) satisfy (3) for $0 \leqq t \leqq L$ and taking notice of the comment made after equation (3).

2. The following simple and general theorem motivates the further characterizations. For the sake of brevity, I shall denote by C the class of r.v.'s X such that $\Pr.(X=i)>0$, $i=0, 1, 2, \cdots$, and $\sum_{i=0}^{\infty} \Pr.(X=i)=1$.

THEOREM 3. *Let a_i, b_i, $i=1, 2, \cdots$ be sequences of positive numbers satisfying the following*: (i) $a_1+b_1=1$, $a_i+b_i<1$ *for* $i \geqq 2$, (ii) $\sum_{i=1}^{\infty} \mu_i u^i$ *and* $\sum_{i=1}^{\infty} \mu_i/a_i \cdot u^i$ *converge for $|u| \leqq R$ for some $R>0$, where* $\mu_i = \prod_{j=1}^{i} a_j/b_j$, $i \geqq 1$,

$$\text{(iii)} \qquad \frac{1 + \sum_{i=1}^{\infty} (\mu_i/a_i) u^i}{1 + \sum_{i=1}^{\infty} \mu_i \cdot u^i} = 1 + \sum_{i=1}^{\infty} p_i \cdot u^i$$

for $|u| \leqq R$, where $p_i>0$ for $i \geqq 1$. Then there exist independent r.v.'s X, Y in the class C such that

$$(13) \qquad \begin{aligned} \Pr(X = t \mid X + Y = t) &= a_t \\ \Pr(X = t - 1 \mid X + Y = t) &= b_t \qquad t \geqq 1. \end{aligned}$$

Also $f(t) = \Pr(X=t) = \theta^t\mu_t / \{1 + \sum_{t=1}^{\infty} \theta^t \mu_t\}$, $t \geqq 1$, $0<\theta \leqq R$, $f(0)=\Pr(X=0) = 1/\{1+\sum_{t=1}^{\infty} \theta^t\mu_t\}$ *and $g(t) = \Pr(Y=t)$ is given by the relations*

$$\frac{G(u)}{g(0)} = \frac{f(0) + \sum_{t=1}^{\infty} (f(t)/a_t) \cdot u^t}{F(u)}$$

and $\sum_{t=0}^{\infty} g(t)=1$, *where* $F(u)=\sum_{t=0}^{\infty} f(t)u^t$, $G(u)=\sum_{t=0}^{\infty} g(t)u^t$. *Conversely if there exist independent r.v.'s* X, Y *in class* C *satisfying equation* (13) *then* $\{a_t\}$, $\{b_t\}$ *must satisfy conditions* (i), (ii) *and* (iii).

Proof. Relations (13) imply for $t \geqq 1$

$$\frac{f(t)g(0)}{\sum_{i=0}^{t} f(i)g(t-i)} = a_t, \qquad \frac{f(t-1)g(1)}{\sum_{i=0}^{t} f(i)g(t-i)} = b_t.$$

Hence $f(t)=[g(1)/g(0)]\cdot[a_t/b_t]\cdot f(t-1)$, i.e., $f(t)=\theta^t\cdot\mu_t\cdot f(0)$, where $\theta=g(1)/g(0)$. Since from (13) $\sum_{i=0}^{t} f(i)g(t-i)=[f(t)\cdot g(0)]/a_t$, $t\geqq 1$, one has

$$\frac{G(u)}{g(0)} = \frac{1}{F(u)}\left\{f(0)+\sum_{t=1}^{\infty}(f(t)/a_t)\cdot u^t\right\}.$$

The above computations obviously prove Theorem 3.

Theorem 3 and the formula

$$\Pr(X=t \mid X+Y=t) = \Pr(Y=0 \mid X+Y=t)$$

yield Corollaries 1 and 2.

COROLLARY 1. *Let* X, Y *be independent r.v.'s from* C. *If*

$$\Pr(X=t \mid X+Y=t)=p^t, \qquad t\geqq 1, \qquad 0<p<1,$$
$$\Pr(X=t-1 \mid X+Y=t)=t\cdot p^{t-1}(1-p)$$

then X, Y *are Poisson variables with means* $\theta p/(1-p)$ *and* θ *respectively where* $\theta>0$ *is arbitrary.*

COROLLARY 2. *Let* X, Y *be independent r.v.'s from* C. *If*

$$\Pr(X=0 \mid X+Y=t)=(1-p)^t, \qquad t\geqq 1$$
$$\Pr(X=1 \mid X+Y=t)=tp(1-p)^{t-1}, \qquad 0<p<1$$

then X, Y *are Poisson variables with means* $\theta\cdot p/(1-p)$ *and* θ *respectively where* $\theta>0$ *is arbitrary.*

3. In this section, I shall discuss the possibility of characterizing Poisson distributions by weakening the assumptions of Corollary 1 to

$$\begin{aligned} &\Pr(X=t \mid X+Y=t)=p_t^t && t\geqq 1\\ &\Pr(X=t-1 \mid X+Y=t)=t\cdot p_t^{t-1}(1-p_t) && 0<p_t<1. \end{aligned} \tag{14}$$

A simple computation shows that equations (14) do not characterize the Poisson distributions. As a matter of fact, if there exist r.v.'s X, Y satisfying (14) for a specified sequence $\{p_t\}$, $t\geqq 1$ (conditions which do guarantee the existence of

X, Y are as in Theorem 3) then one has (from Theorem 3)

$$f(t) = \Pr(X = t) = \frac{\theta^t}{t!} \cdot \prod_{i=1}^{t} \alpha_i \cdot f(0) \qquad t \geqq 1,$$

where $\alpha_i = p_i/(1-p_i)$, $i \geqq 1$ and $\theta > 0$ is chosen anyway whatsoever as long as the series $\sum_{t=1}^{\infty} f(t)$ remains convergent. Also $g(t) = \Pr(Y=t)$, $t \geqq 0$ are given by

$$\frac{G(u)}{g(0)} = \frac{1}{F(u)} \left\{ f(0) + \sum_{t=1}^{\infty} (f(t)/p_t^t) \cdot u^t \right\},$$

where $G(u)$, $F(u)$ are as in Theorem 3. Examples of non-Poissonian distributions ratisfying (14)-type equations can now be easily obtained by suitable choice of p_t. For example $p_t = 1 - 1/(t+1)$, $t \geqq 1$, give

$$f(t) = (1-\theta)\theta^t, \qquad t \geqq 0, \qquad 0 < \theta < 1,$$

$$g(t) = \theta^t \cdot \left\{ \left(1 + \frac{1}{t}\right)^t - \left(1 + \frac{1}{t-1}\right)^{t-1} \right\} \cdot \Big/ \left\{ (1-\theta) \cdot \sum_{t=0}^{\infty} \theta^t \cdot \left(1 + \frac{1}{t}\right)^t \right\}, \qquad t \geqq 1$$

$$g(0) = 1 \Big/ \left\{ (1-\theta) \sum_{t=0}^{\infty} \theta^t \cdot \left(1 + \frac{1}{t}\right)^t \right\} \qquad (\text{define } \infty^0 = 1)$$

non-Poissonian distributions satisfying (14).

However, one can obtain the following generalization of Corollary 1 and Theorem 1:

THEOREM 4. *Let X and Y be independent r.v.'s from C. If*

$$\begin{aligned} \Pr(X = t \mid X + Y = t) &= p_t^t \\ \Pr(X = t-1 \mid X + Y = t) &= t p_t^{t-1}(1-p_t) \\ \Pr(X = t-2 \mid X + Y = t) &= \frac{t(t-1)}{2} p_t^{t-2}(1-p_t)^2 \end{aligned} \tag{15}$$

for $t \geqq 2$ and $0 < p_t < 1$ then $p_t \equiv p$ with $t \geqq 2$, and X and Y are Poisson r.v.'s with means $\theta p/(1-p)$ and θ respectively where θ is an arbitrary positive real number.

Proof. Let $p_1 = \Pr(X=1 \mid X+Y=1)$, $f(t) = \Pr(X=t)$, $t \geqq 0$, $g(t) = \Pr(Y=t)$. Then one has from (15) (as before in the proof of Theorem 1)

$$\begin{aligned} f(t) &= \frac{g(1)}{g(0)} \cdot \frac{1}{t} \frac{p_t}{1-p_t} \cdot f(t-1) \\ f(t-1) &= \frac{2g(2)}{g(1)} \cdot \frac{1}{t-1} \cdot \frac{p_t}{1-p_t} \cdot f(t-2) \qquad (t \geqq 2). \end{aligned} \tag{16}$$

Comparing the two equations in (16) one has $\alpha_t=(\theta/\phi)\cdot\alpha_{t-1}$, $t\geqq 2$, where $\theta=[g(1)/g(0)]$, $\phi=2g(2)/g(1)$ and $\alpha_t=p_t/(1-p_t)$.

Hence $\alpha_t=(\theta/\phi)^{t-1}\cdot\alpha_1$, $p_t=\alpha_1/\{\alpha_1+(\phi/\theta)^{t-1}\}$. Now from (16)

$$f(t) = \frac{\theta^t}{t!}\cdot\prod_{i=1}^{t}\alpha_i\cdot f(0)$$

$$= \frac{(\theta\alpha_1)^t}{t!}\left(\frac{\theta}{\phi}\right)^{t(t-1)/2}\cdot f(0) \qquad t\geqq 1.$$

Since $\sum_{t=1}^{\infty} f(t)$ has to converge, $\theta/\phi\leqq 1$.

Using notation introduced in the previous section, one has

$$\frac{G(u)}{g(0)} = \frac{1}{F(u)}\left\{f(0)+\sum_{t=1}^{\infty}(f(t)/p_t^t)\cdot u^t\right\}. \tag{17}$$

The series in the numerator of the r.h.s. of (17) is equal to

$$\sum_{t=1}^{\infty}\frac{\theta^t}{t!}\cdot\left(\frac{\theta}{\phi}\right)^{t(t-1)/2}\cdot\left(\alpha_1+\left(\frac{\phi}{\theta}\right)^{t-1}\right)^t\cdot f(0)\cdot u^t,$$

which converges for some interval of values of u only if $\phi/\theta\leqq 1$. Hence $\phi=\theta$. This proves that $p_t\equiv p_1=p$. The rest of Theorem 4 now follows easily.

It is easily seen that conditions like those in Theorem 4, if modified to cover the conditional probabilities of $X=0, 1, 2$ respectively given that $X+Y=t$, also characterize the Poisson distributions.

In conclusion, I should like to point out that lots of discrete distributions can be characterized by the above methods. For example, an argument as in Theorem 3 will show that if

$$\Pr(X=t\mid X+Y=t) = \frac{1}{t+1}$$

$$\Pr(X=t-1\mid X+Y=t) = \frac{1}{t+1} \qquad t\geqq 1$$

then X, Y are distributed according to the geometric distribution with arbitrary parameter θ, $0<\theta<1$, i.e., $\Pr(X=t)=\Pr(Y=t)=\theta^t(1-\theta)$, $t\geqq 0$.

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REDUCIBILITY PROPERTIES OF POLYNOMIALS OVER THE RATIONALS

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It is a consequence of the prime number theorem that "almost all integers are composite." This statement is made precise in Section 1. In Section 2 it is shown that for polynomials over the rationals the opposite is true, namely "almost all nth degree polynomials over the rationals (for any fixed positive n) are irreducible."

1. The prime number theorem. Let $\pi(x)$ be the number of primes less than or equal to x, and let

$$\rho(x) = \frac{\pi(x)}{x} \cdot \tag{1}$$

Clearly, $\rho(x)$ is the density of primes among the positive integers less than or equal to x. According to the prime number theorem (see Chapter 2 of [1])

$$\lim_{x\to\infty} \rho(x) \log x = 1. \tag{2}$$

It follows that the density of primes among all the positive integers is

$$\lim_{x\to\infty} \rho(x) = 0, \tag{3}$$

which is equivalent to the statement that almost all integers are composite.

2. Reducibility of polynomials over the rationals. Let T be the set of all nth degree polynomials over the rationals. Let $S = \{S_1, S_2, S_3, \cdots, S_L, \cdots\}$ be a sequence of finite subsets of T such that

$$\begin{gathered} S_L \subset S_{L+1}; \qquad L = 1, 2, 3, \cdots \\ \bigcup_{L=1}^{\infty} S_L = T. \end{gathered} \tag{4}$$

Let ρ_L be the density of reducibles in S_L, and let

$$\rho = \lim_{L\to\infty} \rho_L \tag{5}$$

if the limit exists. We shall say that ρ is the *density of reducibles in T, relative to the sequence S.*

Since all linear polynomials over the rationals are irreducible, ρ is identically zero for all sequences S when n is equal to one. When n is greater than one, it is clear that a sequence S can be chosen to yield any preassigned value for ρ in the interval $0 \leqq \rho \leqq 1$, simply by making explicit reference to reducibility in the definition of the subsets S_L. However, it is not unreasonable to hope that ρ is independent of S for a large class of "unbiased" choices of the sequence S. We shall present one such sequence S and prove that ρ is zero for it.

Before proceeding, let us review the definition of irreducibility [2]. A polynomial over an integral domain D is irreducible if and only if its only divisors (among the polynomials over D) are units (divisors of unity) and associates (unit multiples). Note that if D is a field, every nonzero element of D is a unit.

Our task is greatly simplified if we consider only the subclass of integer-representing polynomials from each associate class. An integer-representing polynomial is one whose values at integers are integers. Our discussion here is a summary of Section 35 of [1]. Clearly, every integral polynomial is integer-representing, but there exist integer-representing polynomials which are not integral. For example,

$$\binom{x}{n} \equiv \frac{x(x-1)\cdots(x-n+1)}{n!}; \qquad n > 0 \tag{6}$$

is an integer-representing polynomial of degree n whose leading coefficient is $1/n!$. It is easily shown that every nth degree polynomial $P(x)$ over the rationals has a unique expansion of the form

$$P(x) = c_0 + \sum_{i=1}^{n} c_i \binom{x}{i} \tag{7}$$

with rational coefficients c_i. If the values $P(0), \cdots, P(n)$ are integers, then the coefficients c_i are integers (the proof is by induction on i) and therefore, $P(x)$ is integer-representing. Finally, we note that there is a one-one correspondence between integer-representing polynomials $P(x)$, of degree at most n and $(n+1)$-tuples of integers $[P(0), \cdots, P(n)]$ so it is permissible to speak of "the polynomial $[P(0), \cdots, P(n)]$."

We are now ready to translate the qualitative statement that "almost all nth degree polynomials over the rationals (for any fixed positive n) are irreducible" into a precise form:

THEOREM. *For fixed positive $n \geqq 2$, let T be the set of all nth degree polynomials over the rationals. Let S_L be the set of integer-representing nth degree polynomials $P(x)$ such that*

$$|P(i)| \leqq L; \qquad i = 0, \cdots, n, \tag{8}$$

and such that the first nonzero $P(i)$ is positive. Let ρ_L be the density of reducibles in S_L. Then

$$\lim_{L\to\infty} \rho_L = 0. \tag{9}$$

Proof. First, let us find a lower bound on $N(S_L)$, the number of elements of S_L. Clearly, the number of polynomials of degree n or less satisfying (8) is $(2L+1)^{n+1}$. Of these, the number having degree less than n is at most $(2L+1)^n$, so the number having degree precisely n is at least $2L(2L+1)^n$. For precisely

half of these, the first nonzero $P(i)$ is positive, so

$$N(S_L) \geqq L(2L+1)^n > 2^n L^{n+1}. \tag{10}$$

Let R_L be the subset of reducible polynomials in S_L. Let $R_L^{(k)}$ be the subset of R_L such that each polynomial in $R_L^{(k)}$ has an irreducible factor of degree k and none of lower degree. Clearly, k must lie in the interval

$$0 < k \leqq \left[\frac{n}{2}\right], \tag{11}$$

where the bracket denotes the integer part. Our next task is to get an upper bound on the number of elements in $R_L^{(k)}$ for each k in this interval.

If a polynomial $P(x)$ in S_L satisfies $P(i)=0$ for some i in the interval $0 \leqq i \leqq n$, then $(x-i)$ divides $P(x)$ so $P(x)$ is in $R_L^{(1)}$. It is clear that the number of such polynomials is bounded above by

$$n(2L+1)^n. \tag{12}$$

In the remainder of our discussion we shall replace (8) by the stronger requirement

$$1 \leqq |P(i)| \leqq L; \qquad i = 0, \cdots, n; \tag{13}$$

so at the end we shall have to add (12) to the bound on $N(R_L^{(1)})$.

Now let $P(x)$ be any polynomial in $R_L^{(k)}$. Since $P(x)$ is integer-representing, the coefficients in (7) are integers, so the polynomial

$$F(x) = n!P(x) \tag{14}$$

is integral. Therefore by Theorem 15 of [2] we can write

$$F(x) = G(x)H(x), \tag{15}$$

where $G(x)$ and $H(x)$ are integral polynomials of degree k and $n-k$, respectively. From (13)–(15) we find

$$1 \leqq |G(i)|, |H(i)| \leqq |G(i)H(i)| \leqq L' = n!L; \qquad i = 0, \cdots, k, \tag{16}$$

and (if $n > 2k$)

$$1 \leqq |H(i)| \leqq L'; \qquad i = k+1, \cdots, n-k. \tag{17}$$

Actually, (16) holds for $i = 0, \cdots, n$, but we shall not need the extra information. Also since $P(0)$ is positive, we can require

$$G(0) > 0, H(0) > 0. \tag{18}$$

We have shown that for every element of $R_L^{(k)}$ there is at least one pair of integer-representing polynomials

$$\begin{aligned} G &= [G(0), \cdots, G(k)] \\ H &= [H(0), \cdots, H(n-k)] \end{aligned} \tag{19}$$

satisfying (16)–(18). Therefore, the number of elements of $R_L^{(k)}$ is bounded by the number of such pairs.

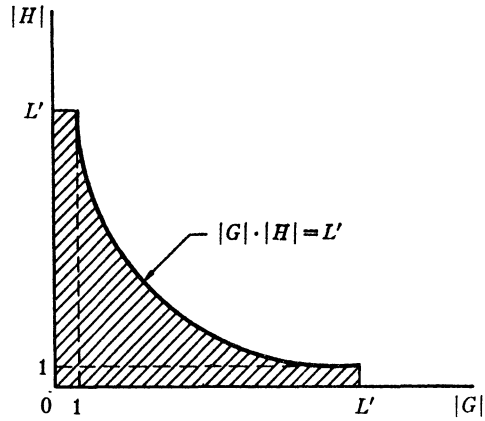

FIG. 1. For $0 \leqq i \leqq k$, the number of allowed pairs $(|G(i)|, |H(i)|)$ is less than the area of the shaded region.

For $0 \leqq i \leqq k$ the number of allowed pairs $[|G(i)|, |H(i)|]$ is equal to the number of integral points in the shaded region of Fig. 1, not counting points on either axis. This is clearly less than the area of the region which is $L'(1+\log L')$. Therefore, the number of allowed pairs $[G(i), H(i)]$ is bounded above by $L'(1+\log L')$ for $i=0$ and by $4L'(1+\log L')$ for $1 \leqq i \leqq k$. For i in the interval $k+1 \leqq i \leqq n-k$ (if $n>2k$) there are $2L'$ values of $H(i)$ consistent with (17). Therefore, the total number of pairs $[G, H]$ consistent with (16)–(18) is bounded above by

$$4^k[L'(1+\log L')]^{k+1}(2L')^{n-2k} = 2^n(n!L)^{n-k+1}(1+\log n!L)^{k+1}.$$

If $k=1$, we must add (12) to this. Therefore,

$$\begin{aligned} N(R_L^{(1)}) &< 2^n(n!L)^n(1+\log n!L)^2 + n(2L+1)^n \\ N(R_L^{(k)}) &< 2^n(n!L)^{n-k+1}(1+\log n!L)^{k+1}; \qquad 2 \leqq k \leqq \left[\frac{n}{2}\right]. \end{aligned} \tag{20}$$

For large L these expressions can be expanded to yield

$$N(R_L^{(k)}) < 2^n(n!L)^{n-k+1}(\log L)^{k+1}\left\{1+O\left(\frac{1}{\log L}\right)\right\}; \qquad 1 \leqq k \leqq \left[\frac{n}{2}\right]. \tag{21}$$

Summing over k we find

$$N(R_L) < 2^n(n!L)^n(\log L)^2\left\{1+O\left(\frac{1}{\log L}\right)\right\}. \tag{22}$$

Dividing by (10) gives

$$\rho_L < (n!)^n \frac{(\log L)^2}{L}\left\{1 + O\left(\frac{1}{\log L}\right)\right\}, \tag{23}$$

and therefore

$$\lim_{L\to\infty} \rho_L = 0 \tag{24}$$

as asserted.

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INHERITED PROPERTIES OF FUNCTIONS IN QUOTIENT SPACES

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1. Introduction. Let X and X^* be topological spaces with equivalence relations R and R^* respectively. Let X/R and X^*/R^* be given the quotient topologies and suppose that $f: X\to X^*$ is a single valued transformation with the property that xRy always implies $f(x)R^*f(y)$. Then a single valued transformation $f^*: X/R\to X^*/R^*$ is induced and we ask the following question: If $f: X\to X^*$ has property Q, does $f^*: X/R\to X^*/R^*$ also have property Q? In most of the cases discussed in the paper, the answer is in the negative. In this case, a sufficient condition will usually be given to get an affirmative answer.

2. Preliminaries.

DEFINITION 1. *Let X and X^* be sets with equivalence relations R and R^* respectively and let $f: X\to X^*$ be single valued. We term f equivalence-preserving relative to R and R^*, written henceforth as* EP(R, R^*), *iff xRy always implies $f(x)R^*f(y)$.*

DEFINITION 2. *Let $f: X\to X^*$ be* EP(R, R^*). *Then $f^*: X/R\to X^*/R^*$ is defined as follows: $f^*(R[x]) = R^*[f(x)]$.*

LEMMA 1. *If $f: X\to X^*$ is* EP(R, R^*), *then $f^*: X/R\to X^*/R^*$ is single valued.*

The proof is left as an exercise for the reader.

DEFINITION 3. *Let R be an equivalence relation on X and $A \subset X$. A is termed admissible (see [1]) iff A is a union of equivalence classes in X.*

LEMMA 2. *Let* $f: X \to X^*$ *be* EP(R, R^*) *and* A^* *an admissible subset of* X^*. *Then* $f^{-1}[A^*]$ *is an admissible subset of* X.

Proof. Let $a \in f^{-1}[A^*]$ and aRb. Then $f(a)R^*f(b)$ and $f(a) \in A^*$. Then $f(b) \in A^*$ and thus $b \in f^{-1}[A^*]$.

DEFINITION 4. *Let* $f: X \to X^*$ *be single valued with* R *and* R^* *equivalence relations on* X *and* X^* *respectively. We term* f *to be* 1-1(R, R^*) *iff for* $xR'y$, *then* $f(x)R^{*\prime}f(y)$ ($'$ *denotes negation*).

LEMMA 3. *Let* $f: X \to X^*$ *be* 1-1(R, R^*) *and onto. If* A *is an admissible subset of* X, *then* $f[A]$ *is an admissible subset of* X^*.

Proof. Let $a^* \in f[A]$ and $a^*R^*b^*$. Then $f(a) = a^*$ for some $a \in A$ and $f(b) = b^*$ for some $b \in X$. Then aRb and thus $b \in A$. Hence $b^* = f(b) \in f[A]$.

The reader will easily construct an example to show that "onto" is vital to Lemma 3.

LEMMA 4. *Let* $f: X \to X^*$ *be* EP(R, R^*) *and* $P: X \to X/R$ *and* $P^*: X^* \to X^*/R^*$ *the projections. Then* $P^*f = f^*P$ *on* X.

Proof. For $x \in X$, $P^*f(x) = P^*(f(x)) = R^*[f(x)] = f^*(R[x]) = f^*(P(x)) = f^*P(x)$.

LEMMA 5. *Let* $f: X \to X^*$ *be* EP(R, R^*) *and* $\mathcal{A} \subset X/R$. *Then* $f^*[\mathcal{A}] = P^*fP^{-1}[\mathcal{A}]$.

Proof. $f^*[\mathcal{A}] = f^*PP^{-1}[\mathcal{A}]$ since $P: X \to X/R$ is onto
$= P^*fP^{-1}[\mathcal{A}]$ by Lemma 4.

LEMMA 6. *Let* $f: X \to X^*$ *be* EP(R, R^*) *and* $\mathcal{A}^* \subset X^*/R^*$. *Then* $f^{*-1}[\mathcal{A}^*] = Pf^{-1}P^{*-1}[\mathcal{A}^*]$.

The proof follows from Lemma 5.

LEMMA 7. *Let* $f: X \to X^*$ *be* EP(R, R^*), 1-1(R, R^*) *and onto. Then* $f^*: X/R \to X^*/R^*$ *is* 1-1 *and onto*.

The proof is omitted.

LEMMA 8. *Let* $P: X \to X/R$ *be closed and* A *and* B *admissible subsets of* X. *If* A *and* B *are separated, then* $P[A]$ *and* $P[B]$ *are separated*.

Proof. Suppose A and B are separated admissible subsets of X. Let c denote the closure operator in X and in X/R. Now $A \cap cB = \emptyset$ and since A is admissible, $P[cB] \subset \mathcal{C}P[A]$. But $cP[B] \subset P[cB]$ since P is closed, and hence $cP[B] \subset \mathcal{C}P[A]$. Thus $P[A] \cap cP[B] = \emptyset$. Similarly, $P[B] \cap cP[A] = \emptyset$.

Example 1. *$P: X \to X/R$ closed cannot be omitted from Lemma* 8. Let X be the reals with the usual topology and R the equivalence relation induced by the decomposition $A = (-\infty, 0)$, $B = [0, 1)$, and $C = [1, \infty)$. Then A and C are clearly admissible and separated, but $P[A]$ and $P[C]$ are not separated.

3. Continuation. Let X and X^* be topological spaces with equivalence relations R and R^* respectively. For the remainder of the paper X/R and X^*/R^* will always be given the quotient topology and $P: X \to X/R$ and $P^*: X^* \to X^*/R^*$ will denote the projections.

THEOREM 1. *Let $f: X \to X^*$ be* EP(R, R^*) *and continuous, X and X^* being topological spaces. Then $f^*: X/R \to X^*/R^*$ is continuous.*

Proof. Let $\mathcal{O}^*$ be open in X^*/R^*. Then $f^{*-1}[\mathcal{O}^*] = Pf^{-1}P^{*-1}[\mathcal{O}^*]$ by Lemma 6. But $P^{*-1}[\mathcal{O}^*]$ is open and admissible and thus $f^{-1}P^{*-1}[\mathcal{O}^*]$ is open and admissible by Lemma 2 and thus $Pf^{-1}P^{*-1}[\mathcal{O}^*]$ is open.

Example 2. *$f^*: X/R \to X^*/R^*$ does not inherit sequential continuity from $f: X \to X^*$.* Let X be the unit interval with topology consisting of the empty set together with all sets whose complements are countable. Let R be the equivalence relation induced by the decomposition $[0, \frac{1}{2}]$, $(\frac{1}{2}, 1]$ of X. Let X^* be the unit interval with the usual topology and let $R^* = R$. Let $f: X \to X^*$ be the identity transformation. Clearly $f: X \to X^*$ is EP(R, R^*) and is sequentially continuous (if $x_n \to x_0$ in X, then $x_n = x_0$ for $n \geqq N$). Now $X/R = \{R[0], R[1]\}$ with the indiscrete topology and $X^*/R^* = \{R^*[0], R^*[1]\}$ with open sets $\varnothing$, $\{R^*[1]\}$ and X^*/R^*. Then $f^*: X/R \to X^*/R^*$ is clearly not sequentially continuous.

THEOREM 2. *Let $f: X \to X^*$ be* EP(R, R^*), 1-1(R, R^*) *and onto, X and X^* being topological spaces. If $f: X \to X^*$ is open* (*closed*), *then $f^*: X/R \to X^*/R^*$ is open* (*closed*).

Proof. Let $f: X \to X^*$ be open and $\mathcal{O}$ an open subset of X/R. Then $f^*[\mathcal{O}] = P^*fP^{-1}[\mathcal{O}]$ by Lemma 5. Then $P^{-1}[\mathcal{O}]$ is admissible and open and hence $fP^{-1}[\mathcal{O}]$ is open and admissible by Lemma 3; thus $P^*fP^{-1}[\mathcal{O}]$ is open in X^*/R^*. The closed case is proved by substituting closed for open above.

Example 3. *Onto cannot be dropped from Theorem* 2. Let $X = \{a, b\}$ with topology τ: $\varnothing$, $\{a\}$, $\{b\}$, X and $X^* = \{c, d, e\}$ with topology τ^*: $\varnothing$, $\{c\}$, $\{d\}$, $\{c, d\}$, X^*. Suppose R is ordinary equality in X, and R^* is the equivalence relation induced by the decomposition $\{c, e\}$, $\{d\}$ of X^*. Define $f: X \to X^*$ as follows: $f(a) = c$; $f(b) = d$. Then $f: X \to X^*$ is open, EP(R, R^*), 1-1(R, R^*), but not onto. $R[a]$ is open in X/R, but $f^*(R[a]) = R^*[c] = \{c, e\}$ which is not open in X^*/R^*.

THEOREM 3. *Let $f: X \to X^*$ be a homeomorphism and* EP(R, R^*). *Then $f^*: X/R \to X^*/R^*$ is a homeomorphism iff $f: X \to X^*$ is* 1-1(R, R^*).

Proof. Use Theorem 2 and Lemma 7.

THEOREM 4. *Let $f: X \to X^*$ be* EP(R, R^*), *X and X^* being topological spaces. If X^* has the quotient topology* (A^* *is open in* X^* *iff* $f^{-1}[A^*]$ *is open in* X), *then X^*/R^* has the quotient topology determined by $f^*: X/R \to X^*/R^*$.*

Proof. If $\mathcal{O}^* \subset X^*/R^*$ is open, then $f^{*-1}[\mathcal{O}^*]$ is open since f^* is continuous by Theorem 1. Conversely, let $f^{*-1}[\mathcal{O}^*]$ be open. Then $P^{-1}f^{*-1}[\mathcal{O}^*]$ is open and thus $f^{-1}P^{*-1}[\mathcal{O}^*]$ is open by Lemma 4. Then $P^{*-1}[\mathcal{O}^*]$ is open since X^* has the quotient topology and thus $\mathcal{O}^*$ is open.

Example 4. *$f^*: X/R \to X^*/R^*$ does not inherit "compact preserving" from $f: X \to X^*$.* Let X be the reals with the usual topology and X^* the integers with the discrete topology. Let R^* be ordinary equality in X^* and R the equivalence relation induced by the decomposition $\{[n, n+1)\}_{-\infty}^{\infty}$ on X. Suppose $f: X \to X^*$ is defined as follows: $f(x) = n$, where $n \leqq x < n+1$. Then $f: X \to X^*$ is EP(R, R^*), 1-1(R, R^*), onto, and takes compact sets into finite sets in X^* and thus is "compact preserving." We show now that $f^*: X/R \to X^*/R^*$ is not "compact preserving." Let $\mathcal{C} = \{R[n] \mid n = 0, -1, -2, \cdots\}$ in X/R. Then $f^*[\mathcal{C}] = \{R^*[n] \mid n = 0, -1, -2, \cdots\}$ which is certainly not compact in X^*/R^*. But $\mathcal{C}$ is compact in X/R, for if $R[0] \in \mathcal{O} \subset X/R$, where $\mathcal{O}$ is open, then $R[0] \subset P^{-1}[\mathcal{O}]$ or $[0, 1) \subset P^{-1}[\mathcal{O}]$. Since $P^{-1}[\mathcal{O}]$ is open in X, $(-\epsilon, \epsilon) \subset P^{-1}[\mathcal{O}]$ for some $0 < \epsilon < 1$. Then $R[-1] = P(-1) = P(-\epsilon/2) \in PP^{-1}[\mathcal{O}] = \mathcal{O}$. By induction $R[n] \in \mathcal{O}$ for $n = 0, -1, -2, \cdots$. Note that in this example, $P: X \to X/R$ is not a closed mapping, since $\{0\}$ is closed in X, but $P(0) = R[0] = [0, 1)$ is not closed in X/R.

THEOREM 5. *Let $f: X \to X^*$ be* EP(R, R^*), *X being a compact topological space and X^* a topological space. Suppose further that X/R is Hausdorff. If $f: X \to X^*$ is "compact preserving," then $f^*: X/R \to X^*/R^*$ is "compact preserving."*

Proof. Let $\mathcal{C} \subset X/R$ be compact. Since $f^*[\mathcal{C}] = P^*fP^{-1}[\mathcal{C}]$ by Lemma 5, it suffices to show that $P^{-1}[\mathcal{C}]$ is compact. Now $\mathcal{C}$ is closed since X/R is Hausdorff and thus $P^{-1}[\mathcal{C}]$ is closed since $P: X \to X/R$ is continuous. Since X is compact, $P^{-1}[\mathcal{C}]$ is compact, which completes the proof.

Example 5. *$f^*: X/R \to X^*/R^*$ does not inherit "connected preserving" from $f: X \to X^*$.* Let X be the rationals with the relative topology and $X^* = \{a^*, b^*\}$ with the discrete topology. Let R be the equivalence relation on X defined as follows: xRy iff $x, y \leqq 0$ or $x, y > 0$. Let R^* be ordinary equality in X^*. Let $f: X \to X^*$ be defined as follows:

$$f(x) \equiv \begin{cases} a^* & \text{if} \quad x \leqq 0 \\ b^* & \text{if} \quad x > 0. \end{cases}$$

Clearly $f: X \to X^*$ is EP(R, R^*), 1-1(R, R^*), onto, and "connected preserving" since X is totally disconnected. But $f^*: X/R \to X^*/R^*$ is not "connected preserving" since X/R is connected, but X^*/R^* is not.

THEOREM 6. *Let $f: X \to X^*$ be* EP(R, R^*), 1-1(R, R^*), *and onto, X and X^* being topological spaces. If $P: X \to X/R$ is closed and $f^{-1}[A^*]$, $f^{-1}[B^*]$ are separated whenever A^* and B^* are separated in X^*, then $f^*: X/R \to X^*/R^*$ is "connected preserving."*

Proof. Let $\mathcal{C} \subset X/R$ be connected. Suppose $f^*[\mathcal{C}] = \mathcal{A}^* \cup \mathcal{B}^*$, where $\mathcal{A}^*$ and $\mathcal{B}^*$ are nonempty separated sets. Then $\mathcal{C} = f^{*-1}f^*[\mathcal{C}]$ since f^* is 1-1 by Lemma 7, and

$$\begin{aligned} \mathcal{C} &= f^{*-1}[\mathcal{A}^*] \cup f^{*-1}[\mathcal{B}^*] \\ &= Pf^{-1}P^{*-1}[\mathcal{A}^*] \cup Pf^{-1}P^{*-1}[\mathcal{B}^*] \end{aligned}$$

by Lemma 6. Since $f: X \to X^*$ is onto, it follows that $Pf^{-1}P^{*-1}[\mathcal{A}^*]$ and $Pf^{-1}P^{*-1}[\mathcal{B}^*]$ are nonempty. Now $P^{*-1}[\mathcal{A}^*]$ and $P^{*-1}[\mathcal{B}^*]$ are clearly admissible separated sets and thus $f^{-1}P^{*-1}[\mathcal{A}^*]$ and $f^{-1}P^{*-1}[\mathcal{B}^*]$ are admissible separated sets by Lemma 2. Then $Pf^{-1}P^{*-1}[\mathcal{A}^*]$ and $Pf^{-1}P^{*-1}[\mathcal{B}^*]$ are separated by Lemma 8 and thus $\mathcal{C}$ is disconnected, a contradiction.

Example 6. *$f^*: X/R \to X^*/R^*$ does not inherit "separation preserving" from $f: X \to X^*$.* Let $X = \{a, b\}$ with the discrete topology and R be ordinary equality. Let $X^* = \{a, b\}$ with discrete topology and xR^*y for all x, y in X^*. Let $f: X \to X^*$ be the identity map. Certainly $f: X \to X^*$ is EP(R, R^*) and separation preserving, but $f^*: X/R \to X^*/R^*$ is not separation preserving. (Note further that $P^*: X^* \to X^*/R^*$ is closed and that $f: X \to X^*$ is a homeomorphism, but that f is not 1-1(R, R^*).)

THEOREM 7. *Let $f: X \to X^*$ be* EP(R, R^*), 1-1(R, R^*) *and onto, X and X^* being topological spaces. If $P^*: X^* \to X^*/R^*$ is closed and $f: X \to X^*$ is "separation preserving," then $f^*: X/R \to X^*/R^*$ is "separation preserving."*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be separated subsets of X/R. Then $P^{-1}[\mathcal{A}]$ and $P^{-1}[\mathcal{B}]$ are admissible separated subsets of X and hence $fP^{-1}[\mathcal{A}]$ and $fP^{-1}[\mathcal{B}]$ are separated admissible subsets of X^* by Lemma 3. Then $P^*fP^{-1}[\mathcal{A}]$ and $P^*fP^{-1}[\mathcal{B}]$ are separated by Lemma 8. It follows then that $f^*[\mathcal{A}]$ and $f^*[\mathcal{B}]$ are separated by Lemma 5.

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AXIOMS FOR COMPLEX NUMBERS

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1. Introduction. We seldom see a direct axiomatic approach to the system of complex numbers. Perhaps the first to attempt such an approach was Huntington. In his paper of 1905 [1] he gave a system with 27 axioms which can be summarized briefly as follows. The complex number system is a system $\langle C, R, +, \cdot, <\rangle$ such that (i) $\langle C, +, \cdot,\rangle$ is a field; (ii) $\langle R, +, \cdot\rangle$ is a subfield of $\langle C, +, \cdot\rangle$; (iii) $\langle R, +, \cdot, <\rangle$ is a complete ordered field (i.e. is the real number system); (iv) $C=R[i]$, where $-i^2$ is the unit of R.

A topological characterization of the complex numbers is to be found in Pontryagin ([2] p. 173) who states that

A connected locally compact topological field F which is second countable is either (isomorphic to) the real field or the complex field.

By adding a suitable condition, say that F is identical with the set of its squares, we get a topological characterization of the complex numbers.

There is also the approach via the Mazur-Gelfand Theorem in Banach algebras. For example (Rickart [3], p. 39):

A complete normed field over the reals is (isomorphic to) the real field or complex field.

It is our purpose in this note to present another characterization of the complex numbers which is more algebraic in nature than that of Pontryagin and yet does not assume the prior existence of the notion of the real number system as is inherent in the axiom system of Huntington or in the Mazur-Gelfand approach. It is clear that a purely algebraic definition does not seem possible, at least if one regards completeness as a nonalgebraic notion. And we can hardly expect any definition which doesn't have the real number system lurking rather closely in the background. We present the following system as being perhaps somewhat better than other systems on these two points.

2. Definition of the complex number system. We consider a system, $\langle F, +, \cdot, '\rangle$ satisfying the following requirements.

(i) $\langle F, +, \cdot\rangle$ is a field such that $F=\{x^2: x\in F\}$;

(ii) $'$ is a function from F to F such that (1) $(a+b)'=a'+b'$; (2) $(ab)'=a'b'$; (3) $a''=a$; (4) for each nonzero a, b there exists $c\neq 0$ such that $aa'+bb'=cc'$; (5) $a=a'$ implies that there exists c such that $a^2=(cc')^2$;

(iii) if $R=\{x: x=x'\}$ and $P=\{x: \exists y(y\neq 0)\&x=yy'\}$ and A, B are nonempty subsets of R such that $A-B=\{a-b: a\in A \;\&\; b\in B\}\subseteq P$, then there exists c in R such that $A-c\subseteq\overline{P}$ and $c-B\subseteq\overline{P}$, where $\overline{P}=P\cup\{0\}$.

Before giving the demonstration of the sufficiency of this set of conditions we point out the following. The additional condition in (i) is to insure that we get the complex numbers rather than the reals. The operation $'$ of (ii) is obviously *complex conjugation*. The condition (iii) is the needed completeness condition and is essentially the one used by Veblen [4] in a characterization of the real number system.

3. Adequacy of the definition. We prove the following.

THEOREM 1. *R is a subfield of F.*

Proof. By (1), (2) of (ii) R is closed with respect to addition and multiplication. Using (3) we have $0'=(00')'=0'0''=0'0=0$ and $a1'=(a'1)'=a''=a$. So 0, 1 are in R. Furthermore $a+(-a)=0$ implies $a'+(-a)'=0$ and $aa^{-1}=1$ implies $a'(a^{-1})'=1$. Thus if $a\in R$ then $-a=(-a)'$ and $(a^{-1})'=a^{-1}$; therefore $-a$ and a^{-1} are in R.

THEOREM 2. *P orders R.*

Proof. Using (3) it is easy to show that $P\subseteq R$. Closure of P with respect to addition and multiplication is a consequence of (4) and (2) respectively. Now $yy'=0$ implies $y=0$ or $y'=0$. But $y'=0$ implies $y=y''=0$. So $0\notin P$. Since $x+(-x)=0$ and P is closed with respect to addition, $x\in P$ implies $-x\notin P$. From (5) $x\in R$ implies there exists c such that $x^2=(cc')^2$. Thus $x=cc'$ or $x=-cc'$. If $x\neq 0$, then c, $c'\neq 0$. And so $x\in R$ implies $x\in P$ or $-x\in P$.

THEOREM 3. *$F=R+iR$, where $i^2=-1$.*

Proof. By (i) there is an element i such that $i^2=-1$. Furthermore, since $i^2=-1\notin P$, $i\notin R$. And so from $i^2=(i')^2$ we have $i'=-i$. Also, from $1+1=2$ we get $2'=2$ and $(1/2)'=1/2$. For each $z\in F$ we have

$$z = \tfrac{1}{2}(z + z') + \tfrac{1}{2}(z - z') = \tfrac{1}{2}(z + z') + i\cdot\tfrac{1}{2}(z' - z)i.$$

Now $(\frac{1}{2}(z+z'))'=\frac{1}{2}(z'+z)$ and $(\frac{1}{2}(z'-z)i)'=\frac{1}{2}(z'-z)i$; thus both are in R. Furthermore, if u, v, x, $y\in R$ and $u+iv=x+iy$, then $u-x=i(y-v)\in R$. Therefore $i(y-v)=(i(y-v))'=-i(y-v)$. And so $y-v=-(y-v)$, i.e. $v=y$. Finally $u=x$.

THEOREM 4. *R is complete.*

Proof. One easily gets the usual Dedekind postulate from (iii).

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ON FERMAT'S LAST THEOREM

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In this note we shall prove two theorems which pertain to Fermat's Last Theorem.

THEOREM 1. *If p and $2p+1$ are odd primes and*

$$a^p + b^p + c^p = 0 \tag{1}$$

where a, b, c are nonzero, pairwise prime integers, then precisely one of the integers a, b, c is divisible by $2p+1$.

Proof. Assume that $(abc, 2p+1)=1$; then from Fermat's theorem we have

$$a^{2p} \equiv b^{2p} \equiv c^{2p} \equiv 1 \pmod{2p + 1}.$$

Now (1) can be written in the form $a^p+b^p=-c^p$. Squaring both sides of the above relation and applying Fermat's theorem, we obtain

$$2a^pb^p \equiv -1 \pmod{2p + 1}.$$

Thus, $4a^{2p}b^{2p}\equiv 1 \pmod{2p+1}$ so that $4\equiv 1 \pmod{2p+1}$. Since p and $2p+1$ are odd primes, this congruence cannot be satisfied. Thus, since the integers a, b, c are pairwise prime, exactly one of them must be divisible by $2p+1$, which was to be shown.

THEOREM 2. *Under the same hypothesis as in Theorem 1, precisely one of the integers a, b, c is divisible by p.*

Proof. From Theorem 1, exactly one of the integers a, b, c is divisible by $2p+1$. Assume that $(abc, p)=1$, so that it suffices to consider only one case. Thus, let c be divisible by $2p+1$ so that from (1) we have

$$a^p + b^p \equiv 0 \pmod{2p + 1}.$$

Now,

$$a^p + b^p = (a + b)(a^{p-1} - a^{p-2}b + \cdots - ab^{p-2} + b^{p-1})$$

so that

$$(a + b)(a^{p-1} - a^{p-2}b + \cdots + b^{p-1}) \equiv 0 \pmod{2p + 1}. \tag{2}$$

Legendre has shown (see L. E. Dickson's *History of the Theory of Numbers*, Vol. II, page 734) for the case $(abc, p)=1$ that the integers a, b, c must satisfy the following relations:

(3a)	$a + b = c_0^p$	(4a)	$a^{p-1} - a^{p-2}b + \cdots + b^{p-1} = -c_1^p$
(3b)	$c + b = a_0^p$	(4b)	$c^{p-1} - c^{p-2}b + \cdots + b^{p-1} = -a_1^p$
(3c)	$c + a = b_0^p$	(4c)	$c^{p-1} - c^{p-2}a + \cdots + a^{p-1} = -b_1^p$

where $a=a_0a_1$, $b=b_0b_1$, and $c=c_0c_1$, and where the integers a_0, a_1, b_0, b_1, c_0, c_1 are pairwise prime. Since $(a, 2p+1)=(b, 2p+1)=1$, we see from Fermat's theorem that

$$a_0^{2p} \equiv b_0^{2p} \equiv 1 \pmod{2p+1}.$$

Thus, since $2p+1$ divides c, we have from (3b) and (3c)

$$a^2 \equiv b^2 \equiv 1 \pmod{2p+1} \tag{5}$$

so that

$$(a+b)(a-b) \equiv 0 \pmod{2p+1}.$$

When $a-b$ is divisible by $2p+1$, we see from (2) that

$$2a^p \equiv 0 \pmod{2p+1}.$$

Since p is an odd prime and since $(a, 2p+1)=1$, this congruence cannot be satisfied. Thus, $(a-b, 2p+1)=1$. When $a+b$ is divisible by $2p+1$, c_0 is divisible by $2p+1$ from (3a). In this case, since $(c_0, c_1)=1$, we see from (4a) that

$$pa^{p-1} \equiv pb^{p-1} \equiv -c_1^p \pmod{2p+1}.$$

Squaring both sides of this congruence and making use of (5), we obtain

$$p^2 \equiv 1 \pmod{2p+1}.$$

Since p and $2p+1$ are odd primes, this congruence cannot be satisfied. This contradiction shows that $(abc, p) \neq 1$. Thus, since the integers a, b, c are pairwise prime, exactly one of them must be divisible by p, which was to be shown.

A REMARK CONCERNING PROJECTIONS IN SUMMABILITY DOMAINS

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Let A be a conservative summability matrix, that is, a matrix such that $Ax \in c$ for every $x \in c$, where c is the Banach space of all convergent sequences. Let m be the Banach space of all bounded sequences (with the sup as norm) and let c_A be the subspace of m consisting of all the bounded sequences summed by A. (We say that a sequence x is summed by A if $Ax \in c$.) Clearly c_A is a closed subspace of m containing c. It may happen that $c_A = c$; take for example the identity matrix as A. If $c_A \neq c$ then, as is well known, c_A is not separable and hence it is "much larger" than c. Wilansky [4, problem VI] conjectured that if $c_A \neq c$ there is not even a projection from c_A onto its subspace c. In this remark we show that this is indeed the case. By projection we mean a bounded linear projection.

THEOREM. *Let A be a conservative matrix, and let $c_A = A^{-1}(c) \cap m$. Then either $c_A = c$ or there is no projection from c_A onto c.*

LEMMA. *Let A be a conservative matrix. A sums a bounded divergent sequence if and only if there exists, for every $\epsilon>0$ and every integer n, a sequence $x=(x_1, x_2, \cdots)\in c$ such that*

(i) $\|x\|=1$, $\|Ax\|\leqq\epsilon$,
(ii) *the support $s(x)$ of x (that is, the set of i for which $x_i\neq 0$) is finite,*
(iii) $\{1, 2, \cdots, n\}\cap s(x)=\emptyset$.

The "if" part of the lemma is trivial. The "only if" part follows immediately from the construction given in the proof of Theorem 1 of Berg [1].

Proof of the Theorem. Suppose that $c_A\neq c$. We recall that c_0 denotes the subspace of m consisting of those sequences which converge to zero. By the lemma there exist $x^{(n)}\in c_0$, $n=1, 2, \cdots$ such that

$$s(x^{(n)})\cap s(x^{(m)})=\emptyset \qquad \text{if } n\neq m, \tag{1}$$

$$\|x^{(n)}\|=1, \tag{2}$$

$$\|Ax^{(n)}\|\leqq 2^{-n}. \tag{3}$$

Let X be the subspace of m consisting of all the sequences of the form $\sum_{n=1}^{\infty}\lambda_n x^{(n)}$ with $\sup_n|\lambda_n|<\infty$. By (3) $X\subset c_A$ and by (1) and (2) the mapping $\sum\lambda_n x^{(n)}\rightarrow(\lambda_1, \lambda_2, \cdots)$ is an isometry from X onto m. Let Y be the subspace of X consisting of all the sequences $\sum\lambda_n x^{(n)}$ with $\lambda_n\rightarrow 0$. The isometry from X onto m takes Y onto c_0. Hence, by a well-known result of Phillips [2], there is no projection from X onto Y. Suppose now that there were a projection P from c_A onto c; $c\supset Y$ and Y is isometric to c_0. By a result of Sobczyk [3] there exists a projection, say Q, from c onto Y. QP is thus a projection from c_A onto Y. The restriction of QP to X is therefore a projection from X onto Y and this is a contradiction.

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SOME SUMMATION THEOREMS ON THE HAHN POLYNOMIALS

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This paper is concerned with the summation of a finite series whose terms are Hahn polynomials. The Hahn polynomials form a family of orthogonal polynomials. They were introduced by Hahn [2], discussed by Weber and Erdélyi [4] and further discussed by Karlin and McGregor [3]. They also appear briefly in Erdélyi [1]. The Hahn polynomials may be defined in terms

of the generalized hypergeometric series

$$(1) \qquad {}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{l=0}^{\infty} \frac{(a_1)_l(a_2)_l(a_3)_l z^l}{(b_1)_l(b_2)_l l!},$$

where $(a)_0 = 1$, $(a)_l = a(a+1) \cdots (a+l-1)$ for $l \geqq 1$. The series terminates if one of the a_i is zero or a negative integer. For real $a > -1$, $b > -1$ and for positive integral M, the Hahn polynomials $Q_m(x) = Q_m(x; a, b, M)$, $m = 0, 1, 2, \cdots, M-1$ are defined [4] by

$$(2) \qquad \begin{aligned} Q_m(x) &\equiv Q_m(x; a, b, M) \\ &= {}_3F_2(-m, -x, m+a+b+1; a+1, -M+1; 1) \\ &= \sum_{l=0}^{m} \frac{(-m)_l(-x)_l(m+a+b+1)_l}{(a+1)_l(-M+1)_l l!}. \end{aligned}$$

Several special values of (2) are:

$$(3) \qquad Q_m(0; a, b, M) = 1, \qquad Q_0(x; a, b, M) = 1,$$

$$(4) \qquad Q_m(M-1; a, b, M) = (-1)^m \binom{m+b}{m} \Big/ \binom{m+a}{m}.$$

The Hahn polynomials also satisfy the following difference equation (see [3]),

$$(5) \quad -\omega_m Q_m(x) = D(x)Q_m(x-1) - [B(x) + D(x)]Q_m(x) + B(x)Q_m(x+1),$$

where

$$\begin{aligned} B(x) &= (M-1-x)(a+1+x), \\ D(x) &= x(M+b-x), \\ \omega_m &= m(m+a+b+1), \end{aligned}$$

and (5) is valid for $m = 0, 1, \cdots, M-1$ and all complex values of x. A further discussion and listing of properties of the Hahn polynomials not immediately relevant to the following discussion can be found in [3].

THEOREM 1. *If $Q_m(x) = Q_m(x; a, b, M)$ are the Hahn polynomials of* (2), *$a > -1$, $b > -1$, M is a positive integer, and $m = 0, 1, \cdots, M-1$ then*

$$(6) \qquad \begin{aligned} \sum_{x=x_1}^{x_2} Q_m(x) = \big[&B(x_2)[Q_m(x_2) - Q_m(x_2+1)] + (a+b)(x_2+1)Q_m(x_2) \\ &- B(-1)Q_m(x_2) - \{B(x_1-1)[Q_m(x_1-1) - Q_m(x_1)] \\ &+ (a+b)x_1 Q_m(x_1-1) \\ &- B(-1)Q_m(x_1-1)\}\big]/(m+1)(m+a+b). \end{aligned}$$

Proof. From (5) we note that

$$B(-1) = aM$$

and

$$D(x) = bx + x(M - x) = (a + b)x + B(x - 1) - B(-1);$$

thus we can write the difference equation (5) as

$$\begin{aligned} -\omega_m Q_m(x) &= [(a + b)x + B(x - 1) - B(-1)]Q_m(x - 1) \\ &\quad - [B(x) + (a + b)x + B(x - 1) - B(-1)]Q_m(x) + B(x)Q_m(x + 1), \\ &= B(x - 1)[Q_m(x - 1) - Q_m(x)] - B(x)[Q_m(x) - Q_m(x + 1)] \\ &\quad + (a + b)x[Q_m(x - 1) - Q_m(x)] - B(-1)[Q_m(x - 1) - Q_m(x)]. \end{aligned} \tag{7}$$

Note that $(a+b)x[Q_m(x-1)-Q_m(x)]=(a+b)[xQ_m(x-1)-(x+1)Q_m(x)+Q_m(x)]$, and therefore

$$\begin{aligned} -\omega_m Q_m(x) &= -\Delta\{B(x - 1)[Q_m(x - 1) - Q_m(x)]\} - \Delta[(a + b)xQ_m(x - 1)] \\ &\quad + (a + b)Q_m(x) + B(-1)\Delta Q_m(x - 1). \end{aligned}$$

Hence

$$\begin{aligned} &(\omega_m + a + b)Q_m(x) \\ &= \Delta\{B(x - 1)[Q_m(x - 1) - Q_m(x)] + (a + b)xQ_m(x - 1) - B(-1)Q_m(x - 1)\} \end{aligned}$$

and

$$\begin{aligned} \sum_{x=x_1}^{x_2} Q_m(x) = {} & [B(x_2)[Q_m(x_2) - Q_m(x_2 + 1)] + (a + b)(x_2 + 1)Q_m(x_2) \\ & - B(-1)Q_m(x_2) - \{B(x_1 - 1)[Q_m(x_1 - 1) - Q_m(x_1)] \\ & + (a + b)x_1Q_m(x_1 - 1) \\ & - B(-1)Q_m(x_1 - 1)\}]/(m + 1)(m + a + b), \end{aligned}$$

since $\omega_m+a+b=(m+1)(m+a+b)$.

COROLLARY 2.

$$\begin{aligned} &\sum_{x=0}^{i} Q_m(x) \\ &= \frac{B(i)[Q_m(i) - Q_m(i + 1)] + (a + b)(i + 1)Q_m(i) + B(-1)[1 - Q_m(i)]}{(m + 1)(m + a + b)}. \end{aligned} \tag{8}$$

Proof. The proof follows immediately from Theorem 1 noting from (3) the special value $Q_m(0)=Q_m(0; a, b, M)=1$.

COROLLARY 3.

$$\sum_{x=0}^{M-1} Q_m(x) = \frac{M}{(m+1)(m+a+b)}\left[\frac{b(-1)^m\binom{m+b}{m}}{\binom{m+a}{m}} + a\right]. \tag{9}$$

Proof. From Corollary 2 we have

$$\sum_{x=0}^{M-1} Q_m(x) = \frac{M[bQ_m(M-1)+a]}{(m+1)(m+a+b)}$$

noting from (5) that $B(m-1)=0$, and $B(-1)=aM$. Further from (4) we have the special value $Q_m(M-1;\, a,\, b,\, M)=(-1)^m\binom{m+b}{m}/\binom{m+a}{m}$, and hence the result.

In the above expressions some interesting results are obtained if either a or b or both are zero. Of these, we mention the following:

COROLLARY 4. *If* $a=0$, *then*

$$\sum_{x=0}^{M-1} Q_m(x) = \sum_{x=0}^{M-1} Q_m(x;0,b,M) = \frac{M(-1)^m(b)_m}{(m+1)!}, \tag{10}$$

where $(b)_m=b(b+1)\cdots(b+m-1)$.

Proof. The result follows from Corollary 3 because $(m+b-1)!/(b-1)! = (b)_m$.

If in (10) b also is zero then we have the following further result.

COROLLARY 5. *If* $a=b=0$, *then for* $m\neq 0$,

$$\sum_{x=0}^{M-1} Q_m(x) = \sum_{x=0}^{M-1} Q_m(x;\,0,\,0,\,M) = 0. \tag{11}$$

Proof. The proof follows directly from Corollary 3.

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AN INEQUALITY WHICH INCLUDES THAT OF KANTOROVICH

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Let f be a measurable function such that $0<m\leqq f(x)\leqq M$ on a set of unit measure. Integrating the inequality $(f-m)(f-M)/f\leqq 0$ gives:

$$\int fdx + mM\int (1/f)dx \leqq m + M.$$

To illustrate the use of this theorem, denote the second term on the left-hand side by u; then

$$u\int fdx \leqq (m+M)u - u^2 \leqq (m+M)^2/4,$$

which is the inequality of Kantorovich.

A bibliography on the latter result was given by Peter Henrici: *Two remarks on the Kantorovich Inequality*, this MONTHLY, 68 (1961) 904–906.

REPRESENTATION OF DIHEDRAL GROUPS BY MEANS OF PERSPECTIVITIES

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Let L_0 be a directed line in a Euclidean plane and let any other line L_i of the same plane be directed by the induced direction of L_0 when the latter is rotated around any of its points, positively through an angle α with $0\leqq\alpha\leqq\pi$ until it becomes parallel to L_0. Thus, every line L_i in the plane is directed uniquely.

Let a range of n points $(A_0, A_1, A_2, \cdots, A_{n-1})$ lie on a line L_0. We indicate by $(A_0, \cdots, A_{n-1})$ the *ordered* range of these points ordered compatibly with the direction of L_0. In what follows we consider only ordered ranges with $n\geqq 3$.

If a range of n points $(A_0, A_1, \cdots, A_{n-1})$ on the line L_0 is projected from a point O, not on L_0, and the pencil obtained cut by a secant L_i, not passing through O, then on L_i there is obtained a range of points perspective with the given range. We denote each image point on L_i by the corresponding original letter (see Fig. 1).

The two image ranges on L_i and L_j, in perspective with the range (A_0, A_1, A_2, A_3) on L_0, are respectively (A_0, A_1, A_2, A_3) and (A_2, A_1, A_0, A_3).

We define two ranges of points as being equal if their ordered ranges have the same representation. All ranges on L_i lines are equal if they lie in the same half plane generated by a line parallel to L_0 through O. The ranges on L_i lines in different half planes are reversed, and therefore different.

THEOREM 1. *On the lines parallel to a given line S for a given center of perspectivity O, there exist only two different ranges perspective with a range on a given line L.*

Proof. Consider a range of n points $A_0, \cdots, A_{n-1}$, on a line L, a point O, not on L, and a line S, not through O and not parallel to any OA_k, in the plane. Projecting $A_0, \cdots, A_{n-1}$, from O we obtain on S or any of its parallels a range perspective with the range on L which will be equal to the given range or reversed and hence different from the range on L, depending on the position of O with respect to L. No other perspective ranges are possible. Since O, L, and S were taken arbitrarily, not subject to any conditions, the theorem is proved.

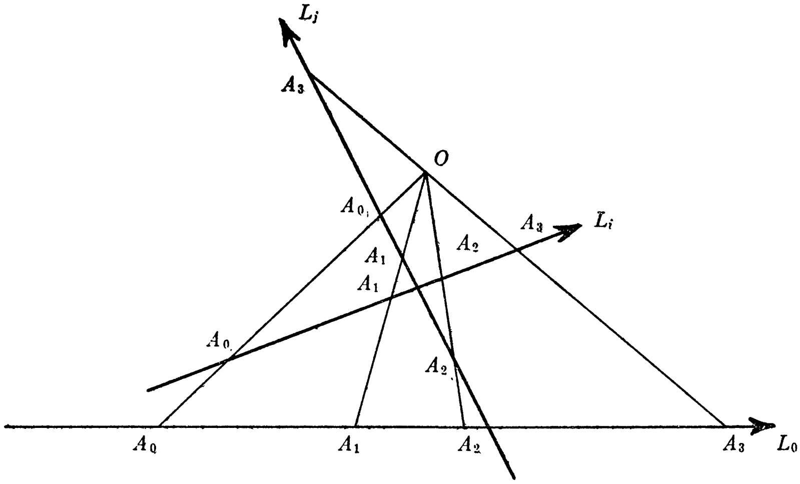

FIG. 1

THEOREM 2. *There are $2n$ different ranges derived by means of perspectivities from a given range of n points.*

Proof. Let S rotate continuously around one of its points, in its rotation intersecting L in the sequence $A_0, A_1, \cdots, A_{n-1}$. To simplify things, without loss of generality, we assume: L is horizontal; the center of perspectivity O, when it lies to the left of S, lies in the upper half plane generated by L and between S and the parallel to it through the next point to the left of S; when O lies to the right of S, it lies in the upper half plane between S and the parallel to it through the next point to the right of S. For each position of S there are, according to Theorem 1, only two different ranges perspective with $A_0, A_1, \cdots, A_{n-1}$. As S crosses the point A_k, the range obtained from the left lying O becomes $A_{k+1}, A_{k+2}, \cdots, A_k$. The range obtained from the right lying O becomes $A_{k+1}, A_k, \cdots, A_{k+2}$. There being n points, we obtain $2n$ different ranges.

Theorem 2 may also be proved in the following way.

DEFINITION. *In a range $(A_0, A_1, \cdots, A_{n-1})$ two points, A_i and A_j are called adjacent if $i=j\pm 1 \pmod n$.*

Clearly, if $n \geqq 3$ every point of $(A_0, A_1, \cdots, A_{n-1})$ has two distinct adjacents. Thus, the adjacents of A_1 are A_0 and A_2. The adjacents of A_{n-1} are A_{n-2} and A_0.

LEMMA 1. *If* $(A_0, A_1, \cdots, A_{n-1})$ *is in perspective with* $(\cdots, A_i, \cdots)$ *then in the latter the adjacents of* A_i *are again* A_j, *where* $i = j \pm 1 \pmod{n}$, *i.e., adjacency is preserved under perspectivity.*

Proof. The proof is trivial for $n = 1, 2$, or 3. For $n > 3$ we prove the Lemma by induction. Thus assuming the Lemma true for $n = p > 3$, we prove it for $n = p + 1$.

Let $(A_0, \cdots, A_p)$ be in perspective with $(\cdots, A_i, \cdots)$; it is enough to show that in $(\cdots, A_i, \cdots)$ the adjacency is preserved with respect to A_p.

According to our induction hypothesis adjacency is preserved between

$$(A_0, \cdots, A_{p-1}) \quad \text{and} \quad (\cdots, A_i, \cdots) - (A_p),$$

where in the above $(\cdots, A_i, \cdots) - (A_p)$ indicates that the point A_p is deleted from $(\cdots, A_i, \cdots)$. On the other hand, again by our induction hypothesis, adjacency is preserved between

$$(A_1, \cdots, A_p) \quad \text{and} \quad (\cdots, A_i, \cdots) - (A_0). \tag{1}$$

Therefore, in $(\cdots, A_i, \cdots)$ neither A_0 nor A_p can have both of their adjacents belonging to the set $\{A_1, \cdots, A_{p-1}\}$, hence one of the adjacents of A_p must be A_0, and since (1) shows that A_{p-1} is an adjacent of A_p, consequently the two adjacents of A_p in $(\cdots, A_i, \cdots)$ are A_0 and A_{p-1}, as desired. Thus Lemma 1 is proved.

LEMMA 2. *If adjacency is preserved between* $(A_0, A_1, \cdots, A_{n-1})$ *and* $(A_{k_0}, A_{k_1}, \cdots, A_{k_{n-1}})$ *then* $(A_0, A_1, \cdots, A_{n-1})$ *is perspective with an ordered range equal to* $(A_{k_0}, A_{k_1}, \cdots, A_{k_{n-1}})$.

Proof. Let O be a point not on the line passing through A_0 and A_1. It is always possible to find a secant line which cuts the pencil $OA_0, OA_1, \cdots, OA_{n-1}$ in such a way that A_{k_0} and A_{k_1} are respectively the images of A_0 and A_1 and where the direction from A_{k_0} to A_{k_1} coincides with the direction of the secant L. Since adjacency is preserved between $(A_0, A_1, \cdots, A_{n-1})$ and $(A_{k_0}, A_{k_1}, \cdots, A_{k_{n-1}})$ it is obvious that the range obtained on the secant line L is equal to $(A_{k_0}, A_{k_1}, \cdots, A_{k_{n-1}})$. Thus Lemma 2 is proved.

From Lemmas 1 and 2 it follows that:

THEOREM 1′. *A range of* $(A_0, A_1, \cdots, A_{n-1})$ *is perspective with a range equal to* $(A_{k_0}, A_{k_1}, \cdots, A_{k_{n-1}})$ *if and only if adjacency is preserved between the two ranges.*

THEOREM 2′. *There are only* $2n$ *distinct ranges perspective with a given range* $(A_0, A_1, \cdots, A_{n-1})$.

Proof. Since there are n choices for the first letter of a range perspective with $(A_0, A_1, \cdots, A_{n-1})$ and since for $n \geqq 3$ there are two distinct adjacents to each A_i, hence by Theorem 1′ there are altogether $2n$ distinct ranges perspective with $(A_0, A_1, \cdots, A_{n-1})$, Q.E.D.

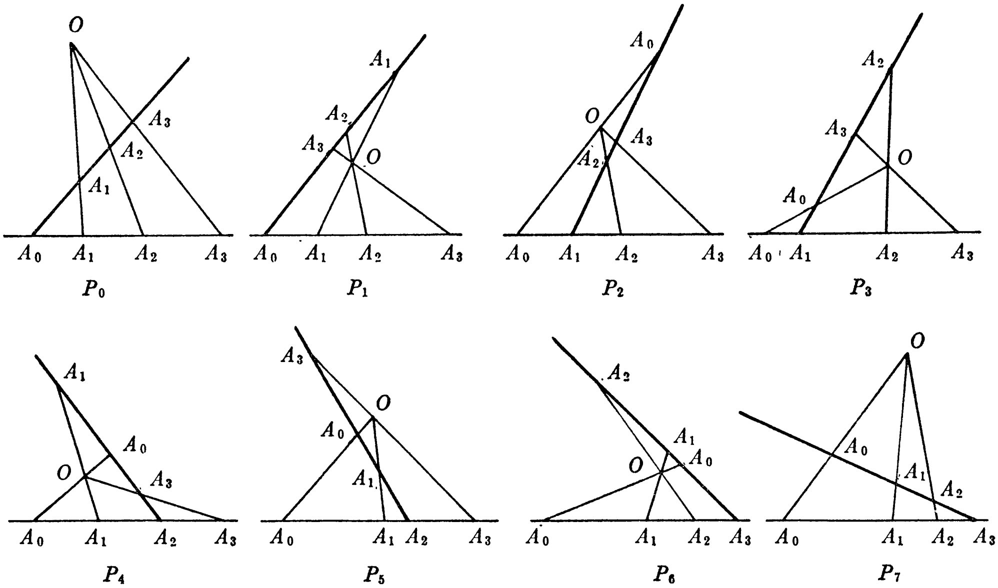

FIG. 2

Figure 2 gives eight different ranges obtained from the given range (A_0, A_1, A_2, A_3).

Product of two ranges. Each perspectivity determines uniquely the range obtained on the secant line. Consequently, we may identify the range determined by the perspectivity with the perspectivity itself, and denote them by the same letter, P_i.

The $2n$ ranges obtained by $2n$ perspectivities are:

$$
\begin{aligned}
P_0 &= A_0, A_1, \cdots, A_{n-1} \\
P_1 &= A_0, A_{n-1}, \cdots, A_1 \\
&\cdots\cdots\cdots\cdots \\
P_{2n-2} &= A_{n-1}, A_n, \cdots, A_{n-2} \\
P_{2n-1} &= A_{n-1}, A_{n-2}, \cdots, A_0
\end{aligned}
\tag{2}
$$

in which the ranges P_{2m} and P_{2m+1} are given by

$$P_{2m} = A_m, A_{m+1}, \cdots, A_{m+n-1} \tag{3}$$

$$P_{2m+1} = A_m, A_{m-1}, \cdots, A_{m-n+1} \tag{4}$$

for $m=0, 1, 2, \cdots, n-1$, and where the subscripts of the P's are integers modulo $2n$ and those of the A's are integers modulo n. (3) and (4) represent two typical ranges.

Now consider P_0 and P_{2m}. The range P_{2m} is the image of the range P_0 by applying to it the perspectivity P_{2m}. Replacing P_0 by any other range as given by (2), (3) represents the image of this other range. In other words, the image of a range P_{2i} as given by (2), applying to it the perspectivity P_{2m} is obtained by adding m to the subscripts of P_{2i}. This may be indicated by

$$P_{2m}(\cdots, A_i, \cdots) = (\cdots, A_{m+i}, \cdots). \tag{5}$$

Similarly, considering P_0 and P_{2m+1}, we may write

$$P_{2m+1}(\cdots, A_i, \cdots) = (\cdots, A_{m-i}, \cdots). \tag{6}$$

Next we define the product of two perspectivities P_j and P_k to be the perspectivity P_h which applied to any range of n points $(\cdots, A_i, \cdots)$ yields the same image which is obtained by first applying to $(\cdots, A_i, \cdots)$ the perspectivity P_j, and then by applying to the latter the perspectivity P_k, i.e., we define $P_h = P_j P_k$ by

$$P_h(\cdots, A_i, \cdots) = P_k(P_j(\cdots, A_i, \cdots)). \tag{7}$$

We consider the four products of the two typical perspectivities given by $P_{2m}P_{2k}$, $P_{2m+1}P_{2k}$, $P_{2m+1}P_{2k+1}$, $P_{2m}P_{2k+1}$.

Taking any range $(\cdots, A_i, \cdots)$ given by (2), we have because of (5) $P_{2m}(\cdots, A_i, \cdots) = (\cdots, A_{m+i}, \cdots)$ and $P_{2k}(P_{2m}(\cdots, A_i, \cdots)) = P_{2k}(\cdots, A_{m+i}, \cdots) = (\cdots, A_{k+m+i}, \cdots)$. Hence

$$P_{2m}P_{2k} = P_{2(m+k)}. \tag{8}$$

We see the product is independent of the range taken, hence it is unique. Similarly, with the aid of (6), it is shown that

$$P_{2m+1}P_{2k} = P_{2(k+m)+1} \tag{9}$$

$$P_{2m+1}P_{2k+1} = P_{2(k-m)} \tag{10}$$

$$P_{2m}P_{2k+1} = P_{2(k-m)+1}. \tag{11}$$

From (8), (9), (10), and (11) it is easy to verify that the $2n$ perspectivities $P_0, \cdots, P_{2n-1}$ form a group under multiplication. (8), (9), and (11) show that P_0 is the neutral (unity) element of the group. From (8) it is seen that the inverse of P_{2m} is P_{-2m} and from (10) it is seen that the inverse of P_{2m+1} is P_{2m+1}. (9) and (11) show that the group is not commutative. Moreover, (8) shows that the set $\{P_{2m} \mid m=0, 1, \cdots, n-1\}$ which has n elements is a commutative subgroup of the entire group.

THEOREM 3. *The group of $2n$ perspectivities $P_0, P_1, \cdots, P_{2n-1}$ of a range of n points $A_0, \cdots, A_{n-1}$ is isomorphic to the dihedral group of order $2n$.*

Proof. The entire group of $2n$ perspectivities, $P_0, P_1, \cdots, P_{2n-1}$ can be generated by P_1 and P_2, since any element P_i is of the form $(P_1)^s(P_2)^t$, where $s=0$ or 1. Indeed, by (8) and (9): $P_{2m}=(P_2)^m$ and $P_{2m+1}=P_1(P_2)^m$, but by (8)

$$(P_2)^n = P_{2n} = P_0, \tag{12}$$

by (10):

$$P_1^2 = P_0, \tag{13}$$

and by (9): $P_1P_2=P_3$, hence

$$(P_1P_2)^2 = P_0. \tag{14}$$

On the other hand, (12), (13), (14) are the defining relations of the dihedral group of order $2n$ ($n \geqq 3$) generated by P_1 and P_2 (the group of symmetries of a regular n-gon).

I wish to express my thanks to Professor Alexander Abian, Ohio State University, and to the referee for their suggestions.

CLASSROOM NOTES

EDITED BY A. L. SHIELDS, University of Michigan

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to A. L. Shields, Mathematics Department, University of Michigan, Ann Arbor, Michigan.

ON IDENTITIES FOR FIBONACCI NUMBERS

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The Fibonacci sequence $\{f_n\}$ is defined by $f_0=0$, $f_1=1$, $f_{n+2}=f_{n+1}+f_n$, $n=0, 1, \cdots$. The paper by Rao [1] has an extensive list of identities satisfied by the Fibonacci numbers. In this note, the following well-known identities,

$$f_nf_{n+2} - f_{n+1}^2 = (-1)^{n+1} \qquad (n = 0, 1, \cdots), \tag{1}$$

$$f_{m+h}f_{m+k} - f_mf_{m+h+k} = (-1)^m f_hf_k, \qquad (h, k, m = 0, 1, \cdots), \tag{2}$$

$$f_m = f_{k+1}f_{m-k} + f_kf_{m-k-1} \qquad (k = 0, 1, \cdots, m-1), \tag{3}$$

(see [2], [3], and [4] for recent references to (1), (2), and (3), respectively) will be used to give *noninductive* proofs for several new Fibonacci identities. Our results are as follows:

Identity I.

$$f_{n+1}^4 - 2f_{n+1}^3f_n - f_{n+1}^2f_n^2 + 2f_{n+1}f_n^3 + f_n^4 = 1 \qquad (n = 0, 1, \cdots).$$

Proof. Writing (1) as $f_n^2-f_{n+1}^2=(-1)^{n+1}-f_nf_{n+1}$, we have

$$(4)\quad f_{n+1}^4+f_n^4=(f_n^2-f_{n+1}^2)^2+2f_n^2f_{n+1}^2=1+2(-1)^nf_nf_{n+1}+3f_n^2f_{n+1}^2,$$

$$(5)\quad \begin{aligned}-2f_{n+1}^3f_n-f_{n+1}^2f_n^2+2f_{n+1}f_n^3&=2f_nf_{n+1}[(-1)^{n+1}-f_nf_{n+1}]-f_n^2f_{n+1}^2\\&=-2(-1)^nf_nf_{n+1}-3f_n^2f_{n+1}^2.\end{aligned}$$

Addition of (4) and (5) yields (I).

Identity II.

$$f_{n+4}^4-4f_{n+3}^4-19f_{n+2}^4-4f_{n+1}^4+f_n^4=-6\qquad(n=0,1,\cdots).$$

Proof. Since $f_{n+4}=3f_{n+1}+2f_n$, $f_{n+3}=2f_{n+1}+f_n$, the left hand side of (II) simplifies to $-6\cdot$[left hand side of (I)]$=-6$.

Identity III.

$$f_{n+5}^4=5f_{n+4}^4+15f_{n+3}^4-15f_{n+2}^4-5f_{n+1}^4+f_n^4\qquad(n=0,1,\cdots).$$

Proof. From (II), we have $f_{n+5}^4=4f_{n+4}^4+19f_{n+3}^4+4f_{n+2}^4-f_{n+1}^4-6$. If we now substitute for -6 the left hand side of (II), we obtain (III).

Identity IV.

$$25\sum_{k=0}^{n}f_k^4=f_{n+3}^4-3f_{n+2}^4-22f_{n+1}^4-f_n^4+6n+9\qquad(n=0,1,\cdots).$$

Proof. Let $V_n=\Delta f_n^4=f_{n+1}^4-f_n^4$. Then replacing n by k, we may rewrite (II) as

$$(6)\quad V_{k+3}-3V_{k+2}-22V_{k+1}-26V_k-25f_k^4=-6.$$

Summing both sides of (6) for $k=0,1,\cdots,n$, and noting that $\sum_{k=0}^{n}V_{k+j}=f_{n+j+1}^4-f_j^4$, $j=0,1,\cdots$, we obtain

$$(7)\quad 25\sum_{k=0}^{n}f_k^4=f_{n+4}^4-3f_{n+3}^4-22f_{n+2}^4-26f_{n+1}^4+6n+15.$$

If we substitute for f_{n+4}^4 in (7), using (II), we obtain (IV), which yields now the following obvious result.

Identity V.

$$f_{n+3}^4-3f_{n+2}^4-22f_{n+1}^4-f_n^4+6n+9\equiv 0\pmod{25}\qquad(n=0,1,\cdots).$$

Identity VI.

$$\begin{aligned}18\sum_{k=0}^{n}(-1)^kf_k^4&=(-1)^n[f_{n+4}^4-6f_{n+3}^4-9f_{n+2}^4+24f_{n+1}^4-f_n^4]\\&=2(-1)^n[-f_{n+3}^4+5f_{n+2}^4+14f_{n+1}^4-f_n^4-3]\qquad(n=0,1,\cdots).\end{aligned}$$

Proof of (VI). Let $W_n=\Delta[(-1)^n f_n^4]=(-1)^{n+1}f_{n+1}^4-(-1)^n f_n^4$. If we multiply both sides of (III) by $(-1)^{n+5}$, we obtain, writing k for n,

$$W_{k+4}+6W_{k+3}-9W_{k+2}-24W_{k+1}-19W_k=18(-1)^k f_k^4. \tag{8}$$

Summing both sides of (8) for $k=0, 1, \cdots, n$, and noting that

$$\sum_{k=0}^{n} W_{k+j}=(-1)^{n+j+1}f_{n+j+1}^4-(-1)^j f_j^4, \qquad j=0, 1, \cdots,$$

we obtain

$$\begin{aligned} 18\sum_{k=0}^{n}(-1)^k f_k^4 &= (-1)^n[-f_{n+5}^4+6f_{n+4}^4+9f_{n+3}^4-24f_{n+2}^4+19f_{n+1}^4] \\ &= (-1)^n[f_{n+4}^4-6f_{n+3}^4-9f_{n+2}^4+24f_{n+1}^4-f_n^4] \quad \text{(using (III))} \\ &= (-1)^n[-2f_{n+3}^4+10f_{n+2}^4+28f_{n+1}^4-2f_n^4-6] \quad \text{(using (II).} \end{aligned} \tag{9}$$

Identity VI yields now the obvious result

Identity VII.

$$f_{n+4}^4-6f_{n+3}^4-9f_{n+2}^4+24f_{n+1}^4-f_n^4\equiv 0 \pmod{18} \qquad (n=0, 1, \cdots).$$

From (VII) we conclude

Identity VIII.

$$f_{n+4}^4-9f_{n+2}^4-f_n^4\equiv 0 \pmod{6} \qquad (n=0, 1, \cdots).$$

From (VIII) we conclude

Identity IX.

$$f_{n+4}^4-f_n^4\equiv 0 \pmod{3} \qquad (n=0, 1, \cdots).$$

The following identity,

$$f_{k+1}f_{k+2}f_{k+4}f_{k+5}=f_{k+3}^4-1 \qquad (k=0, 1, \cdots) \tag{10}$$

which was stated by E. Gelin (1880) and proved by E. Cèsaro (1880) (see [5] p. 401) may now be used to establish the following two results:

Identity X. For $n=0, 1, \cdots,$

$$25\sum_{k=0}^{n} f_{k+1}f_{k+2}f_{k+4}f_{k+5}=26f_{n+3}^4+22f_{n+2}^4+3f_{n+1}^4-f_n^4-19n-66.$$

Identity XI.

$$9\sum_{k=0}^{m}(-1)^k f_{k+1}f_{k+2}f_{k+4}f_{k+5}=(-1)^m[-f_{m+6}^4+5f_{m+5}^4+14f_{m+4}^4-f_{m+3}^4-3]-9g(m),$$

where

$$g(m) = \begin{cases} 0, & \text{if } m = 2n - 1 \qquad (n = 1, 2, \cdots), \\ 1, & \text{if } m = 2n \qquad (n = 0, 1, \cdots). \end{cases}$$

Proof. If we sum both sides of (10) with respect to k, we obtain

$$(11) \qquad 25 \sum_{k=0}^{n} f_{k+1}f_{k+2}f_{k+4}f_{k+5} = 25 \sum_{k=0}^{n} f_{k+3}^4 - 25(n+1),$$

$$(12) \quad 9 \sum_{k=0}^{m} (-1)^k f_{k+1}f_{k+2}f_{k+4}f_{k+5} = 9 \sum_{k=0}^{m} (-1)^k f_{k+3}^4 - 9g(m), \qquad g(m) = \sum_{k=0}^{m} (-1)^k.$$

Since $\sum_{k=0}^{n} f_{k+3}^4 = \sum_{j=0}^{n+3} f_j^4 - 2$, we find, using (IV) with n replaced by $n+3$, that the right hand side of (11) reduces to

$$(13) \qquad f_{n+6}^4 - 3f_{n+5}^4 - 22f_{n+4}^4 - f_{n+3}^4 - 19n - 48.$$

Using (II) to eliminate the terms f_{n+6}^4, f_{n+5}^4, and f_{n+4}^4, respectively, from (13) yields the desired result, (X).

Since $\sum_{k=0}^{m} (-1)^k f_{k+3}^4 = -\sum_{j=0}^{m+3} (-1)^j f_j^4$, we obtain, using (VI) with n replaced by $m+3$, the desired result, (XI), from (12).

If we define $L_n = f_{n-1} + f_{n+1}$, $n = 1, 2, \cdots$, then $\{L_n\}$ is the well-known Lucas sequence defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$, $n = 1, 2, \cdots$.

Identity XII. For $j = 0, 1, \cdots, n+2$, (n may be odd or even), we have

$$f_{n+1-j}f_{n+2-j} + (-1)^{j+1}L_{n+2}f_{n+2-j}f_{j+1} + (-1)^{n+1}f_{j+1}f_j = (-1)^{j+1}f_{n+2}f_{n+3}.$$

Proof. Since $f_0 = 0$, (XII) is readily verified for $j = n+2$, $n+1$. Let $j = 0, 1, \cdots, n$. From (2), with $m = n - j$, $h = j$, and $k = 1$, we obtain

$$(14) \qquad f_n f_{n+1-j} - f_{n-j}f_{n+1} = (-1)^{n-j}f_j = (-1)^{n+j}f_j.$$

Since $f_n = f_{n+2} - f_{n+1}$, (14) simplifies to

$$(15) \qquad f_{n+2}f_{n+1-j} - f_{n+1}f_{n+2-j} = (-1)^{n+j}f_j.$$

From (3), with $m = 2n + 4 - j$ and $k = n+2$, we obtain

$$(16) \qquad f_{2n+4-j} = f_{n+3}f_{n+2-j} + f_{n+2}f_{n+1-j}.$$

If we substitute for $f_{n+2}f_{n+1-j}$ in (15) by means of (16), we obtain

$$(17) \qquad f_{2n+4-j} = L_{n+2}f_{n+2-j} + (-1)^{n+j}f_j,$$

which may be written as

$$(18) \qquad (-1)^{j+1}f_{j+1}f_{2n+4-j} = (-1)^{j+1}L_{n+2}f_{n+2-j}f_{j+1} + (-1)^{n+1}f_{j+1}f_j.$$

From (2), with $m = j+1$, $h = n+1-j$, and $k = n+2-j$, we obtain

$$(19) \qquad f_{n+2}f_{n+3} - f_{j+1}f_{2n+4-j} = (-1)^{j+1}f_{n+1-j}f_{n+2-j}.$$

or

$$(-1)^{j+1}f_{j+1}f_{2n+4-j} = (-1)^{j+1}f_{n+2}f_{n+3} - f_{n+1-j}f_{n+2-j}. \tag{20}$$

The desired conclusion follows upon equating the right hand sides of (18) and (20).

The following identities are readily established by the method of mathematical induction:

$$\text{(XIII)} \quad 2\sum_{k=0}^{n}(-1)^k f_{m+3k} = (-1)^n f_{m+3n+1} + f_{m-2} \qquad (m = 2, 3, \cdots),$$

$$\text{(XIV)} \quad 3\sum_{k=0}^{n}(-1)^k f_{m+4k} = (-1)^n f_{m+4n+2} + f_{m-2} \qquad (m = 2, 3, \cdots),$$

$$\text{(XV)} \quad 11\sum_{k=0}^{n}(-1)^k f_{m+5k} = (-1)^n[5f_{m+5n+1} + 2f_{m+5n}] + 4f_m - 5f_{m-1} \qquad (m = 1, 2, \cdots),$$

$$\text{(XVI)} \quad 4\sum_{k=0}^{n} f_k f_{2k+1} = f_{2n+3}f_n + f_{2n}f_{n+3},$$

$$\text{(XVII)} \quad 3\sum_{k=0}^{n}(-1)^k f_{m+2k}^2 = (-1)^n f_{m+2n}f_{m+2n+2} + f_m f_{m-2} \qquad (m = 2, 3, \cdots),$$

$$\text{(XVIII)} \quad 7\sum_{k=0}^{n}(-1)^k f_{m+4k}^2 = (-1)^n f_{m+4n}f_{m+4n+4} + f_m f_{m-4} \qquad (m = 4, 5, \cdots),$$

$$\text{(XIX)} \quad 2\sum_{k=0}^{n} f_{k+2}f_{k+1}^2 = f_{n+3}f_{n+2}f_{n+1},$$

$$\text{(XX)} \quad 2\sum_{k=0}^{n}(-1)^k f_k f_{k+1}^2 = (-1)^n f_{n+2}f_{n+1}f_n,$$

$$\text{(XXI)} \quad 2\sum_{k=0}^{n}(-1)^k f_{k+1}^3 = (-1)^n[f_{n+1}^2 f_{n+4} - f_n f_{n+2} f_{n+3}] - 1$$

$$= f_{n+4} + (-1)^n f_n f_{n+2}^2 - 1.$$

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BODY FORCE EQUATIONS IN ORTHOGONAL CURVILINEAR COORDINATES

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For cartesian, polar and spherical coordinates, the body force equations or stress equilibrium equations, as they are sometimes called, can be derived easily by various methods. One or two derivations of these equations in orthogonal curvilinear coordinates have been obtained by long intricate methods [1], [2] which, apart from mathematical fineness, completely obscure the physical nature of the stress variations in a system in which the axes themselves undergo a variation from point to point. This note fills this gap. The results have been derived by vectorial methods, which are much simpler than those usually given and also yield more physical insight into the stress variation in a moving system.

If the three orthogonal surfaces are given by $u_1=\text{const.}$, $u_2=\text{const.}$, $u_3=\text{const.}$, they intersect in three curves which are mutually perpendicular. Let $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ be the unit vectors at (u_1, u_2, u_3) in the direction of u_1-increasing, u_2-increasing, u_3-increasing. Further let the elements of lengths in these directions be $h_1\,du_1$, h_2du_2, h_3du_3. Thus the element of an arc ds is given by

$$ds^2 = h_1^2du_1^2 + h_2^2du_2^2 + h_3^2du_3^2. \tag{1}$$

In terms of cartesian coordinates (x, y, z)

$$h_1^2 = \left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2, \tag{2}$$

with similar expression for h_2^2 and h_3^2.

Let $\boldsymbol{P}_N$ be the stress vector on a surface ds whose normal is in the direction N given by $l\mathbf{i}_1+m\mathbf{i}_2+n\mathbf{i}_3$, let $\mathbf{F}=F_{u_1}\mathbf{i}_1+F_{u_2}\mathbf{i}_2+F_{u_3}\mathbf{i}_3$ be the body force per unit mass, and let $\boldsymbol{f}=f_1\mathbf{i}_1+f_2\mathbf{i}_2+f_3\mathbf{i}_3$ be the acceleration. Then for any mass within solid of density ρ

$$\int_s P_N ds + \int_\Omega \rho F d\Omega = \int_\Omega \rho f d\Omega, \tag{3}$$

where $d\Omega$ is an element of volume. The integral on the first term is taken on the bounding surface and on the second and third throughout the volume. Noting that

$$P_N = P_{Nu_1}\mathbf{i}_1 + P_{Nu_2}\mathbf{i}_2 + P_{Nu_3}\mathbf{i}_3, \tag{4}$$

where P_{Nu_1}, P_{Nu_2}, P_{Nu_3} are the respective components of the stress-vector in the $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ direction, and substituting this value of $\boldsymbol{P}_N$ in the first term of (3), the first integral is

$$\int_s P_N ds = \int_s (P_{Nu_1}\mathbf{i}_1 + P_{Nu_2}\mathbf{i}_2 + P_{Nu_3}\mathbf{i}_3)ds. \tag{5}$$

We shall change the surface integral in (3) into a volume integral. We evaluate only the first integral $\int_s P_{Nu_1}\mathbf{i}_1 ds$ on the right hand side of (5). The others can be written by a cyclic permutation. It is known that

$$\begin{aligned} P_{Nu_1} &= lP_{u_1u_1} + mP_{u_1u_2} + nP_{u_1u_3}, \\ &= (P_{u_1u_1}\mathbf{i}_1 + P_{u_1u_2}\mathbf{i}_2 + P_{u_1u_3}\mathbf{i}_3)\cdot(l\mathbf{i}_1 + m\mathbf{i}_2 + n\mathbf{i}_3), \end{aligned}$$

where the dot denotes the scalar product. Using the theorem which states that if $\boldsymbol{a}$, $\boldsymbol{q}$ are two vectors and $\boldsymbol{n}$ denotes a unit normal vector, then

$$\int_s \boldsymbol{a}(\boldsymbol{n}\cdot\boldsymbol{q})ds = \int_\Omega \{\nabla\cdot\boldsymbol{q}\boldsymbol{a} + (\boldsymbol{q}\cdot\nabla)\boldsymbol{a}\}d\Omega,$$

we get

$$\begin{aligned} \int_s P_{Nu_1}\mathbf{i}_1 ds &= \int_s \mathbf{i}_1(P_{u_1u_1}\mathbf{i}_1 + P_{u_1u_2}\mathbf{i}_2 + P_{u_1u_3})\cdot(l\mathbf{i}_1 + m\mathbf{i}_2 + n\mathbf{i}_3)ds \\ &= \int_\Omega \Big\{\mathbf{i}_1 \operatorname{div}(P_{u_1u_1}\mathbf{i}_1 + P_{u_1u_2}\mathbf{i}_2 + P_{u_1u_3}\mathbf{i}_3) \\ &\quad + P_{u_1u_1}\frac{\partial\mathbf{i}_1}{h_1\partial u_1} + P_{u_1u_2}\frac{\partial\mathbf{i}_1}{h_2\partial u_2} + P_{u_1u_3}\frac{\partial\mathbf{i}_1}{h_3\partial u_3}\Big\} d\Omega. \end{aligned}$$

To evaluate the first integral, we use the theorem that if ϕ is a scalar, and $\boldsymbol{a}$ a vector quantity, then $\operatorname{div}(\boldsymbol{a}\phi) = \phi \operatorname{div} \boldsymbol{a} + \boldsymbol{a}\cdot\operatorname{grad}\phi$. Hence

$$\begin{aligned} \int_s P_{Nu_1}\mathbf{i}_1 ds = \int_\Omega \Big\{&\mathbf{i}_1(P_{u_1u_1}\operatorname{div}\mathbf{i}_1 + P_{u_1u_2}\operatorname{div}\mathbf{i}_2 + P_{u_1u_3}\operatorname{div}\mathbf{i}_3 \\ &+ \mathbf{i}_1\cdot\operatorname{grad}P_{u_1u_1} + \mathbf{i}_2\cdot\operatorname{grad}P_{u_1u_2} + \mathbf{i}_3\cdot\operatorname{grad}P_{u_1u_3}) \\ &+ P_{u_1u_1}\frac{\partial\mathbf{i}_1}{h_1\partial u_1} + P_{u_1u_2}\frac{\partial\mathbf{i}_1}{h_2\partial u_2} + P_{u_1u_3}\frac{\partial\mathbf{i}_1}{h_3\partial u_3}\Big\} d\Omega. \end{aligned} \tag{6}$$

Using the theorems that

$$\operatorname{div}\mathbf{i}_1 = \frac{1}{h_1h_2}\frac{\partial h_2}{\partial u_1} + \frac{1}{h_1h_3}\frac{\partial h_3}{\partial u_1},$$

$$\frac{\partial\mathbf{i}_1}{\partial u_1} = -\frac{\mathbf{i}_2}{h_2}\frac{\partial h_1}{\partial u_2} - \frac{\mathbf{i}_3}{h_3}\frac{\partial h_1}{\partial u_3},$$

$$\frac{\partial\mathbf{i}_1}{\partial u_2} = \frac{\mathbf{i}_2}{h_1}\frac{\partial h_2}{\partial u_1},$$

with similar expressions for $\operatorname{div}\mathbf{i}_2$, $\operatorname{div}\mathbf{i}_3$, $\partial\mathbf{i}_1/\partial u_3$ etc., we get by (6)

$$
\begin{aligned}
&\int_s P_{Nu_1}\mathbf{i}_1 ds \\
&= \int_\Omega \Bigg[\Bigg\{ P_{u_1u_1}\left(\frac{1}{h_1h_2}\frac{\partial h_2}{\partial u_1}+\frac{1}{h_1h_3}\frac{\partial h_3}{\partial u_1}\right)+P_{u_1u_2}\left(\frac{1}{h_2h_3}\frac{\partial h_3}{\partial u_2}+\frac{1}{h_1h_2}\frac{\partial h_1}{\partial u_1}\right) \\
&\quad + P_{u_1u_3}\left(\frac{1}{h_3h_1}\frac{\partial h_1}{\partial u_3}+\frac{1}{h_2h_3}\frac{\partial h_2}{\partial u_3}\right)+\frac{\partial P_{u_1u_1}}{h_1\partial u_1}+\frac{\partial P_{u_1u_2}}{h_2\partial u_2}+\frac{\partial P_{u_1u_3}}{h_3\partial u_3}\Bigg\}\mathbf{i}_1 \\
&\quad + \frac{P_{u_1u_1}}{h_1}\left(-\frac{\mathbf{i}_2}{h_2}\frac{\partial h_1}{\partial u_2}-\frac{\mathbf{i}_3}{h_3}\frac{\partial h_1}{\partial u_3}\right)+\frac{P_{u_1u_2}}{h_2}\left(\frac{\mathbf{i}_2}{h_1}\frac{\partial h_2}{\partial u_1}\right) \\
&\quad + \frac{P_{u_1u_3}}{h_3}\left(\frac{\mathbf{i}_3}{h_1}\frac{\partial h_3}{\partial u_1}\right)\Bigg] d\Omega.
\end{aligned} \tag{7}
$$

We obtain similar expressions for $\int_s P_{Nu_2}\mathbf{i}_2 ds$ and $\int_s P_{Nu_3}\mathbf{i}_3 ds$. Further we use the equations

$$
\boldsymbol{F} = F_{u_1}\mathbf{i}_1 + F_{u_2}\mathbf{i}_2 + F_{u_3}\mathbf{i}_3 \quad \text{and} \quad \boldsymbol{f} = f_{u_1}\mathbf{i}_1 + f_{u_2}\mathbf{i}_2 + f_{u_3}\mathbf{i}_3.
$$

We write the value of $\int_s P_N ds$ in (3) with the help of (5) and (7) and two similar expressions in terms of volume integrals. We then bring all terms under one sign of integration since all have been transformed to volume integrals. Now, as this result is true for all volumes that can be conceived of within the solid, we infer that integrand itself is zero and therefore the coefficients of $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ should be separately zero. Setting the coefficient of $\mathbf{i}_1$ equal to zero, we get

$$
\begin{aligned}
&\frac{\partial P_{u_1u_1}}{h_1\partial u_1}+\frac{\partial P_{u_1u_2}}{h_2\partial u_2}+\frac{\partial P_{u_1u_3}}{h_3\partial u_3}+P_{u_1u_1}\left(\frac{1}{h_1h_2}\frac{\partial h_2}{\partial u_1}+\frac{1}{h_1h_3}\frac{\partial h_3}{\partial u_2}\right) \\
&\quad + P_{u_2u_2}\left(-\frac{1}{h_1h_2}\frac{\partial h_2}{\partial u_1}\right)+P_{u_3u_3}\left(-\frac{1}{h_1h_3}\frac{\partial h_3}{\partial u_1}\right) \\
&\quad + P_{u_1u_2}\left(\frac{2}{h_1h_2}\frac{\partial h_1}{\partial u_2}+\frac{1}{h_2h_3}\frac{\partial h_3}{\partial u_2}\right)+P_{u_1u_3}\left(\frac{2}{h_1h_3}\frac{\partial h_1}{\partial u_3}+\frac{1}{h_2h_3}\frac{\partial h_2}{\partial u_3}\right) \\
&\quad + \rho F_{u_1} = \rho f_{u_1}.
\end{aligned} \tag{8}
$$

We shall get two similar equations when we set the coefficients of $\mathbf{i}_2$ and $\mathbf{i}_3$ equal to zero. Equation (8) and two similar equations are the body force equations in orthogonal curvilinear coordinates.

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AN ALTERNATE PROOF FOR THE INVARIANCE OF PARITY OF A PERMUTATION WRITTEN AS A PRODUCT OF TRANSPOSITIONS

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Introduction. The following proof is for the "invariance of parity" of a permutation and seems less cumbersome than the one usually presented (i.e., use of a specially designed polynomial).

Let $Z_n = \{1, 2, 3, \cdots, n\}$. Let $S(Z_n)$ be the set of all "one-one" and "onto" maps from Z_n to Z_n (i.e., all permutations). Let $A_k = \{f \in S(Z_n) \mid f(1) = k\}$.

It is easily seen that:

1) A_1 is a subgroup of $S(Z_n)$ and is isomorphic to $S(Z_{n-1})$.

2) For each $k \neq 1$, there exists one and only one transposition, $t_k \in A_k (2 \leqq k \leqq n)$.

3) For each $k \neq 1$, $A_k = t_k \circ A_1$ (i.e., the A_k's are left cosets of A_1).

LEMMA 1. *Let b, c, be transpositions $\notin A_1$ (hence $b \neq c$). Then $b \circ c = c \circ a$ for some transposition $a \in A_1$. $(b \circ c)(x) = b(c(x))$.*

Proof. Let $b(1) = k$, $b(k) = 1$; $c(1) = m$, $c(m) = 1$; choose $a(m) = k$, $a(k) = m$.

LEMMA 2. *If a is a transposition $\in A_1$, and b is a transposition $\in A_k$ $(k \neq 1)$, then $a \circ b = c \circ a$, where c is a transposition $\in A_m$, $(m \neq 1)$.*

Proof. Let $a(1) = 1$, $a(x) = y$, $a(y) = x$, $b(1) = k$, $b(k) = 1$. If $k \neq x$, $k \neq y$, let $b = c$. If $k = x$, let $c(1) = y$. If $k = y$, let $c(1) = x$.

Note that these lemmas hold for all $n \geqq 3$.

THEOREM. *If h is a permutation $\in S(Z_n)$ and h is written as the product of transpositions in two different ways, then $(-1)^r = (-1)^t$, where r and t are the number of transpositions in the respective products.*

Proof. The proof is by induction on n. Assume the theorem is true for $n = u$ and consider $n = u + 1$. Let h have two arbitrary representations of transposition products such as $h = f \circ g \circ \cdots$ (r trans.) and $h = p \circ q \circ \cdots$ (t trans.). In the first representation, let e be the last transposition in the product such that $e \notin A_1$. Now if the transposition preceding e is in A_1, apply Lemma 2, if not apply Lemma 1. Repeated use of these lemmas enables us to write h as $h = b \circ a'$, where $a' \in A_1$ and each transposition in the product a' is in A_1. Similarly, for the second representation, we may write $h = c \circ \bar{a}$, where $\bar{a} \in A_1$.

Now if $h \in {}_1A$, then b and c are both in A_1, and the result follows from the inductive hypothesis. If $h \notin A_1$, then $b = c$, which implies $a' = \bar{a}$, and $(-1)^{r-1} = (-1)^{t-1}$. To complete the induction, consider the case $n = 2$; but this is the cyclic group of order 2, $\{e, \sigma\}$ with $\sigma^2 = e$ and σ a transposition. Note that for any integer p, $\sigma^{2p} = e$ and $\sigma^{2p+1} = \sigma$; therefore parity is clearly preserved.

ON THE CONGRUENCE $2^p \equiv 2$ (p^2)

MELVIN HAUSNER AND DAVID SACHS, New York University, Washington Square College

It is apparently well known that $2^{1093} \equiv 2$ (1093^2). (See for example problem 1b of Vinogradov's *Elements of Number Theory*, page 58, Dover, 1954.) Further, the general problem $2^p \equiv 2$ (p^2) is related to Fermat's last theorem. Therefore, with the help of IBM 7090, the authors brutally attacked the problem "What primes p satisfy $2^p \equiv 2$ (p^2)?" The result is as follows:

The only primes $p < 10^6$ for which $2^p \equiv 2$ (p^2) are $p = 1{,}093$ and $p = 3{,}511$.

THE IDEMPOTENTS OF A SEMIGROUP

SAM B. NADLER, JR., University of Georgia

It may seem to most students that the methods of a first year calculus course could not be applicable to other facets of mathematics. The following is a theorem concerning the idempotents of a semigroup. Its proof, however, rests completely on the methods of elementary calculus, and could be used to introduce a first year calculus student to topics in abstract algebra.

A semigroup is a pair $(S, \circ)$, where S is a set and $\circ$ is a binary associative operation, i.e., if x, y, and z are in S then $x \circ (y \circ z) = (x \circ y) \circ z$. An element x in S is an idempotent if $x \circ x = x$.

Let I denote the unit interval and let D be the set of all differentiable functions on I into I. Let $\circ$ represent composition and let i in D be the function such that $i(x) = x$ for each x in I. Since composition is associative, $(D, \circ)$ is a semigroup with identity.

The purpose of this note is to find all idempotents in the semigroup $(D, \circ)$, i.e., all differentiable functions f on I into I such that $f \circ f = f$. Evidently, each constant function on $[0, 1]$, as well as the identity i, is an idempotent. The problem reduces to whether or not there are idempotents which are not constant and not the identity.

For example, $g(x) = |x - \frac{1}{2}| + \frac{1}{2}$, satisfies $g(g(x)) = g(|x - \frac{1}{2}| + \frac{1}{2}) = |(|x - \frac{1}{2}| + \frac{1}{2}) - \frac{1}{2}| + \frac{1}{2} = |x - \frac{1}{2}| + \frac{1}{2} = g(x)$; but g is not differentiable at $x = \frac{1}{2}$. We prove the following result:

THEOREM. *Let $J = \{f \mid f$ is constant or $f = i\}$. Then J is the set of all idempotents of $(D, \circ)$.*

Proof. First J is composed of only idempotents. Clearly $(i \circ i)(x) = i(i(x)) = i(x)$. Hence $i \circ i = i$. Let $f \in J$ so that $f(x) = c$ for all $x \in I$. Then $(f \circ f)(x) = f(f(x)) = f(c) = c$. Therefore $f \circ f = f$.

Now suppose $g \notin J$ and yet $g \circ g = g$. Let the image of I under g be $[a, b]$. Since g is not a constant function, $a < b$. Also $[a, b] \subset [0, 1]$ and the inclusion is proper. Suppose, on the contrary, that $g([0, 1]) = [0, 1]$ and $p \in [0, 1]$. Then there is a number $q \in [0, 1]$ such that $g(q) = p$. Hence $g(p) = g(g(q)) = g(q) = p$. Therefore, for all p in $[0, 1]$, $g(p) = p$. Thus $g = i$. This is a contradiction.

Let $x \in [a, b]$; then there is an x_0 such that $g(x_0)=x$. Therefore $g(x)=g(g(x_0))$ $=g(x_0)=x$ and thus $g=i$ on $[a, b]$.

Case I. Suppose $b \neq 1$. Since $g'(b)$ exists, we have

$$\lim_{x\to b^+} \frac{g(x)-g(b)}{x-b} = g'(b) = \lim_{x\to b^-} \frac{g(x)-g(b)}{x-b} = \lim_{x\to b^-} \frac{x-b}{x-b} = 1 > 0.$$

Therefore, for some $x>b$, $g(x)>g(b)$, i.e., $g(x)>b$. This is a contradiction.

Case II. Suppose $b=1$. Then $a\neq 0$. Since $g'(a)$ exists, we have

$$\lim_{x\to a^-} \frac{g(x)-g(a)}{x-a} = g'(a) = \lim_{x\to a^+} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a^+} \frac{x-a}{x-a} = 1 > 0.$$

Therefore, for some $x<a$, $g(x)<g(a)$, i.e., $g(x)<a$. This is a contradiction.

A PROOF OF THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY

BENGT ÅKERBERG, Stockholm, Sweden

Suppose that α is a positive number and n a positive integer greater than 1. It is easy to verify that

$$(\alpha-1)[n-(1+\alpha+\alpha^2+\cdots+\alpha^{n-1})] \leqq 0$$

or

$$\alpha(n-\alpha^{n-1}) \leqq n-1, \tag{1}$$

and the inequality is strict unless $\alpha=1$.

If $a_1, a_2, \cdots, a_n$ denote positive numbers and if

$$\alpha = \left\{\frac{a_1}{\dfrac{a_1+a_2+\cdots+a_n}{n}}\right\}^{1/(n-1)}$$

then it follows from (1) that

$$\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^n \geqq a_1\left(\frac{a_2+\cdots+a_n}{n-1}\right)^{n-1}.$$

By repeated applications of this formula we obtain

$$\begin{aligned}\left(\frac{a_1+\cdots+a_n}{n}\right)^n &\geqq a_1\left(\frac{a_2+\cdots+a_n}{n-1}\right)^{n-1}\\ &\geqq a_1a_2\left(\frac{a_3+\cdots+a_n}{n-2}\right)^{n-2}\\ &\cdots\cdots\cdots\cdots\\ &\geqq a_1a_2\cdots a_n,\end{aligned}$$

an inequality which asserts that the arithmetic mean of $a_1, a_2, \cdots, a_n$ is not less than their geometric mean.

The inequality in the first line is strict unless $\alpha=1$ or $a_1=(a_1+\cdots+a_n)/n$, and in the second line unless this holds and also $a_2=(a_2+\cdots+a_n)/(n-1)$, or $a_1=a_2$.

It is clear that we may repeat this argument until we find that the inequality in the final step is strict unless $a_1=a_2=\cdots=a_n=(a_1+\cdots+a_n)/n$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
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A FRESHMAN MATHEMATICS SEMINAR

LUCILLE F. MAIER, Rosary Hill College, Buffalo

During the fall semester of 1962–63, Rosary Hill College of Buffalo, N. Y., offered twenty-one entering freshmen the opportunity to enroll in a mathematics seminar. They were all students who had demonstrated above-average ability during four years of high school mathematics. The major purpose of the seminar was to increase their mathematical maturity so that they will be equipped to work independently, and perhaps creatively, at an early stage in their college careers.

The work was planned so that the initial unit (Determinants and Matrices) was planned by the instructor and directed by students. The more difficult mathematical tasks were assigned to those who already had some knowledge of the topic but the entire group gathered information from various sources. Results were discussed and were then presented in a formal, 54 page, dittoed report.

The second unit (Coordinate and Vector Geometry) followed the same pattern, except that it was completely planned and directed by students. Upon completion of this project, each student selected a topic for individual study. Their papers presented the history of the topic, its practical applications and a bibliography of related material, as well as a rigorous mathematical treatment of the topic studied. Each student was given the opportunity to present an interesting phase of her work to the group. Questions and discussion were encouraged. The emotional response was remarkable. When the work was going well, everyone was very, very happy; when they hit a snag, the waves of gloom and discouragement were almost tangible.

The depth and intensity of mathematical understanding revealed by these

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine
COLLABORATING EDITOR, C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 1631. *Proposed by Roy Feinman, Rutgers University*

Let a, b, c, d be integers, with $a\neq 0$. Can $axy+bx+cy+d=0$ have infinitely many solutions in integral x and y?

E 1632. *Proposed by Michael Fried, Bell Aerosystems, Wheatfield, N. Y.*

Show that there is no polynomial which assumes prime values for all prime values of the argument.

E 1633. *Proposed by Helen M. Marston, Douglass College*

A common problem in elementary calculus is: "An open box is made by cutting squares of side x from the corners of an a by b rectangle and folding up the sides. For what value of x is the volume a maximum?" When $a=b$, the answer to this problem is $a/6$. For what integers a and b, $a\neq b$, $(a, b)=1$, is the answer rational, and how many such a by b rectangles are there of length less than 50?

E 1634. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

Let a_i, $i=1, 2, \cdots, n$, be the sides of a convex polygon with area K. Prove that $\sum_{i=1}^{n} a_i^2\geqq 4K \tan(\pi/n)$.

E 1635. *Proposed by H. D. Ruderman, Hunter College High School*

Let $\{s_1, s_2, \cdots, s_n\}$ be a sequence of n plus and minus ones. Determine, as a function of n, the number of such sequences for which $\sum_{i=1}^{j} s_i \geqq 0$ for $j=1, 2, \cdots, n$.

E 1636. *Proposed by J. D. Cloud, North American Aviation, Inc.*

Given a positive integer N, how many Fibonacci numbers are there not exceeding N?

E 1637. *Proposed by J. C. G. Boot, Netherlands School of Economics, Rotterdam, Netherlands*

If $a=5$, $b=8$, then the smallest integer N such that for all integers $n\geqq N$,

$n = c_1 a + c_2 b$ for some nonnegative integers c_1 and c_2, is 28. Determine $N = N(a, b)$ for arbitrary positive integers a and b.

E 1638. *Proposed by Bruce Lercher, Harpur College*

Let F be a one-one continuous mapping of the plane R_2 onto itself having the additional property that for all points P of R_2, $d(P, F(P)) = 1$ (where $d(P, Q)$ denotes the distance between P and Q). Need F be an isometry?

E 1639. *Proposed by Reuben Hersh, Stanford University*

Find all subsets of the complex plane which, like the positive real axis or the unit circle, are (1) subgroups of the multiplicative group of complex numbers, and (2) continuous connected curves.

E 1640. *Proposed by C. W. Trigg, Los Angeles City College*

The edges of a triangular pyramid and a quadrilateral pyramid are all equal. Show that the two pyramids may be dissected into pieces which may be reassembled into a single cube.

SOLUTIONS

Sum of the Elements of a Matrix

E 1535 [1962, 809; 1963, 570]. *Proposed by S. W. Golomb, Jet Propulsion Laboratory, California Institute of Technology, and A. W. Hales, King's College, Cambridge, England.*

An $n \times n$ array (matrix) of nonnegative integers has the property that for any zero entry, the sum of the row plus the sum of the column containing that entry is at least n. Show that the sum of all elements of the array is at least $n^2/2$.

Remarks by S. W. Golomb. The published solution of this problem in the May, 1963 issue is erroneous. At one stage of that solution a $(k-1)\times(k-1)$ submatrix is obtained and it is assumed that the elements of this submatrix sum to at least $(k-1)^2/2$. This step is not valid, for the submatrix does not necessarily satisfy the original condition—that the sum of the row plus the sum of the column through a zero entry is at least $k-1$. This is illustrated by the identity matrix of order 2; if we delete the row and column of the 0 in the upper right corner, the remaining submatrix fails to satisfy the row-plus-column-sum condition.

The generalization of this problem from 2-dimensional arrays to r-dimensional arrays is of great combinatorial significance, and has not yet (as far as this writer is aware) been completely proved. The conjecture is:

If M is an $n \times n \times \cdots \times n$ array of nonnegative integers (r-dimensional) such that for every 0 entry $a_{i_1 j_1 k_1 \cdots s_1}$ the sum

$$\sum_i a_{i j_1 k_1 \cdots s_1} + \sum_j a_{i_1 j k_1 \cdots s_1} + \cdots + \sum_s a_{i_1 j_1 k_1 \cdots s} \geqq n,$$

then the sum over all entries of M is at least n^r/r.

Following are two valid solutions of Problem E 1535.

II. *Solution by A. W. Hales.* Let M be an $n\times n$ matrix satisfying the hypotheses of the problem. Compute the sum of each row and of each column, and let s be the minimum sum. Suppose the ith row has this minimum sum. (The argument is the same if it is a column; or, we could transpose the matrix to make it a row.) There are at most s nonzero elements on the ith row (since the row sum is s), so that there are at least $n-s$ zeros on this row. Pick $n-s$ columns which are 0 on the ith row. The sum of each such column is at least $n-s$, since the row-plus-column sum for the 0 in that column on the ith row is at least n, and the row contributes only s. Each of the remaining s columns has a sum of at least s, since this is the minimum column (or row) sum. The sum over M is the sum over all its columns; $n-s$ columns have sums of at least $n-s$, and the remaining s columns have sums of at least s, for a total of at least $(n-s)^2+s^2$, which has its minimum at $s=n/2$, where it equals $n^2/2$.

III. *Solution by J. N. Franklin, California Institute of Technology.* The conditions of the problem remain unchanged if rows are permuted with rows, or columns with columns. Given an $n\times n$ matrix satisfying the hypotheses of the problem, permute the rows and columns so as to obtain a maximum number of zeros down the main diagonal. Call this matrix N. If there are no zeros down the diagonal, then N contains no zeros at all, and its sum is at least n^2. If the zeros go all the way down the diagonal, then each of these n zeros has a row-plus-column sum of at least n, for a total of at least n^2 for all the rows plus all the columns—that is, twice the sum over M is at least n^2, and the sum over M is at least $n^2/2$.

In general there are k zeros down the diagonal of M, with $0\leqq k\leqq n$. We must still treat the cases $0<k<n$. Partition the matrix into four submatrices by lines between the kth and $(k+1)$st columns and between the kth and $(k+1)$st rows. The sum of the elements of the lower-right submatrix is at least $(n-k)^2$, since it can contain no zeros. (Otherwise we could put more zeros on the diagonal.) The row-plus-column sum for the k zeros on the diagonal is at least kn. This sum counts the upper-left submatrix twice, and the upper-right and lower-left submatrices once each. However, the sum of the elements of the upper-right plus the lower-left submatrices is at least $k(n-k)$, because if $a_{ij}=0$ in one submatrix, we cannot have $a_{ji}=0$ in the other. (The reason for this is that if $a_{ij}=a_{ji}=0$ we could interchange the ith and jth columns, retaining a zero at the (i, i) position, but adding another zero at the (j, j) position, violating the assumption that M already has a maximum number of zeros on the main diagonal.) Hence, twice the sum over M is at least $2(n-k)^2+kn+k(n-k)$, and the sum over M is at least $n^2/2+(n-k)^2/2\geqq n^2/2$.

Tossing a Stick into a Hatbox

E 1556 [1963, 92]. *Proposed by R. A. Cunninghame-Green, Leo Computers Ltd., England*

A man throws a rough stick 13 inches long into an upright cylindrical hatbox a foot high and 10 inches in diameter. What is the probability that he cannot now close the lid?

Solution by Cornelius Groenewoud, Buffalo, N. Y. Assume: (1) the stick is representable by a straight line segment, (2) every point of the bottom of the box is equally likely to be the position of the bottom end of the stick, (3) every direction from the vertical through the bottom end of the stick is equally likely for the stick to take in falling against the side of the box, (4) the stick remains in the position it has when it first touches the side of the box.

The probability that the bottom end of the stick lands a distance r from the center of the bottom of the box in an element of area $rdrd\theta$ is

$$(1/25\pi)rdrd\theta.$$

The conditional probability that the stick will lean against the side of the box so as to protude is

$$1 - (1/\pi) \cos^{-1} (r/10).$$

The desired probability is then the integral of the product of these probabilities, integrated over the base of the box. The value of the integral is

$$1/3 + \sqrt{3}/2\pi.$$

Also solved by Jack Abad, B. L. Blankenship, A. W. Brunson, Flor Cartuyvels, D. I. A. Cohen and S. P. Cohen (jointly), Michael Gemignani, Michael Goldberg, S. H. Greene, R. T. Hood, A. R. Hyde, R. A. Jacobson, Winton Laubach, W. R. McEwen, Coline Makepeace, D. C. B. Marsh, C. A. Oster, J. F. Ramaley, B. E. Rhoades, G. S. Rogers, Max Rosenberg, S. J. Ryan, D. A. Vivian, K. L. Yocom, and Walter Zayachkowski. Not all these solutions agreed with the published one.

Limit of a Sequence

E 1557 [1963, 92]. *Proposed by D. R. Hayes, Duke University*

If $S_n = (1/n^2) \sum_{k=0}^{n} \log \binom{n}{k}$, evaluate $S = \lim_{n\to\infty} S_n$.

I. *Solution by Stanton Philipp, Seal Beach, Calif.* It may be directly verified that

$$S_n = (2/n) \sum_{k=1}^{n} (k/n) \log (k/n) + [(n + 1)/n^2] \log (n^n/n!).$$

Applying the definition of an integral to the first term on the right, and Stir-

ling's theorem to the second, we obtain

$$S = 2\int_0^1 x \log x dx + 1 = 1/2.$$

II. *Solution by Perry Scheinok, R. C. A., Moorestown, N. J.* We have

$$\begin{aligned} S_n &= (1/n^2) \sum_{k=0}^{n} \log \binom{n}{k} \\ &= (1/n^2) \sum_{k=0}^{n-1} (n - 2k - 1) \log (n - k) \\ &= (1/n) \sum_{k=0}^{n-1} [1 - (2k + 1)/n] \log (1 - k/n). \end{aligned}$$

A closer scrutiny of the last sum shows that this is the Riemann sum approximation to

$$\int_0^1 (2x - 1) \log x dx = 1/2.$$

Thus $\lim_{n\to\infty} S_n = 1/2$.

III. *Solution by D. I. A. Cohen, Princeton University.* The editorial note to E 1464 [1961, 1009] states that

$$\log \prod_{i=0}^{n} \binom{n}{i} = n^2/2 + O(n \log n).$$

Therefore $S_n = 1/2 + O[(\log n)/n]$, and $S = 1/2$.

Also solved by G. E. Bardwell, P. T. Bateman, Joseph Beer, W. G. Brady, J. L. Brown, Jr., Flor Cartuyvels, Robert Cohen, N. J. Fine, James Foster, H. W. Gould, Ralph Greenberg, Cornelius Groenewoud, Emil Grosswald, Erwin Just and Norman Schaumberger (jointly), J. K. Mackenzie, D. C. B. Marsh, H. J. Ricardo, T. P. Schwartzbauer, Arnold Singer, Rory Thompson, Oswald Wyler, David Zeitlin, and the proposer.

Several solvers found an equivalent problem (with answer) in Polya-Szegö, *Aufgaben und Lehrsätze aus der Analysis* (Problem 51, p. 45, vol. I).

Balanced Numbers

E 1558 [1963, 92]. *Proposed by M. V. Subbarao, University of Missouri*

A positive integer n is called a *balanced number* if it satisfies the equation $\sigma(n)/t(n) = n/2$, where $\sigma(n)$ and $t(n)$ denote as usual the sum and number of divisors of n respectively. Prove that 6 is the only balanced number.

Solution by Erwin Just and Norman Schaumberger, Bronx Community College. Let the divisors of n be $d_1, d_2, \cdots, d_k$. Then $\sigma(n)/t(n) = n/2$ implies $\sigma(n)/n = t(n)/2$, or

$$(1) \qquad 1/d_1 + 1/d_2 + \cdots + 1/d_k = k/2.$$

Since the d_i are distinct it is easily found that the only solution to (1) occurs for $k=4$, in which case the divisors are 1, 2, 3, and 6. Therefore the only balanced number is 6.

Also solved by Jack Abad and Paul Abad (jointly), G. E. Bardwell, Joseph Beer, J. L. Brown, Jr., Leonard Carlitz, D. I. A. Cohen, M. T. Cohen, Ralph Greenberg, M. S. Klamkin, A. E. Livingston, Lindsay McManus, C. F. Marion, D. C. B. Marsh, J. W. Moon, J. B. Muskat, Stanton Philipp, T. L. Ray, D. L. Silverman, Guy Torchinelli, L. J. Warren, Raymond Whitney, John Wood, and Oswald Wyler.

An Impossible Group Property

E 1559 [1963, 92]. *Proposed by Omar Khayyam, Jr., University of California at Berkeley*

Prove that there exists no finite group every nonidentity element of which commutes with exactly half of the elements of the group.

Solution by E. L. Spitznagel, Jr., Xavier University, Cincinnati, Ohio. The group of order one is, of course, a counterexample. If we eliminate this one trivial case, however, the proposition is true; for then it is obvious that the group must be of even order. But it is almost as obvious that it is of odd order, too, since the number of conjugates of an element is equal to the index of its normalizer and thus the nonidentity elements are divided into equivalence pairs. Hence no such group exists.

One can generalize the result slightly by replacing the word "half" with "$1/n$, $b=2, 3, \cdots$." The proof is an immediate generalization of the above.

Also solved by Richard Aron and H. E. Bell (jointly), K. F. Bailie, Joseph Beer, J. B. Brennan, A. W. Brunson, Ira Ewen, C. M. Frye, A. W. Fuller, H. A. Grindler, R. D. Horowitz, G. A. Kandall, Seymour Kass, C. R. MacCluer, R. L. McFarland, Stephen Montague, Shimon ben Moshe, M. G. Murdeshwar, T. L. Ray, H. J. Ricardo, H. W. E. Schwerdtfeger, Donna J. Seaman, John Stuelpnagel, Dennis Travis, W. L. Werner, John Wood, Oswald Wyler, and J. J. Zeltmacher, Jr.

Two Intersecting Ellipses

E 1560 [1963, 92]. *Proposed by Helen M. Marston, Douglass College*

An equilateral triangle PQR of side s has vertex Q on the x-axis and R on the y-axis. What is the locus of P?

I. *Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.* It is assumed that the vertex P may lie on either side of QR and hence the trace of P can be found in each quadrant. If we assume that the coordinates of P, Q, R are respectively (x, y), $(a, 0)$, $(0, b)$, then the locus of P is found by eliminating a and b among the equations

$$x^2 + (y - b)^2 = (x - a)^2 + y^2 = a^2 + b^2 = s^2,$$

which leads to the quartic

$$16(x^4 - x^2y^2 + y^4) - 8(x^2 + y^2)s^2 + s^4 = 0.$$

This quartic factors into

$$[4(x^2 + \sqrt{3}xy + y^2) - s^2][4(x^2 - \sqrt{3}xy + y^2) - s^2] = 0,$$

representing two congruent ellipses centered at the origin and having their major axes on the lines $x = y$ and $x = -y$.

II. *Solution by Kit Hanes, Orange Coast College.* This problem is a special case of a more general problem where PQR is any triangle. If O is the origin and T is the foot of the altitude from P to QR, let $\measuredangle OQR = \phi$, $QR = s$, $QT = a$, and $PT = b$. Then parametric equations for the locus of P are

$$x = (s - a)\cos\phi + b\sin\phi,$$
$$y = a\sin\phi + b\cos\phi.$$

Upon eliminating ϕ, the following equation is obtained:

$$(a^2 + b^2)x^2 - 2bsxy + [b^2 + (s - a)^2]y^2 = [a(s - a) - b^2]^2.$$

This is the equation of an ellipse except when $\measuredangle QPR = 90°$, in which case $a(s-a) - b^2 = 0$ and the locus is a segment on the line $ax = by$.

Also solved by H. L. Abbott and M. G. Murdeshwar (jointly), Louisa R. Alger, E. F. Allen, R. H. Anglin, Matthew Audibert, Joseph Basile, Joseph Beer, D. A. Blaeuer, B. L. Blankenship, Walter Bluger, W. G. Brady, John Brillhart, Brother L. F. Zirkel, Flor Cartuyvels, F. L. Celauro, H. L. Chow, D. I. A. Cohen and S. P. Cohen (jointly), J. T. Cornpone, Frank Dapkus, Gus DiAntonio, J. F. Dillon, R. Feinman, S. T. Fisk, Michael Goldberg, S. H. Greene, Florence N. Greville, Cornelius Groenewoud, Raymond Huck, A. R. Hyde, W. R. McEwen, D. C. B. Marsh, Beckham Martin, Gus Mavrigian, R. R. Muhl, R. A. Newcomb, P. S. Pappas, Stanton Philipp, Perry Scheinok, E. M. Scheuer, Sister M. Stephanie, W. K. Viertel, D. A. Vivian, Hazel S. Wilson, Samuel Wolf, Roscoe Woods, Oswald Wyler, K. L. Yocom, and the proposer.

The problem was located in Smith, Gale, and Neeley, *New Analytic Geometry*, Ginn and Co. (1928), p. 194, and, slightly modified, in Woods, *Analytic Geometry*, Macmillan (1939), p. 168.

A Minimal Determinant

E 1561 [1963, 92]. *Proposed by R. B. Killgrove, San Diego State College*

Find an explicit example of a positive nth order determinant whose elements are ones and minus ones and which has minimum possible value.

I. *Solution by R. A. Jacobson, South Dakota State College.* Add the first row to each of the remaining rows of the determinant. Since a 2 can be factored out of each of these rows, the minimum positive value must be at least 2^{n-1}. A determinant with this value is $|a_{ij}|$; $a_{ij} = 1$, $i \leqq j$; $a_{ij} = -1$, $i > j$.

II. *Solution by D. A. Moran, University of Chicago.* An nth order determinant of ones and minus ones can be regarded as $n!$ times the content of an n-dimensional simplex with one vertex at the origin, the others being chosen from the

vertex set of the n-cube with side two units long and center at the origin (in n-space). This simplex will have minimum content when its vertices (other than the origin) are chosen so as to lie on the star of a single vertex in the one-dimensional skeleton of one of the $(n-1)$-dimensional faces of the n-cube. Thus, for example, the minimum positive value is realized by the determinant of the $n\times n$ matrix with ones in the first column and along the main diagonal, and minus ones elsewhere. This value is 2^{n-1}.

Also solved by E. R. Barnes, Joseph Beer, Cornelius Groenewoud, M. S. Klamkin, D. C. B. Marsh, T. L. Ray, Arnold Singer, K. L. Yocom, and the proposer.

Klamkin called attention to S. W. Golomb and L. D. Baumert, *The search for Hadamard matrices*, this MONTHLY, Jan. 1963, p. 12.

A Two-Triangle Inequality

E 1562 [1963, 92]. *Proposed by D. Pedoe, Purdue University*

If ABC and $A'B'C'$ are any two triangles, a, b, c and a', b', c' their respective sides, and K, K' their respective areas, prove that

$$a^2(-a'^2+b'^2+c'^2)+b^2(a'^2-b'^2+c'^2)+c^2(a'^2+b'^2-c'^2)\geqq 16KK',$$

with equality if and only if the two triangles are similar.

Solution by the proposer. If one of the triangles is equilateral, the inequality becomes one attributed to Weitzenboeck:

$$a^2+b^2+c^2\geqq 4\sqrt{3}K,$$

with equality if and only if the triangle is equilateral.

To prove the two-triangle inequality, erect on BC, and on the same side of BC as the vertex A, a triangle $A''BC$ similar to triangle $A'B'C'$. Then the distance AA'' is zero if and only if the triangles ABC and $A'B'C'$ are similar. By the cosine rule we have

$$\begin{aligned}a'^2(AA'')^2 &= a'^2b^2+b'^2a^2-2aa'bb'\cos(C'-C)\\ &= a'^2b^2+b'^2a^2-2aa'bb'\cos C\cos C'-2aa'bb'\sin C\sin C'\geqq 0.\end{aligned}$$

Therefore $a'^2b^2+b'^2a^2-2aa'bb'\cos C\cos C'\geqq 8KK'$, or

$$a'^2b^2+b'^2a^2-(b^2+a^2-c^2)(b'^2+a'^2-c'^2)/2\geqq 8KK',$$

or

$$a^2(-a'^2+b'^2+c'^2)+b^2(a'^2-b'^2+c'^2)+c^2(a'^2+b'^2-c'^2)\geqq 16KK'.$$

For other proofs see D. Pedoe, "An inequality for two triangles," *Proc. Cambridge Philos. Soc.*, vol. 38, part 4, p. 397.

Also solved by A. N. Aheart, Joseph Beer, Leonard Carlitz, S. T. Fisk, R. A. Jacobson, D. C. B. Marsh, and Oswald Wyler.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College
COLLABORATING EDITORS: L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; and A. WILANSKY, Lehigh University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and with name of contributor on each sheet. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editors. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

5141. *Proposed by C. W. Kohls, Syracuse University*

An ideal I in a commutative ring is said to be primary if, whenever $ab\in I$, ither $a\in I$ or $b^n\in I$ for some positive integer n. Evidently a prime ideal is prinary. It is known that in any ring $C(X)$ of continuous real-valued functions, every prime ideal is absolutely convex (see Gillman and Jerison, *Rings of Continuous Functions*, p. 69). Show that, in fact, every primary ideal in $C(X)$ is absolutely convex.

5142. *Proposed by J. G. Mauldon, Oxford University, England*

Do there exist five coplanar oriented circles, not necessarily real, mutually inclined at the same angle?

5143. *Proposed by J. B. Roberts, Reed College*

Let $n_1, n_2, \cdots$ be a sequence of integers each greater than unity. Put $p_0=1$, $p_j=n_1 \cdots n_j$ for $j\geqq 1$. Let ϕ be a function of period 1 satisfying

$$\sum_{j=1}^{n_j}\phi\left(x+\frac{j-1}{n_j}\right)=0 \qquad j=1,2,\cdots.$$

Then the sequence $\{\phi_n\}$ is orthogonal on (0,1) when $\phi_n(x)=\phi(p_nx)$. (This generalizes a problem, p. 43 of Kaczmarz and Steinhaus, *Theorie der Orthogonal Reihen.*)

5144. *Proposed by Reuben Hersh, Fairleigh Dickinson University, Teaneck, N. J.*

If $F_0, A_i, B_i, i=1, \cdots, k$ are given constant $n\times n$ matrices, solve the matrix differential equation

$$\frac{dF}{dt}=\sum_{i=1}^{k}A_iFB_i, \qquad F(0)=F_0.$$

5145. *Proposed by Seth Warner, Duke University*

Prove that if E is a nonzero unitary module over a semi-simple ring K with identity, then every set of generators of E contains a basis of E if and only if K is a division ring.

5146. *Proposed by R. J. Cormier, University of Missouri*

In the Euclidean plane, denote the distance between two points p and q by pq and let k_1, k_2, k_3 be any three positive real numbers less than 1. Let p_1, q_1, r_1 be three noncollinear points and let p_n, q_n, r_n be defined inductively as follows for $n>1$:

$$p_{n-1}r_n = k_1 \cdot p_{n-1}q_{n-1}, \qquad p_{n-1}r_n + r_nq_{n-1} = p_{n-1}q_{n-1},$$

$$q_{n-1}p_n = k_2 \cdot q_{n-1}r_{n-1}, \qquad q_{n-1}p_n + p_nr_{n-1} = q_{n-1}r_{n-1},$$

$$r_{n-1}q_n = k_3 \cdot r_{n-1}p_{n-1}, \qquad r_{n-1}q_n + q_np_{n-1} = r_{n-1}p_{n-1}.$$

Let t_n be any point interior to the triangle with vertices p_n, q_n, r_n. Does the sequence $\{t_n\}$ converge? If so, what is the position of the limit point relative to the points p_1, q_1, and r_1?

5147. *Proposed by Eric D. Nix, New York City*

Prove or disprove the proposition: Every noncompact metrizable space admits an unbounded metric.

5148. *Proposed by Hewitt Kenyon, George Washington University*

Let us agree that F is eventually in a set A if and only if F is a filter and $B \subset A$ for some member B of F. Let us agree further that F is equivalent to G if and only if F and G are such filters that each is eventually in every member of the other.

Suppose that h is a function mapping the set X onto the set Y; suppose that F is an ultrafilter eventually in X, and let G be the ultrafilter consisting of maps by h of subsets of X belonging to F. Show that F is equivalent to G if and only if F is eventually in the set of fixed points of h.

In the terminology of Kenyon and Morse, *Runs*, Pacific Journal of Mathematics 8 (4), 1958, the problem may be rephrased: Suppose that f is a function, x is a full run, and x is eventually in the domain of f. Show that $f:x$ runs the same as x if and only if x is eventually in the set of fixed points of f.

5149. *Proposed by Michael Goldberg, Washington, D. C.*

Find r, the radius of the largest sphere on the surface of which two circles of given radius a, and two circles of given radius b, can be placed so that every point on the surface is within at least one of the regions (less than hemispheres) bounded by the four given circles.

5150. *Proposed by J. R. Clay, University of Washington*

Let $Z_n = \{0, 1, 2, \cdots, n-1\}$. There are n^{n^2} distinct closed binary operations that can be defined on Z_n. One of these is $+$, addition modulo n. How many of the remaining are left distributive over $+$?

SOLUTIONS

A Diophantine Equation

4638 [1955, 370]. *Proposed by Paul Erdös, University College, London*

Let $k > 2$. Does the following equation have any solutions in integers:

$$n(n+1)\cdots(n+k-1) = 2m(m+1)\cdots(m+k-1), \quad m+k-1 < n?$$

Comment by Andrzej Makowski, Warsaw, Poland. In *A note on pyramidal numbers* [1962, 637–8] S. L. Segal obtains results which imply that there are no solutions for $k = 3$.

It is interesting to note that without the condition $m+k-1 < n$, there would be infinitely many solutions when $m = n-1$, also when $m = n-2$, but none at all when $m = n-3$.

Switching Circuit

5026 [1962, 438; 1963, 580]. *Proposed by A. A. Mullin, University of Illinois*

Consider the following Boolean function of four variables

$$f(a, b, c, d) = b'(a + c) + b(ac' + a'd').$$

Show that every switching circuit realization of f contains at least seven contacts and that a circuit exists which contains precisely seven contacts.

II. *Comment by S. R. Petrick, Air Force Research Laboratories, Bedford, Mass.* There are a number of errors in the published solution [1963, 580]. In the first place, the prime implicant listed as $a'cd$ should be $a'cd'$. Determination of the minimal and irredundant normal forms is trivial for this small function. The unique normal form is $b'c \vee ac' \vee a'bd'$ and the only other irredundant form is $b'c \vee ac' \vee a'cd' \vee bc'd'$. The minimal form is listed in the published solution as (2), but (1) is not even a realization of the given truth function as it does not imply the developed normal form clauses $a'b'cd'$ and $a'b'cd$.

The figure given is a switching circuit representation of solution (2) and is not a representation of (1) at all. (The central vertical bar is unnecessary.)

The original problem was, in part, to show that every switching circuit realization of f contains at least seven contacts. The given solution assumes that a minimal circuit can be obtained by finding the simplest normal form truth function. It is not in general true, however, that the minimal contact circuit is given by a minimal truth function, let alone by a minimal normal form. Consequently, although Quine's algorithm can be used in establishing that

$b'c\vee ac'\vee a'bd'$ is the unique minimal normal form, this does not preclude the possibility that there exists a simpler truth function which is not a normal form, or even that there is no truth function which is isomorphic to a minimal switching circuit. In the present simple case it is easy to prove by means of the Residue Test that all seven literals a, a', b, b', c, c' and d' must be included in any representation of the given function. Thus, in the present case, the circuit corresponding to the minimal normal form is a minimal circuit, but this is not true in general.

Similar criticism by R. Jones, and R. L. Stanley.

Maximal Nonsingular Subspaces

5027 [1962, 438; 1963, 580]. *Proposed by A. J. Goldman, National Bureau of Standards*

Let $M_n(F)$ be the set of $n\times n$ matrices over the field F, considered as an n^2-dimensional vector space over F. Call a vector subspace of $M_n(F)$ nonsingular if all its nonzero members are nonsingular matrices. Find maximal nonsingular subspaces of $M_n(F)$.

II. *Solution by the Proposer.* The published solution [1963, 580] is unsatisfactory in that it assumes AB^{-1} has at least one eigenvalue λ in F. This is unjustified since the field F of scalars was not given as algebraically closed. I believe the following is free from this defect.

The set of all scalar multiples of some fixed nonsingular matrix forms a nonsingular *line*, and all nonsingular lines are obtained in this way. These lines are maximal, for if there existed nonsingular $n\times n$ matrices A, B which spanned a nonsingular *plane*, then B would not be a scalar multiple of A and so $xA+B$ would be nonsingular for every scalar x; hence, for any nonsingular C, $C^{-1}A^{-1}(xA+B)C=xI+C^{-1}A^{-1}BC$ would be nonsingular for all x. Choose C so that $C^{-1}(A^{-1}B)C$ is superdiagonal with leading term d_{11}; then $xI+C^{-1}A^{-1}BC$ is superdiagonal and is singular and nonzero for $x=-d_{11}$, a contradiction.

The error was reported also by George Bergman, John Pryce, and Earl J. Taft.

Normal Subgroup

5055 [1962, 926]. *Proposed by Peter Yff, American University, Beirut, Lebanon*

If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal.

I. *Solution by David Carlson, University of Wisconsin.* Let H be any subgroup of index p. Our proof depends upon:

(*) If $x\notin H$, then $x^i\notin H$ for $i=1, 2, \cdots, p-1$.

Suppose (*) were not true, i.e., $x^j \in H$ for some $j<p$, but $x^i \notin H$ for $i=1, \cdots, j-1$. Let m be the order of x, and n the order of G. As m divides n, and $j<p$, the smallest prime factor of n, we see that j does not divide m, and also $j<p\leqq m$. Thus $m=qj+r$ for some q and $0<r<j$. Now if e is the identity element, $e=x^m=(x^j)^q x^r$; and as $(x^j)^q \in H$, we must have $x^r \in H$, which contradicts our choice of j. Thus (*) must be true.

Now suppose that H is not normal in G. Then for some $x \notin H$ and $h \in H$, $y=xhx^{-1} \notin H$. By (*) we see that $1 \leqq i<j<p$ implies $x^iH \neq x^jH$; otherwise for some $k_1 \in H$, $x^ik_1=x^j$ and $x^{j-i}=k_1 \in H$. Thus $\{H, xH, \cdots, x^{p-1}H\}$, and similarly $\{H, yH, \cdots, y^{p-1}H\}$ form complete sets of cosets. Now for some $0<i<p$, $y^iH=xH$. But $y^i=xh^ix^{-1}$, and so for some $k_2 \in H$, $xh^ix^{-1}=xk_2$. Solving for x, we get $x=k_2^{-1}h^i \in H$, a contradiction. Thus H must be normal in G.

II. *Solution by H. G. Jacob, University of Massachusetts, and J. D. Reid, Amherst College.* Let H be a subgroup of index p in G. Then the number of cosets Hy of H in the double coset HxH is $m=[H: H \cap x^{-1}Hx]$; and since $[G: H]=p$, we have $m \leqq p$. But m is a divisor of the order of G, so that $m=1$ or $m=p$. In the former case, $x^{-1}Hx=H$, while in the latter, $x \in G=HxH$, hence $Hx=xH$. Thus H is normal.

Also solved by W. H. Bonney, A. Brumer and P. Sally, D. Ž. Djoković, Robert Gregorac, C. V. Holmes, Ossie J. Huval, G. J. Janusz, Seymour Kass, Donald Knuth, Barbara L. Osofsky, Veselin Perić, H. Schwerdtfeger, C. Seguin, Neal Zierler, and the proposer.

Editorial Note. An equivalent exercise is found in Burnside, *Theory of Groups of Finite Order*, Dover, ex. 5, p. 45.

Local Compactness under Open Mapping

5056 [1962, 926]. *Proposed by C. W. Kohls and M. E. Mahowald, Syracuse University*

In the book *Topology* by J. G. Hocking and G. S. Young, a space is defined to be locally compact if each point belongs to an open set whose closure is compact. On page 72, it is asserted that local compactness is invariant under open mappings. Give an example to show that this need not be the case if the spaces involved are not Hausdorff spaces.

I. *Solution by C. H. Cunkle's Class in Topology, Utah State University.* Let S be the set of positive integers with a basis the collection of all sets consisting of the two points $2n-1$ and $2n$. S is easily seen to be locally compact since each neighborhood $\{2n-1, 2n\}$ is compact. Let T be the same set with a basis the singleton 1 and the collection of all sets consisting of the two points 1 and n. T is not locally compact since the closure of each neighborhood is T, which is not compact. Define a mapping f of S onto T to be $f(2n-1)=1, f(2n)=n$. Every nonnull open set A in S contains a point $2n-1$ so that $f(A)$ contains 1 and is thus open in T.

II. *Solution by E. L. Spitznagel, Jr., Xavier University, Cincinnati, Ohio.* Let X be the real line and consider the following topologies for X:

$$\tau_1 = \text{the indiscrete topology,}$$
$$\tau_2 = \{(-\infty, a) \mid a \text{ real}\} \cup \{X\} \cup \{\phi\},$$
$$\tau_3 = \{X \sim \bigcup_{\text{finite}} \{a_i\} \mid a_i \text{ real}\} \cup \{\phi\},$$
$$\tau_4 = \{(-\infty, a) \sim \bigcup_{\text{finite}} \{a_i\} \mid a, a_i \text{ real}\} \cup \{X \sim \bigcup_{\text{finite}} \{a_i\} \mid a_i \text{ real}\} \cup \{\phi\}.$$

Then, since (X, τ_1) and (X, τ_3) are locally compact and (X, τ_2) and (X, τ_4) are not, and since the identity mappings

$$I: \quad (X, \tau_1) \to (X, \tau_2),$$
$$I: \quad (X, \tau_3) \to (X, \tau_4)$$

are open, we have two counterexamples. The first of these is perhaps the most obvious example available, since every mapping from an indiscrete space is open. The second, a slight modification of the first, is a bit more interesting, since both spaces satisfy the T_1 separation axiom.

Also solved by David Andrew, Joseph Beer, M. D. Cox, Helen F. Cullen, M. J. Lempel, R. L. McKinney, Barbara L. Osofsky, Dennis Sjerve, R. H. Sorgenfrey, J. D. Thomas, J. V. Whittaker, Alexander Zabrodsky, and the proposers.

Row Sums of a Matrix

5057 [1962, 926]. *Proposed by F. M. Hornyak, Virginia Polytechnic Institute*

An analytic expression $f(m, n)$ is sought for the sum of the m elements in the nth row of a rectangular array of numbers formed as follows: The elements of the first row are all unity. Each of the m elements in the second row is obtained by adding all elements in the first row directly above and to the left; hence, these are the integers 1 through m. Each element of the third row is the sum of all elements in the second row directly above and to the right of the element; hence we have the elements $\frac{1}{2}m(m+1)$ through m. The array is completed by continuing similar additions alternately to the left and to the right.

Solution by L. Carlitz, Duke University. Let $a_{r,s}$ denote the element in the rth row and sth column of the array and put

$$f(r) = f(m, r) = \sum_{s=1}^{m} a_{r,s}.$$

Then we have $a_{2r,s} = \sum_{j=1}^{s} a_{2r-1,j}$, $a_{2r+1,s} = \sum_{j=s}^{m} a_{2r,j}$. It follows that

$$\begin{aligned} a_{2r+1,s} &= f(2r) - \sum_{j=1}^{s-1} a_{2r,j} = f(2r) - \sum_{j=1}^{s-1} \sum_{k=1}^{j} a_{2r-1,k} \\ &= f(2r) - \sum_{k=1}^{s} (s-k) a_{2r-1,k}. \end{aligned} \tag{1}$$

Applying this recurrence, we get

$$a_{2r+1,s} = f(2r) - \sum_{k=1}^{s} (s-k)\left\{f(2r-2) - \sum_{j=1}^{k} (k-j)a_{2r-3,j}\right\}$$
$$= f(2r) - \binom{s+1}{2} f(2r-2) + \sum_{j=1}^{s} a_{2r-3,j} \sum_{k=j}^{s} (s-k)(k-j).$$

Since $\sum_{k=j}^{s} (s-k)(k-j) = \binom{s-j+1}{3}$, we have

$$a_{2r+1,s} = f(2r) - \binom{s+1}{2} f(2r-2) + \sum_{j=1}^{s} \binom{s-j+1}{3} a_{2r-3,j}.$$

Continuing in this way we are led to

$$a_{2r+1,s} = \sum_{j=0}^{r} (-1)^j \binom{s+j-1}{2j} f(2r-2j), \tag{2}$$

where $f(0)=1$. This is obvious for $r=0$. Assuming that (2) holds for the value r, it follows from (1) that

$$a_{2r+3,s} = f(2r+2) - \sum_{k=1}^{s} (s-k) \sum_{j=0}^{r} (-1)^j \binom{k+j-1}{2j} f(2r-2j)$$
$$= f(2r+2) - \sum_{j=0}^{r} (-1)^j f(2r-2j) \sum_{k=1}^{r} (s-k) \binom{k+j-1}{2j}.$$

But

$$\sum_{k=1}^{s} (s-k)\binom{k+j-1}{2j} = (s-j-1)\sum_{k=1}^{s}\binom{k+j-1}{2j}$$
$$- (2j+1)\sum_{k=1}^{s}\binom{k+j-1}{2j+1}$$
$$= \binom{s+j}{2j+2},$$

and the induction is complete.

From (2) and $a_{2r,s} = \sum_{k=1}^{s} a_{2r-1,k}$, we get

$$a_{2r,s} = \sum_{j=0}^{r-1} (-1)^j \binom{s+j}{2j+1} f(2r-2j-2). \tag{3}$$

Summing with respect to s, (2) and (3) give

$$f(2r+1) = \sum_{j=0}^{r} (-1)^j \binom{m+j}{2j+1} f(2r-2j), \tag{4}$$

$$\sum_{j=0}^{r} (-1)^j \binom{m+j}{2j} f(2r-2j) = 0 \qquad (r > 0). \tag{5}$$

It does not seem possible to get a simple closed expression for the polynomial $f(r)$ of degree r. Now $f(3)=m(m+1)(2m+1)/6$, $f(4)=m(m+1)(5m^2+5m+2)/24$, $f(5)=m(m+1)(2m+1)(2m^2+2m+1)$, etc. However, if we put

$$F(x) = \sum_{r=0}^{\infty} f(2r)x^r, \qquad G(x) = \sum_{r=0}^{\infty} f(2r+1)x^r,$$

then (5) and (4) are simply

$$F(x) = \left\{ \sum_{j=0}^{m} (-1)^j \binom{m+j}{2j} x^j \right\}^{-1},$$

$$G(x) = F(x) \sum_{j=0}^{m} (-1)^j \binom{m+j}{2j+1} x^j.$$

Also solved by R. A. Jacobson.

Metric Space

5058 [1962, 1012]. *Proposed by Albert Wilansky, Lehigh University*

Must the one point compactification of a locally compact metric space be metrizable?

I. *Solution by W. J. Pervin, Pennsylvania State University.* Any uncountable set with the discrete topology is an example of a locally compact metrizable space whose one point compactification is not metrizable. In particular, it would not satisfy the first axiom of countability at the "ideal" point since the complement of any open set containing that point must be finite. Then, although the intersection of the members of a countable base at the "ideal" point should be just the "ideal" point, its complement would only be countable.

This space was first described by M. K. Fort, this MONTHLY, 62 (1955) p. 372 in answer to another problem (4577) by Wilansky.

II. *Solution by Zulette Gordon, University of California at Santa Barbara.* If M is a nonseparable locally compact metric space then the one point compactification, $\hat{M}$, will not be metrizable. For if $\hat{M}$ is a compact metric space then it is second countable. Now second countability is a hereditary property, and hence M must have a countable base and therefore be separable, contrary to hypothesis.

This is the proof outlined in Hocking and Young, *Topology*, p. 76.

III. *Solution by Joel Pitcairn, Huntingdon Valley, Pa.* The answer can be found in Bourbaki, *Topologie Générale*, chapter IX, no. 2, ex. 22. The one point compactification of a locally compact metric space E is metrizable if and only if E has a countable base.

Also solved by D. R. Anderson, W. G. Andrews, Ralph Bennett, W. H. Bonney, Marshall Cohen, Helen F. Cullen, Orrin Frink, M. A. Geraghty, A. F. Gibbard, Seymour Goldberg, W. J.

Gray, Charles Himmelberg, B. E. Johnson, M. D. Mavinkurve, Keiko Minaguchi, D. A. Moran, J. M. H. Olmsted, Helmut Salzmann, J. J. Schäffer, Dennis Sjerve, J. R. Swenson, Dennis Travis, Seth Warner, W. C. Waterhouse, Bro. T. C. Wesselkamper, A. B. Willcox, and A. Zabrodsky.

A limit holding for all continuous functions

5060 [1962, 1013]. *Proposed by H. S. Shapiro, New York University*

Let $f(x)$ be continuous for $0 \leqq x \leqq 1$. Prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{k}{n}\right) \Big/ 2^n = 0.$$

I. *Solution by D. Ž. Djoković, University of Belgrade, Yugoslavia.* We remark that $f(x)$ is uniformly continuous on $[0, 1]$. Hence, for arbitrary given $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $|f(k/n) - f((k+1)/n)| < \epsilon$, $(n > N;\ k = 0, 1, \cdots, n-1)$. For $n > N$, we have

$$S_n = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{k}{n}\right) = \frac{1}{2^n} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left[f\left(\frac{k}{n}\right) - f\left(\frac{k+1}{n}\right) \right],$$

$$|S_n| < \frac{\epsilon}{2^n} \sum_{k=0}^{n-1} \binom{n-1}{k} = \epsilon/2,$$

from which the desired result follows.

II. *Solution by P. T. Bateman, University of Illinois.* Given any positive number ϵ, we can find a polynomial function P such that $|f(x) - P(x)| \leqq \epsilon$ for all x in $[0, 1]$. This is the Weierstrass Approximation Theorem. If n is greater than the degree of P, then by the theory of finite differences

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P\left(\frac{k}{n}\right) = 0.$$

Thus, if n is sufficiently large,

$$\left| 2^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{k}{n}\right) \right| = \left| 2^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left\{ f\left(\frac{k}{n}\right) - P\left(\frac{k}{n}\right) \right\} \right|$$

$$\leqq 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \epsilon = \epsilon.$$

Since ϵ is arbitrary, the assertion of the problem is established.

III. *Solution by Gordon Crawford, A. W. Marshall, and Frank Proschan, Boeing Scientific Laboratories, Seattle.* Let

$$L_n = \sum_{k=0}^{n} (-1)^k f\left(\frac{k}{n}\right) \binom{n}{k} 2^{-n}.$$

Then

$$\left| L_n \right| = \left| L_n - f(\tfrac{1}{2}) \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{-n} \right| \leqq \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) - f(\tfrac{1}{2}) \right| \binom{n}{k} 2^{-n},$$

which converges to $|f(\frac{1}{2})-f(\frac{1}{2})|=0$ by Bernstein's theorem. (See G. G. Lorentz, *Bernstein Polynomials*, pp. 5–6.)

One can similarly prove the same result with 2^{-n} replaced by $p^k(1-p)^{n-k}$, $0<p<1$. Furthermore, the continuity of f may be replaced by the assumption that f is bounded and has right and left hand limits at p.

Also solved by R. L. Adler, I. N. Baker, M. T. Bird, Robert Breusch, J. L. Brown, Jr., L. Carlitz, G. Di Antonio, S. I. Drobnies, F. A. Ficken, N. J. Fine, Harley Flanders, J. Foster, Joseph Gayda, A. J. Goldman and Joel Levy, S. H. Greene, William Harkness and S. G. Mrowka, J. C. Hickman, R. H. Hines, Jr., J. M. Horner, J. B. Kelly, J. P. King, J. Koekoek, A. E. Livingston, J. S. MacNerney, M. D. Mavinkurve, Brockway McMillan, W. L. Murdock, F. J. Papp, Jr. Stanton Philipp, Ronald Pyke, John Rausen, Sylvester Reese, John V. Ryff, Paul Schaefer, F. G. Schmitt, Jr., Z. Šidák, Michael Skalsky, W. C. Waterhouse, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, Carleton College and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. Programmed Materials: K. O. May, Carleton College, Northfield, Minn. Films: E. P. Vance, Oberlin College, Oberlin, Ohio.

Statistical and Inductive Probabilities. By Hugues Leblanc. Prentice-Hall, Englewood Cliffs, N. J., 1962. xii+148 pp. $5.00.

There has been considerable concern in the past few decades with the meaning of probability. Most statisticians take "probability" to be a term with an empirical meaning, a meaning which is closely related to, if not identical with, long run frequencies. Many philosophers insist that "probability" also be used in the logical sense of partial implication; it is in this sense that we refer to the probabilities of particular events. Hugues Leblanc has attempted to bring these two notions of probability closer together and to provide some connection between them.

The general framework of his book is a family of sublanguages L^N (each containing the names of N distinct individuals) of a language L^∞ (containing the names of an infinite number of individuals). The novel feature of the two chapters on statistical probabilities is that statistical probabilities, instead of being defined as class ratios, or the limits of class ratios, are defined in terms of the *weights* of individuals. The illustrations he gives of this, however, make it

clear that it is never *necessary* to assign unequal weights to the individuals in a universe of discourse: if we do not want to assign equal weights to beans in an urn, it is because we are going to assign equal weights to beans *drawn from* an urn. In general, if we want to assign rational weights $w_1 \cdot \cdot \cdot w_N$ to the individuals of L^N, we can instead always assign *equal* weights to the individuals of L^M, where w_1M individuals of L^M have just the same properties as the first individual of L^N, w_2M individuals of L^M have just the same properties as the second individual of L^N, and so on. (And surely there can be no pressing need for irrational weights.)

So far as inductive probabilities go, he follows the lead, with some modifications, of most writers on logical probability. Probabilities are numbers assigned to the sentences of L^N or L^∞, subject to conditions embodied in the usual axioms of probability. (But Leblanc explores very carefully the relations among various possible sets of axioms—definitions, in his system—of inductive probabilities.) A probability function assigning these numbers to sentences (or to pairs of sentences) is simply presumed to be on hand. Various ways of specifying this function further are mentioned: Carnap's method, in terms of equiprobable state descriptions (giving rise to c^*); personalistic probabilities; habits of inference.

The rapprochement between statistical probabilities and inductive probabilities takes place through the idea that inductive probabilities can be taken as estimates of truth values, and thus, in a derivative sense, of statistical probabilities. Since there is no extended discussion of the notion of an "estimate," and since he imposes only the most general restrictions on the inductive probability function, it is difficult to see how *this* notion of an estimate provides an interesting connection between frequencies and inductive probabilities.

The book is very carefully worked out, and although I don't believe it quite accomplishes the reconciliation advertised, it is a good introduction to both statistical and inductive probabilities.

HENRY E. KYBURG, JR., Denver University

Infinite Series. By I. I. Hirschman, Jr. Holt, Rinehart and Winston, New York, 1962. x+173 pp. $4.00.

This "pamphlet on infinite series" consists of seven chapters and a short appendix. The first three chapters deal with the usual convergence tests, Taylor series, and Fourier series; the next two are devoted to uniform convergence, rearrangements, double series, and introductory concepts of summability. In Chapters 6 and 7 a more systematic treatment of power series and real analytic functions and of Fourier series and their summability is given.

The book is a carefully written text. Indeed it is a teachable text which will prove successful in the classroom and can also be used for independent study. The problems included are well-graded and carefully chosen. The book has strong claims for adoption as a textbook for a two-hour course on infinite series designed for engineers to supplement their usual advanced calculus course.

While it certainly is a matter of opinion, the reviewer feels that the logical

place for the concepts of boundedness, bounds of a sequence, the Cauchy criterion for convergence, the definitions of continuity, and uniform continuity, which appear in the appendix, is in the text itself at the appropriate places. These concepts are basic in the development of the subject matter of the book.

The book is attractive and makes very pleasant reading; also it is free from any misprints of consequence.

M. S. RAMANUJAN, University of Madras, India

Ordinary Differential Equations. By L. S. Pontryagin. Translated by Leonas Kacinskas and Walter B. Counts. Addison-Wesley, Reading, Mass. 1962. vi+298 pp. $8.75.

Professor Lefschetz's laudatory review of the Russian original (MR 23A, No. 362) applies to this excellent translation. The book enforces the notion of rigor. Its proofs are models of clarity. When the general Routh-Hurwitz criterion or the sufficiency of Kirchhoff's Laws to define a network flow is needed but not proved as being too difficult, this fact is carefully stated.

The reviewer feels that the book is most appropriate for a year course on the Junior level which emphasizes ideas rather than standard tricks.

The teacher must be prepared to add problems for the student. The student must have an understanding of uniformity in analysis, vector spaces, norms and matrices in algebra, or the course must be taught so that this understanding can be obtained. As such notions occur, they can be used as motivation for extra work—a quality too often lacking in the instruction of engineers or even of mathematicians. Some material for supplementary work is included as an appendix on linear algebra in the present book.

There is an adequate index, the format is good, if crowded, and the reviewer found few typographical errors: on page 111, "$t-m_2$" for "$t\to m_2$," and on page 238, line 7, "hens" for "then." Errors in English are also minimal, remarkable in a translation from Russian. On page 58 "Rauss" should be "Routh"; on page 65, first sentence of subsection (C), punctuation is lacking and "where" should be replaced by "in which" and "which are" by "is"; on page 174, line 6 "$k=1$" is inferior to "k has the value 1"; and on page 201, line 15, "only if" is inferior to "whenever." These are small matters.

The price is $8.75; that for the Russian original, according to *Mathematical Reviews*, is 0.69 rubles, i.e., about 70 cents. This illustrates a problem applicable to almost all textbooks published in the United States which should be given serious thought by all concerned.

HARRY GOHEEN, Oregon State University

An Elementary Introduction to the Theory of Probability. By B. V. Gnedenko and A. Ya. Khinchin. Freeman, San Francisco and London, 1961. 140 pp. $1.75.

This paperback is a translation of a book which appeared in its first Russian edition in 1945 and has been reprinted without major changes through four editions. It is intended for readers familiar with high school algebra and such concepts as the absolute value of a real number and the use of the summation symbol.

The concept of probability is developed intuitively, in terms of limits of relative frequencies. The presentation always begins with specific examples, and proceeds to general statements for discrete random variables. The theory covered goes as far as Bayes' Rule, Bernoulli's Theorem, the mathematical expectation (called "mean value"), variance and standard deviation, Chebyshev's inequality and the law of large numbers, and a qualitative discussion of normal distributions.

The selection of examples on which the basic concepts and theorems are illustrated is quite limited; about half of them are taken from marksmanship, performance of artillery pieces and similar military applications. This choice of examples may well be due to the time when the book was originally written.

The translation is rather literal and it frequently looks as if the translator had chosen from a dictionary the incorrect one of several English counterparts of an original word (e.g. "magnitude" is consistently used instead of "quantity"). Also the mathematical meaning must have often been obscure for the translator who then failed to make it clear to the reader, (e.g. the italicized statement III on page 122 can hardly be understood).

The book contains no exercises and is clearly not intended to serve as a textbook, but as an introduction to the theory of probability for readers with limited mathematical background.

Z. W. BIRNBAUM, University of Washington

Recent Developments in General Relativity. Pergamon Press, New York, 1962. 472 pp. $8.00.

This book, dedicated to L. Infeld in connection with his 60th birthday, refers wholly to the recent developments in the classical Einstein formulation of the general relativity. It does not contain any remarks on the recent trends of revising completely the fundamentals of both theories of relativity (nonconstancy of the speed of light, generalization of the metric, paradoxes in time and length dilatations, etc.). The book consists of many papers full of the most brilliant ideas and fascinating mathematical formalism; but, because of the trends mentioned above, it is impossible to predict their value or how long they will preserve their validity. The book is divided into two parts. Part I consists of seven longer papers reviewing the actual status of the basic problems in general relativity: Bazanski (The problem of motion), Bergmann and Komar (Quantization of the gravitational field), Bondi (Relativity and cosmology), Ginzburg (Experiments), Lichnerowicz and Fourès-Bruhat (Mathematical

problems in relativity), Pirani (Gravitational radiation theory) and Tonnelat (Unified field theory). Part II contains 38 original contributions: Anderson (Absolute change in general relativity), Arnowitt, Deser and Misner (Canonical analysis), Bazanski (Equations of motion), Belifante (Two kinds of Schroedinger equations), Bonnor (Birkhoff's theorem), Costa de Beauregard (Asymmetry of the tensor), DeWitt (Invariant commutators for the gravitational field), Dirac (Gravitational and spinor fields), Ehlers (Relativistic hydrodynamics), Fock (Einsteinian statics), Gold (The arrow of time), Goldberg, I. (Quantum theory of gravitation), Goldberg, J. N. (Dynamic variables), Gupta (Quantum theory of gravitation), Havas (Equations of motion of point particles), Hoffman (Extension of Birkhoff's theorem), Jordan ((a) Dirac hypothesis, (b) Ambarzumian's conception), Klein (Mach principle), Kundt and Hoffmann (Gravitational standard time), Kundt (Equivalence problem), Lanczos (Riemann-Christoffel tensor), Laurent (Covariant quantum theory), Marder (Isometric space-time), Ozsvath and Schuecking (Anti-Mach metric), Papapetrou and Teeder (Shock waves), Peres (Motion of pole particles), Peres and Rosen (Boundary conditions), Petrov ((a) Classification of gravitational fields, (b) Field geometry), Rosen (Rainich geometry), Sachs (Asymptotic behavior of waves), Schild (Conservative gravitational theories), Sciama (Charge and spin), Synge (Chronometry), Taub (Incompressible fluids), Trautman (Waves), and Tulczyjew, B. and W. (Multipole formalism). Mathematicians and theoretical physicists who stay away from the recent revolt against restrictions, paradoxes, inconsistencies and lack of determined systems in the classical relativistic theories and follow firmly the Einsteinian formulations, however idealistic they may be, will enjoy reading the book.

M. Z. v. KRZYWOBLOCKI, Michigan State University

Frans van Schooten der Jüngere. By J. E. Hofmann. (Boethius, Texte und Abhandlungen zur Geschichte der exakten Wissenschaften, Band II) F. Steiner Verlag, Wiesbaden, 1962. 54 pp., 4 pictures.

Descartes lived in the Dutch Republic from 1629 to 1649, the period of his greatest productivity. Living a retired bachelor's life, he made his ideas known through highly advanced books and correspondence with his peers. His only mathematical book, the "Géométrie" of 1637, was sketchy and at many places obscure. The triumph of Cartesian ideas in mathematics, the creation of a Cartesian school of thought, both in Holland and to a certain extent also abroad, is in no small degree due to the Leyden professor Frans van Schooten (1615–1660).

Van Schooten, grandson of a Flemish refugee baker, son of another Frans who taught mathematics at the Leyden engineering school, succeeded his father in 1635. During 1641–43 he travelled in France and England, where he met leading mathematicians; after his return he continued teaching at Leyden until his untimely death at the age of 45. He belonged to the small circle with whom Descartes had personal connections and to which the elder Huygens also be-

Differential Equations. By H. S. Bear. Addison-Wesley, Reading, Mass., 1962. 207 pp. $7.50.

This book has little to distinguish it from dozens of other texts designed for an introductory course in differential equations following the calculus. There is the usual material dealing with first order equations and special methods, nth order linear equations and special methods (including operational methods and the Laplace transform), Picard's existence and uniqueness theorem, and a brief discussion of systems of differential equations. There is nothing new or interesting about the problems. The book could easily have been written in the year 1890.

Differential equations is not after all such a cut and dried subject, and this old fashioned type of course is neither of interest to mathematicians nor is it a good introduction to the subject for engineers and scientists. It is possible at this level to introduce some modern ideas and more up-to-date theory, to recognize the existence of computing machines, to contrast differences between linear and nonlinear systems, and to look at some of the problems which are of genuine mathematical and practical interest. A thorough revision in the teaching of introductory courses in differential equations is long overdue in this country and elsewhere.

J. P. LaSalle, RIAS, Baltimore, Md.

NEWS AND NOTICES

Edited by Raoul Hailpern, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo 14, New York.

PERSONAL ITEMS

State University of Iowa: Dr. R. E. Peinado, University of Nebraska, has been appointed Assistant Professor; Assistant Professor J. C. Hickman has been promoted to Associate Professor.

Wesleyan University: Professor W. H. Gottschalk, University of Pennsylvania, has been appointed Professor; Dr. Robert Ellis, University of Pennsylvania, has been appointed Visiting Professor; Dr. Morton Lowengrub, University of Glasgow, Scotland, has been appointed Assistant Professor.

Dr. J. E. Allen, Oklahoma State University, has been appointed Assistant Professor at North Texas State University.

Assistant Professor H. J. Arnold, Princeton University, has been appointed Associate Professor at Bucknell University.

Mr. P. J. Arpaia, Clarkson College, has been promoted to Assistant Professor.

Associate Professor E. S. Ashcraft, Stetson University, has been promoted to Professor.

Associate Professor E. W. Averill, Parsons College, has been promoted to Professor of Statistics.

Associate Professor B. J. Ball, University of Georgia, has been promoted to Professor and appointed Head of the Mathematics Department.

Mrs. Ruth S. Beck, University of Pennsylvania, has been appointed Assistant Professor at Northeastern University.

Mr. J. A. Beekman, University of Minnesota, has been appointed Assistant Professor at Ball State Teachers College.

Dr. C. P. Bell, University of Georgia, has been appointed Assistant Professor at Wofford College.

Dr. R. T. Blackburn, Shimer College, has been appointed Dean of Faculty.

Assistant Professor Ranko Bojanic, University of Notre Dame, has been appointed Associate Professor at Ohio State University.

Dean A. H. Bowker, Stanford University, has been appointed Chancellor of the City University of New York.

Dr. L. E. Boyer, Pennsylvania Department of Public Instruction, has been appointed Acting Director of the Bureau of State Colleges.

Dr. D. R. Brown, Louisiana State University, has been appointed Assistant Professor at the University of Tennessee.

Dr. W. G. Brown, University of Toronto, has been appointed Assistant Professor at the University of British Columbia.

Professor I. H. Brune, Iowa State Teachers College, has been appointed Professor of Education at Bowling Green State University.

Associate Professor J. R. Byrne, Portland State College, has been promoted to Professor.

Mr. F. P. Callahan, General Atronics, Bala Cynwyd, Pennsylvania, has accepted a position as Senior Systems Engineer with the Radio Corporation of America, Moorestown, New Jersey.

Dr. D. H. Carlson, University of Wisconsin, has been appointed Assistant Professor at Oregon State University.

Dr. P. L. Chessin, International Business Machines, New York, has accepted the position of Manager of the Advanced Professional Development Department, International Business Machines, Bethesda, Maryland.

Mr. J. L. Chittenden, Duke University, has been appointed Professor at Manatee Junior College.

Professor W. G. Clark, Mount Union College, has been appointed Chairman of the Mathematics Department.

Dr. C. L. Coates, Jr., General Electric Research Laboratories, Scotia, New York, has been appointed Professor of Electrical Engineering at the University of Texas.

Dr. I. G. Connell, McGill University, has been appointed Assistant Professor.

Mr. D. A. Crothamel, University of Illinois, has been appointed Assistant Professor at Bloomsburg State College.

Associate Professor C. H. Cunkle, Utah State University, has been appointed Professor and Chairman of the Mathematics Department at Clarkson College.

Mr. W. F. Cutlip, Northern Michigan University, has been appointed Assistant Professor.

Dr. A. E. Danese, Union College, has been appointed Associate Professor at the State University of New York at Buffalo.

Assistant Professor B. R. Davis, University of Toledo, has been appointed Chairman of the Mathematics Department.

Associate Professor H. A. Dennis, Lamar State College of Technology, has been promoted to Professor.

Dr. R. D. Depew, International Business Machines, Kingston, New York, has been promoted to Advisory Mathematician.

Dr. Betty C. Detwiler, University of Illinois, has been appointed Assistant Professor at Western Kentucky State College.

Mr. P. D. Drees, Eastern Michigan University, has been appointed Coordinator of Mathematics for Birmingham Public Schools, Birmingham, Michigan.

Assistant Professor Ruth L. Erckmann, Iowa State Teachers College, has been appointed Assistant Professor at Eastern Illinois University.

Dr. A. B. Farnell, Convair, San Diego, California, has been appointed Professor at Colorado State University.

Assistant Professor Chester Feldman, Monmouth College, has been appointed Assistant Professor at Kent State University.

Associate Professor W. R. Ferrante, University of Rhode Island, has been awarded a Fulbright Fellowship for the 1963 academic year and will lecture at Al-Hikma University, Baghdad, Iraq.

Associate Professor D. C. Fielder, Georgia Institute of Technology, has been promoted to Professor.

Professor M. K. Fort, Jr., University of Georgia, has been named Barrow Professor of Mathematics.

Assistant Professor T. S. Frank, Le Moyne College, has been appointed Chairman of the Mathematics Department.

Assistant Professor Natalie Frazis, University of Detroit, has been appointed Assistant Professor at Mercy College of Detroit.

Assistant Professor J. M. Gary, New York University, has accepted a position as staff member with the National Center for Atmospheric Research, Boulder, Colorado.

Dr. Walter Gautschi, Oak Ridge National Laboratories, Oak Ridge, Tennessee, has been appointed Professor at Purdue University.

Dr. Dorothy C. Geddes, Hunter College High School, has been appointed Assistant Professor at Brooklyn College.

Dr. L. J. Gerende, Yale University, has been appointed Assistant Professor of Preventive Medicine at the University of Pittsburgh.

Mr. S. K. Gibson, General Electric Co., Utica, N. Y., has accepted the position of Principal Maintainability Engineer with the Republic Aviation Corporation, Farmingdale, Long Island.

Assistant Professor H. R. Gillette, Los Angeles State College, has accepted a position as Supervisor of Numerical Methods with North American Aviation, Downey, California.

Dr. R. W. Gilmer, Jr., University of Wisconsin, has been appointed Assistant Professor at Florida State University.

Associate Professor G. R. Glabe, Sacramento State College, has been appointed Chairman of the Mathematics Department.

Mr. Marvin Goodman, Jamaica High School, Queens, New York, has been appointed Assistant Professor at Monmouth College.

Dr. N. R. Grabois, Lafayette College, has been appointed Assistant Professor at Williams College.

Dr. A. A. Grau, on leave of absence from Oak Ridge National Laboratories, Oak Ridge, Tennessee, has been appointed Professor of Mathematics and Engineering Sciences at Northwestern University.

Assistant Professor D. S. Greenstein, Northwestern University, has been promoted to Associate Professor.

Associate Professor W. C. Guenther, University of Wyoming, has been promoted to Professor.

Professor G. A. Heuer, Concordia College, has been appointed Chairman of the Mathematics Department.

Professor J. E. Houle, Seton Hall University, has been appointed Professor and Chairman of the Mathematics Department at Pace College.

Associate Professor Raymond Huck, Marietta College, has been promoted to Professor.

Dr. R. F. Jolly, University of Texas, has been appointed Assistant Professor at the University of California at Riverside.

Mrs. Dorothy A. Kennedy, Sweet Home Central High School, Amherst, New York, has been appointed Assistant Professor at the State University College at Buffalo.

Dr. R. B. Kirchner, Carleton College, has been promoted to Assistant Professor.

Assistant Professor Celia E. Klotz, Washington State University, has been promoted to Associate Professor.

Dr. D. E. Knuth, California Institute of Technology, has been promoted to Assistant Professor.

Dr. R. M. Krause, National Science Foundation, has been appointed Associate Program Director of Mathematics.

Dr. G. J. Kurowski, Duke University, has been appointed Assistant Professor at the University of California at Davis.

Dr. H. E. Kyburg, Jr., University of Denver, has been appointed Associate Professor of Philosophy at Wayne State University.

Mr. G. N. Landes, Convair, Fort Worth, Texas, has accepted a position as Group Engineer with General Dynamics, Forth Worth, Texas.

Associate Professor R. D. Larsson, Clarkson College, has been appointed Dean of the College at Mohawk Valley Community College.

Associate Professor E. A. Leonhardt, Union College, has been promoted to Professor.

Dr. D. R. Lick, Purdue University, has been appointed Assistant Professor at New Mexico State University.

Associate Professor C. H. Lindahl, Iowa State University, has been promoted to Professor.

Professor L. F. McAuley, University of Wisconsin, has been appointed Professor at Rutgers, The State University.

Mr. L. B. McKee, Shell Chemical Corporation, Deer Park, Texas, has accepted a position as Process Engineer with the Rheem Manufacturing Company, Richmond, California.

Professor M. St. J. MacPhail, Carleton University, has been appointed Dean of the Faculty of Graduate Studies.

Associate Professor Mark E. Mahowald, Syracuse University, has been appointed Professor at Northwestern University.

Mr. W. W. Malone, Louisiana State University at Alexandria, has been appointed Assistant Professor.

Mr. R. A. Martin, Duke University, has been appointed Assistant Professor at Rider College.

Associate Professor G. H. Meisters, University of Nebraska, has been appointed Associate Professor at the University of Colorado.

Associate Professor E. P. Merkes, Marquette University, has been appointed Visiting Associate Professor at the University of Cincinnati.

Assistant Professor F. T. Metcalf, University of Maryland, has been appointed Assistant Research Professor at the Institute for Fluid Dynamics and Applied Mathematics.

Mr. D. R. Miller, University of Toronto, has been appointed Assistant Professor at the University of Western Ontario.

Assistant Professor D. W. Miller, University of Nebraska, has been promoted to Associate Professor.

Mr. R. H. Miller, Wisconsin State College, has been appointed Assistant Professor at Nebraska State College.

Assistant Professor W. B. Miller, Moravian College, has been appointed Associate Professor at Worcester Polytechnic Institute.

Assistant Professor C. L. Miracle, University of Minnesota, has been promoted to Associate Professor.

Assistant Professor A. C. Moeller, Marquette University, has been appointed Acting Dean of the College of Engineering.

Dr. Hastings Moore, University of Colorado, has been appointed Assistant Professor.

Dr. K. B. Morgan, Mount Kisco High School, Mount Kisco, New York, has been appointed Assistant Professor at Pace College.

Associate Professor Leo Moser, University of Alberta, has been appointed Associate Head of the Mathematics Department.

Mr. T. N. Murphy, Thornleigh College, England, has been appointed Assistant Professor at Memorial University of Newfoundland.

Associate Professor R. H. Niemann, Colorado State University, has been promoted to Professor.

Associate Professor E. A. Nordhaus, Michigan State University, has been promoted to Professor.

Mr. J. M. O'Neil, Auburn University, has been appointed Assistant Professor at Northeast Louisiana State College.

Mr. Dick Osborne, Rensselaer Polytechnic Institute, has been appointed Assistant Professor at San Diego State College.

Assistant Professor M. E. Paradise, Nebraska State Teachers College, has been appointed Associate Professor and Acting Chairman of the Division of Science and Mathematics at Chadron State College.

Assistant Professor Jane Pease, Brooklyn College, has been appointed Assistant Professor at Manhattanville College.

Mr. J. C. Pelzl, Washington State University, has been appointed Assistant Professor at Mankato State College.

Mr. M. P. Perriens, Technical Operations, Washington, D. C., has accepted a position as Senior Associate Systems Analyst with International Business Machines, Rockville, Maryland.

Dr. Barth Pollak, Institute for Defense Analyses, has been appointed Associate Professor at the University of Notre Dame.

Dr. N. J. Pullman, McGill University, has been appointed Assistant Professor.

Assistant Professor J. D. Reid, Amherst College, has been appointed Assistant Professor at Syracuse University.

Assistant Professor R. B. Reisel, Loyola University, has been promoted to Associate Professor.

Professor C. L. Riggs, Texas Technological College, has been appointed Associate Professor at Occidental College.

Dr. G. M. Rosenstein, Jr., Duke University, has been appointed Assistant Professor at Western Reserve University.

Mr. David Rothman, Harvard University, has accepted a position as Specialist-Research with Rocketdyne, Canoga Park, California.

Assistant Professor J. J. Rotman, University of Illinois, has been promoted to Associate Professor.

Dr. J. A. Rudnick, Overbrook High School, Philadelphia, Pennsylvania, has been appointed Associate Professor at Glassboro State College.

Mr. R. A. Schwabe, Mohawk Valley Technical Institute, has been appointed Assistant Professor at the University of Redlands.

Dr. John Seldin, Jr., University of Georgia, has been appointed Assistant Professor at the University of Kentucky.

Dr. L. E. Sigler, Columbia University, has been appointed Assistant Professor at Hunter College.

Assistant Professor L. T. Smith, Arizona State University, has been promoted to Associate Professor.

Dr. D. A. Sprecher, University of Maryland, has been appointed Assistant Professor at Syracuse University.

Associate Professor W. F. Steele, Heidelberg College, has been promoted to Professor.

Professor R. F. Steward, Western Carolina College, has been appointed Professor at Rhode Island College.

Mr. R. K. Stump, Muhlenberg College, has been appointed Assistant Professor.

Assistant Professor T. E. Sydnor, Pasadena City College, has been promoted to Associate Professor.

Associate Professor Selmo Tauber, Portland State College, has been promoted to Professor.

Associate Professor R. F. Tidd, Canisius College, has been promoted to Professor.

Rev. H. J. Vandort, Ph.D., University of Michigan, has been appointed Associate Professor of Education at Wisconsin State College at Superior.

Assistant Professor D. T. Walker, Memphis State University, has been promoted to Associate Professor.

Dr. H. N. Ward, Brown University, has been appointed Assistant Professor.

Mr. K. D. Weaver, Chance Vought Aircraft, Dallas, Texas, has accepted a position as Manager of Systems and Programming with Ling Temco Vought, Dallas, Texas.

Dr. K. W. Weston, University of Wisconsin, has been appointed Assistant Professor at the University of Notre Dame.

Mr. Edgar Wilford, University of California at Davis, has been appointed Assistant Professor at Armstrong College.

Dr. P. R. Young, Massachusetts Institute of Technology, has been appointed Assistant Professor at Reed College.

Assistant Professor Emerita Grace M. Bareis, Ohio State University, died on June 15, 1962. She was a charter member of the Association.

Professor A. E. Lampen, Hope College, died on May 25, 1963. He was a member for 45 years.

Professor Morgan Ward, California Institute of Technology, died on June 26, 1963. He was a member of the Association for 36 years.

Mr. C. L. Weaver, New England Mutual Life Insurance Company, Boston, Massachusetts, died on April 24, 1963. He was a member of the Association for 30 years.

POLYA HALL DEDICATED AT STANFORD

Stanford University dedicated its new Computation Center on August 9, with addresses by Professor George Polya of Stanford, and Dr. Richard Hamming of the Bell Telephone Laboratories. One of the two buildings has been named Polya Hall, in honor of Professor Polya, who has recently celebrated his 75th birthday.

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FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York 53, N. Y.

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NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo, New York 14214

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

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THE SPECTRAL MAPPING THEOREM FOR A HERMITIAN OPERATOR

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The teacher of an introductory course in Hilbert space is confronted with the following question: What to say about "the spectral theorem"? The answer depends on the students' background in analysis. For the spectral theorem for completely continuous normal operators, an adequate background is a mastery of the Weierstrass-Bolzano theorem. This is too easy. For the spectral theorem for normal operators, one needs sophisticated tools such as the general theory of measure and integration, or else the advanced weaponry of Banach algebras. This is too hard. In between, there is the spectral theorem for a Hermitian operator; the deepest analytical tool one needs (aside from the fundamental theorem of algebra) is the Weierstrass polynomial approximation theorem. This is just right. Even the approximation theorem itself can easily be presented in a lecture to the student who has never met it, provided he is prepared and willing to add it to his repertory. Granted this, one can do justice, latish in a semester course, to the spectral theorem for a Hermitian operator; the first aim of the present note is to sketch such a presentation, with details in full at the critical points. Notation and terminology are those of [1].

The key to F. Riesz's formulation and proof [2] of the spectral theorem for a Hermitian operator A is the following result:

THEOREM 1. *If A is a Hermitian operator, with the bounds*

$$m = \mathrm{GLB}\{(Ax, x) \colon \|x\| = 1\}$$
$$M = \mathrm{LUB}\{(Ax, x) \colon \|x\| = 1\},$$

and if p is a real polynomial such that $p \geqq 0$ on the closed interval $[m, M]$, then $p(A) \geqq 0$.

For an elegant proof based on the theory of square roots of positive operators, one may consult [3, Section 106]. Since $mI \leqq A \leqq MI$, it is easy to see (cf. Lemma 1 below) that the spectrum $\Lambda(A)$ of A is contained in $[m, M]$; the following result is therefore an improvement on Theorem 1:

THEOREM 1′. *If A is a Hermitian operator, and p is a real polynomial, then $p(A) \geqq 0$ if and only if $p \geqq 0$ on $\Lambda(A)$.*

Indeed, since the operator $B = p(A)$ is also Hermitian, and since $\Lambda(B) = p(\Lambda(A))$ [1, 33.1], the proof of Theorem 1′ is reduced to the following

LEMMA 1. *If B is a Hermitian operator, then $B \geqq 0$ if and only if $\Lambda(B) \geqq 0$.*

Proof. In any case, $\Lambda(B)$ is real [1, 34.1]. If $B \geqq 0$, that is if $(Bx, x) \geqq 0$ for every vector x, then $\Lambda(B) \geqq 0$ results from the fact that $B + \alpha I$ is invertible for

every $\alpha>0$, a fact which is easily deduced from [1, 21.3]. Suppose conversely that $\Lambda(B)\geqq 0$, and let

$$k = \mathrm{GLB}\{(Bx, x): \|x\| = 1\};$$

it is to be shown that $k\geqq 0$. Choose a sequence of unit vectors x_n such that $(Bx_n, x_n)\to k$, that is, $((B-kI)x_n, x_n)\to 0$. Since $B-kI\geqq 0$, it follows easily from the generalized Schwarz inequality that $\|(B-kI)x_n\|\to 0$ [3, Section 104], and so $k\in\Lambda(B)$ [1, 31.2]. In particular, $k\geqq 0$.

The second aim of the present note is accomplished: a proof of Theorem 1′ which does not require the theory of square roots. The final aim is to apply this result in proving the "spectral mapping theorem" for continuous functions of the Hermitian operator A (Theorem 2).

The correspondence $p\to p(A)$, defined for real polynomials p, may be extended by uniform approximation to a correspondence $f\to f(A)$ defined for all continuous real valued functions f on $[m, M]$. For the sake of completeness, we sketch the well-known proof. Let us write $\mathcal{C}$ for the algebra of all such functions f; the *norm* of f is defined by $\|f\|_\infty=\mathrm{LUB}\ \{|f(\lambda)| : m\leqq\lambda\leqq M\}$, while the *spectral norm* of f is defined by $\|f\|_{sp}=\mathrm{LUB}\ \{|f(\lambda)| : \lambda\in\Lambda(A)\}$. Obviously $\|f\|_{sp}\leqq\|f\|_\infty$. For any real polynomial p, one has $\|p(A)\|=\|p\|_{sp}$ [1, 34.3]. Given any f in $\mathcal{C}$, the operator $f(A)$ is defined as follows. By the Weierstrass theorem, choose a sequence p_n of real polynomials converging to f uniformly on $[m, M]$. The sequence of operators $p_n(A)$ converges uniformly to a suitable limit operator, as is shown by the following calculation: $\|p_m(A)-p_n(A)\|=\|(p_m-p_n)(A)\| =\|p_m-p_n\|_{sp}\leqq\|p_m-p_n\|_\infty\leqq\|p_m-f\|_\infty+\|f-p_n\|_\infty\to 0$ as $m, n\to\infty$. A similar calculation shows that the limit operator is the same for every approximating sequence of polynomials. Thus one may unambiguously define $f(A)$ to be the unique (Hermitian) operator such that $\|p_n(A)-f(A)\|\to 0$. Properties such as $(f+g)(A)=f(A)+g(A)$ and $(fg)(A)=f(A)g(A)$ are easily deduced from the corresponding properties for polynomials. Incidentally, the correspondence $f\to f(A)$ is a bounded linear mapping, as is shown by the calculation $\|f(A)\|=\lim\|p_n(A)\| =\lim\|p_n\|_{sp}=\|f\|_{sp}\leqq\|f\|_\infty$.

If $f\in\mathcal{C}$ and $f\geqq 0$ on $[m, M]$, then $f(A)\geqq 0$; for one has $f=g^2$, where $g=f^{1/2}$, and so $f(A)=(g(A))^2\geqq 0$. However, we shall need the following sharper result:

LEMMA 2. *If f is a continuous real valued function on $[m, M]$ such that $f\geqq 0$ on $\Lambda(A)$, then $f(A)\geqq 0$.*

Proof. Given any $\epsilon>0$ it will suffice to show that $f(A)+\epsilon I\geqq 0$. Thus if $g=f+\epsilon 1$, we have $g\geqq\epsilon$ on $\Lambda(A)$, and it is to be shown that $g(A)\geqq 0$. Choose a sequence p_n of real polynomials such that $\|p_n-g\|_\infty\to 0$. We may assume $\|p_n-g\|_\infty\leqq\epsilon/2$ for all n. Then $p_n\geqq\epsilon/2$ on $\Lambda(A)$, hence $\Lambda(p_n(A))=p_n(\Lambda(A))\geqq\epsilon/2$, and it follows from Lemma 1 that $p_n(A)\geqq(\epsilon/2)I$. Since $\|p_n(A)-g(A)\|\to 0$, we conclude that $g(A)\geqq(\epsilon/2)I$.

Incidentally, it follows at once from Lemma 2 that if $f\in\mathcal{C}$ vanishes everywhere on $\Lambda(A)$, then $f(A)=0$. This result is of course superseded by the following "spectral mapping theorem":

THEOREM 2. *If f is any continuous real valued function on $[m, M]$, then* $\Lambda(f(A))=f(\Lambda(A))$.

Proof. Let us show first that $f(\Lambda(A))\subset\Lambda(f(A))$. Given any $\lambda_0\in\Lambda(A)$, it is to be shown that $f(A)-f(\lambda_0)I$ is singular (i.e. not invertible, in the sense of [1]). Choose any sequence p_n of real polynomials such that $\|p_n-f\|_\infty\to 0$. Then also $p_n-p_n(\lambda_0)1\to f-f(\lambda_0)1$ uniformly on $[m, M]$, hence $p_n(A)-p_n(\lambda_0)I\to f(A)-f(\lambda_0)I$ uniformly. Since each of the operators $p_n(A)-p_n(\lambda_0)I$ is singular [1, 33.1], their uniform limit $f(A)-f(\lambda_0)I$ is also singular [1, 32.1].

Conversely, assuming $\mu\notin f(\Lambda(A))$, it is to be shown that $\mu\notin\Lambda(f(A))$. In other words, assuming that the function $g=f-\mu 1$ never vanishes on $\Lambda(A)$, it is to be shown that $g(A)$ is invertible. Choose $\epsilon>0$ so that $|g|\geqq\epsilon$ on $\Lambda(A)$; this is possible by the continuity of g and the compactness of spectrum [1, 32.2]. Since $g^2\geqq\epsilon^2$ on $\Lambda(A)$, we have $(g(A))^2\geqq\epsilon^2 I$ by Lemma 2, and it follows that $g(A)$ is invertible (see, for example, Lemma 1).

It follows at once from Theorem 2 that $\|f(A)\|=\|f\|_{sp}$ for every f in $\mathcal{C}$ (as we knew already before Lemma 2), and that $f(A)\geqq 0$ if and only if $f\geqq 0$ on $\Lambda(A)$.

The above results are often recast in terms of functions defined on the spectrum. Let $\mathcal{C}_0$ be the algebra of all continuous real valued functions defined on $\Lambda(A)$. Every $f_0\in\mathcal{C}_0$ has at least one extension $f\in\mathcal{C}$, by the Tietze theorem; if also $g\in\mathcal{C}$ extends f_0, then $f-g=0$ on $\Lambda(A)$, and so $f(A)=g(A)$ by the remark following Lemma 2. Thus we may unambiguously define $f_0(A)=f(A)$. The correspondence $f_0\to f_0(A)$ enjoys the formal algebraic properties possessed by the correspondence $f\to f(A)$. Moreover, $\|f_0(A)\|=\|f(A)\|=\|f\|_{sp}=\text{LUB}\{|f_0(\lambda)|:\lambda\in\Lambda(A)\}$, hence the correspondence $f_0\to f_0(A)$ is an *isometric* mapping of the algebra $\mathcal{C}_0$ into the algebra of operators.

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SOME PROPERTIES OF GROUPOIDS

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Introduction. A groupoid is a nonvoid set on which is defined a single valued binary operation meeting only the requirement of closure. In [4], the following topics are developed for groupoids: factor groupoid, homomorphism and isomorphism theorems, direct product and sum, subdirect product, and the solu-

tion of simultaneous congruence relations. This theory is largely due to Garrett Birkhoff ([1], pp. 85–92). Borůvka devotes fifty-four pages of his book [2] to groupoids. Among other things, he generalizes the isomorphism theorems of group theory to groupoids and studies ideals in groupoids. Many of the recent results in the area, an extensive bibliography, and several problems are given by Bruck [3]. In fact, Bruck begins with a more general system, the half-groupoid.

A quasigroup is a groupoid G such that a and b in G implies that there exist unique x and y in G such that $ax=b$ and $ya=b$. A quasigroup with an identity is called a loop. These special nonassociative groupoids have been studied very extensively [3].

A topological groupoid G is a groupoid which is a Hausdorff topological space such that if a, b are in G and $V(ab)$ is any neighborhood of ab, there exist neighborhoods $V(a)$ and $V(b)$ of a and b respectively with $V(a)V(b)\subset V(ab)$, i.e. the multiplication is continuous in both variables.

Topological groupoids have been considered by Warne [21]. His main result has recently been extended by Mostert [14]. Let I_1 denote the unit interval under ordinary multiplication, and I_2 the interval $[\frac{1}{2}, 1]$ with multiplication $x \circ y = \max(xy, \frac{1}{2})$. Mostert's theorem states: Let I be a connected, compact, linearly ordered set with endpoints 0 and 1. Suppose I is a topological groupoid, and 1 acts as an identity and 0 as a zero. Suppose further that the power associative elements contain 1 as a limit point and that there are no idempotents other than 0 and 1 in I. Then I is isomorphic to I_1 or I_2.

Šulka [16–19] considers continuous homomorphic mappings of one topological groupoid G onto another, and their realization in the usual way by dissections of G. He defines a topological factoroid on G as a decomposition of G corresponding to an open continuous homomorphism of G and considered itself as a topological groupoid. He shows that a decomposition of a topological factoroid on a topological groupoid G is a topological factoroid if and only if it generates a topological factoroid on G (with which it is, in such a case, necessarily isomorphic). He gives conditions under which the maximal common refinement, or respectively the minimal common covering of two topological factoroids on a topological groupoid is a topological factoroid. (Šulka just requires the T_1– axiom in general.) The concept of (algebraic) factoroid is used extensively by Borůvka [2].

A loop, which is a topological space such that the operations of multiplication and right and left division are continuous in both variables, is called a topological loop. These structures have been investigated by Hofmann [6–11].

In section 1, we discuss the structures of groupoids in terms of three mutually exclusive and exhaustive nonvoid subsets which we call subgroupoids, antigroupoids, and expansive sets. Next, we investigate the problem of the homomorphic embedding of groupoids in semigroups and groups mentioned in Bruck [3].

In section 2, we generalize a result of Wallace [22] to topological groupoids

and give some topological results about subgroupoids, antigroupoids, etc. We then note some of the "associative like" properties of g-threads [21] induced by the topology. We then pose two problems.

Section 1. Groupoids.

DEFINITION 1.1. *A nonvoid subset A of a groupoid G is called a subgroupoid if and only if* $AA \subset A$. *It is called an antigroupoid if and only if* $AA \subset G-A$ *(the set theoretic complement of A). All other nonvoid subsets are said to be expansive.*

DEFINITION 1.2. *An element e of a groupoid is an idempotent if and only if* $e^2=e$.

THEOREM 1.1. *A groupoid G contains no antigroupoid if and only if every element of G is an idempotent.*

DEFNITION 1.3. *A groupoid G is cyclic if and only if there exists an a in G such that G is the collection of all words in a.*

THEOREM 1.2. *A groupoid containing no proper subgroupoid is cyclic.*

THEOREM 1.3. *Every nonvoid subset of a groupoid G is a subgroupoid or an antigroupoid if and only if each nonvoid subset of cardinality less than or equal to five is a groupoid or antigroupoid.*

Proof. Let A be an expansive subset of G. Then there exist elements a_1, a_2, a_3, a_4 of A such that $a_1\, a_2$ is in A and a_3a_4 are in $G-A$. Then $B=(a_1, a_2, a_1a_2, a_3, a_4)$ is an expansive set of cardinality less than or equal to five contained in A.

THEOREM 1.4. *Every nonvoid subset of a groupoid G is a subgroupoid if and only if* $xy=x$ *or* y *for all* x, y *in G.*

THEOREM 1.5. *Every proper subset of a groupoid G is an antigroupoid if and only if G has exactly two elements* x, y *with* $x^2=y$ *or* $y^2=x$ *or G is the one element groupoid.*

Proof. Suppose G has two distinct elements x, y. Then, since y is in $G-\{x\}$, $y^2=x$. Similarly, $x^2=y$. If q is in $(G-\{x\})\cap(G-\{y\})$, then $q^2=x=y$.

We note that a nontrivial groupoid cannot just contain proper expansive sets.

THEOREM 1.6. *If* $(A_k\colon k \text{ in } K)$ *is a collection of expansive sets,* $\bigcup(A_k\colon k \text{ in } K)$ *is a subgroupoid or expansive set. If* $(A_k\colon k \text{ in } K)$ *is a collection of antigroupoids, then* $\bigcap(A_k\colon k \text{ in } K)$ *is empty or an antigroupoid.*

THEOREM 1.7. *A semigroup (associative groupoid) which is the union of antigroupoids is infinite.*

Proof. Every finite semigroup has an idempotent [**15**].

We now give some examples:

Example 1.1. Let G be the integers under subtraction, $a \circ b = a - b$. The odd integers are an antigroupoid while the even integers are a subgroupoid and the odd integers $\cup \{0\}$ are expansive. If G is the integers under ordinary multiplication, the negative integers form an antigroupoid.

Example 1.2. Let G be a topological groupoid which contains a nonidempotent element g that is not an open set. Then every sufficiently small neighborhood of g is an antigroupoid.

Example 1.3. Let G be the positive integers under the multiplication

$$ab = 1 \qquad \text{if } a \neq b$$
$$aa = a + 1.$$

G is a cyclic groupoid with generator 1 and has no proper subgroupoids since every subgroupoid contains 1 (see Theorem 1.2). G is abelian and $(11)\,1 = 1(11)$. Clearly, G is nonassociative.

Example 1.4. Consider $\{(x, y) : x^2 + y^2 \leqq 1\}$, where (x, y) is a point in E^2. Define

$$(x_1, y_1) \circ (x_2, y_2) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Then G is nonassociative and abelian. Since every element of G is an idempotent, G has no antigroupoids. Every convex subset of G is a subgroupoid. Every closed subgroupoid is convex. Every closed nonconvex subset is expansive. G does not have the two-element groups as its homomorphic image (see Theorem 1.9).

We will now consider the problem of homomorphic embedding [**3**, pp. 32–34]. If G is a groupoid, when does there exist a nontrivial semigroup or a nontrivial group G^* such that G^* is the homomorphic image of G?

DEFINITION 1.4. *An element a of a groupoid G is generally prime if and only if there exist no b and c in G such that $a = bc$.*

THEOREM 1.8. *A groupoid G which contains a generally prime element has a nontrivial abelian semigroup as its homomorphic image.*

Proof. Clearly, $GG \neq G$. Let $P = G - GG$, the set of generally prime elements of G. We note that $PG \subset G - P$ and $GP \subset G - P$ and that $G - P$ is a subgroupoid. Let w be any element not in G, and let $G^* = P \cup w$. Define $x^*y = w$ for all x, y in G^*. Then G^* is a nontrivial abelian semigroup. If we define

$$f(x) = x \qquad \text{for } x \text{ in } P$$
$$f(x) = w \qquad \text{for } x \text{ in } GG,$$

then f is a homomorphism of G onto G^*.

Example 1.5. Let G be the interval $[0, \frac{1}{2}]$ of real numbers under the multiplication $a \circ b = ab^2$ for all a, b in G. The set P of generally prime elements is $(\frac{1}{8}, \frac{1}{2}]$. Hence, $G^* = (\frac{1}{8}, \frac{1}{2}] \cup w$ with $x^*y = w$ for all x, y in G^*. Thus, G is a non-

associative and non-abelian groupoid which has a nontrivial abelian semigroup as its homomorphic image.

DEFINITION 1.5. *A subset A of a groupoid G is a partial ideal if and only if A is an antigroupoid and $(G-A)A\subset A$ and $A(G-A)\subset A$.*

THEOREM 1.9. *A necessary and sufficient condition for a groupoid G to have the two element group as its homomorphic image is that G contain a partial ideal I and that $G-I$ be a subgroupoid.*

Proof. Let the 2-group G^* consist of 0 and 1 with operation $+$, i.e., $0+0=0$, $1+0=0+1=1$, and $1+1=0$. Define $f(x)=1$ for x in I and $f(x)=0$ for x in $G-I$. It is easily shown that f is a homomorphism of G onto G^*. Conversely, suppose that f is a homomorphism of G onto G^*. Then, $I=f^{-1}(1)$ is a partial ideal of G and $f^{-1}(0)$ is a subgroupoid.

THEOREM 1.10. *If I is a partial ideal of a semigroup S, $S-I$ is a semigroup. If S is a group, $S-I$ is a group of index two in S.*

Proof. Suppose a, b are in $S-I$ and ab is in I. If c is in I, $(ab)c$ is in $S-I$ and $a(bc)$ is in I contradicting associativity. If S is a group with identity e, e is in $S-I$ and hence a in $S-I$ implies a^{-1} is in $S-I$ and $x(S-I)=I$ for all x in I.

THEOREM 1.11. *A necessary and sufficient condition for a semigroup S to have the 2-group as its homomorphic image is that S contain a partial ideal.*

Proof. This is an immediate consequence of Theorems 1.9 and 1.10.

Section 2. Topological groupoids. If A is a subset of a topological groupoid G, then A^* is the topological closure of A.

THEOREM 2.1. *Let G be a topological groupoid and suppose that S is a subsemigroup of G. Then S^* is a closed subsemigroup of G.*

We use Theorem 2.1 to extend a result of Wallace [20] to topological groupoids.

THEOREM 2.2. *Let A be an algebraic subgroup of a topological groupoid G. If A^* is compact, A^* is a topological group. If G is compact, each maximal algebraic subgroup is a compact topological group.*

Proof. Since Wallace establishes the first part of the result for topological semigroups (associative topological groupoids), we just note that A is an algebraic subgroup of the compact topological semigroup A^* by Theorem 2.1. The second part is then immediate.

THEOREM 2.3. *Every compact topological groupoid G has a minimal closed subgroupoid. If G contains a nonidempotent element, it has a minimal closed antigroupoid.*

Proof. We will prove the second part. The first part is similar. Let A be the

family of all closed antigroupoids of G. Since the nonidempotent element g is a closed antigroupoid, A is not empty. Let $N=(A_k\colon k \text{ in } K)$ be any nest [**12**, page 32] in A. Since N has the finite intersection property, $\bigcap(A_k\colon k \text{ in } K)$ is not empty and hence is a closed antigroupoid by Theorem 1.6. Thus, A has a minimal member by the minimal principle [**12**, page 33].

THEOREM 2.4. *The set of idempotents of a topological groupoid is closed.*

THEOREM 2.5. *Let G be a topological groupoid with a zero,* 0 $(0x=x0=0$ *for all x in G), and a right identity, $u(xu=x$ for all x in $G)$, or a left identity, v, $(vx=x$ for all x in $G)$. Then, if A is a partial ideal of G, A is connected, provided $G-A$ is connected.*

Proof. If we have a right identity, $A=\bigcup(x(G-A)\colon x \text{ in } A)$. Since each $x(G-A)$ is connected and contains 0, A is connected. The case of the left identity is similar.

We now consider a topological groupoid which has rather striking "associative-like" properties although no such properties are assumed in the definition.

DEFINITION 2.1. *A g-thread is a system $G(\circ, <)$ which has the following properties:*

(1) *G is a groupoid with respect to $\circ$.*
(2) *$<$ is a total (i.e. linear or simple) order relation on G.*
(3) *The mapping $P(x, y)=x\circ y$ of $G\times G$ into G is continuous in the order topology.*
(4) *G is connected in the order topology.*
(5) *G has a zero,* 0, *and an identity,* u. 0 *is the least element and u is the greatest element with respect to $<$.*

We denote $a\circ b$ by ab for all a, b in G.

The next theorem is due to one of the authors [**21**].

THEOREM 2.6. *If G be a g-thread satisfying the cancellation law for nonzero elements, then*

(1) *$x<y$ implies $x c<yc$ and $cx<cy$ for all x, y and all $c\neq 0$ in G.*
(2) *$xy\leqq \min (x, y)$.*
(3) *$x\leqq y$ and $w\leqq v$ implies $xw\leqq yv$.*

THEOREM 2.7. *Let G be a g-thread satisfying the cancellation law for nonzero elements. Then $(Ga)b=G(ab)=(ab)G=a(bG)$ for all a, b, in G.*

Proof. We first show that $Gx=[0, x]$ for all x in G. Since Gx is a connected set containing 0 and x, $[0, x]\subset Gx$. $Gx\subset[0, x]$ by Theorem 2.6. Similarly, $xG=[0,x]$. A similar argument shows that $[0, ab]=[0, a]b$ or $G(ab)=(Ga)b$. By the same method, $(ab)G=a(bG)$ and the result follows.

Examples 1.1 and 1.5 are topological groupoids with respect to the usual topologies. The only topology under which example 1.3 becomes a topological groupoid is the discrete topology. This example motivated the following ques-

tion by Clifford: What algebraic properties must a groupoid possess so that it will admit a nondiscrete topology making it into a topological groupoid?

Example 2.1. A topological groupoid is *monothetic* if and only if there exists a in G such that the collection of all finite words in a is dense in G. A compact topological groupoid contains a compact monothetic subgroupoid. Monothetic semigroups have been studied by Koch [**13**].

Problem 1. Example 1.4 is a compact cancellation topological groupoid which is not a quasi-group. It is well known that any finite cancellation groupoid is a quasi-group [**5**, p. 986]. What compact cancellation groupoids are quasigroups?

Problem 2. Let G be a topological groupoid and let Q be an algebraic subquasigroup of G such that Q^* is compact. When is Q^* a topological quasi-group (right and left division are continuous in both variables)?

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ON MULTINOMIAL COEFFICIENTS

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1. Introduction. Let N be a positive integer; we consider the expression

$$P(N, n) = (x_1 + x_2 + \cdots + x_n)^N, \tag{1}$$

and seek an expansion of the form

$$P(N, n) = \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} \cdots \sum_{k_n=0}^{N} \binom{N}{k_1, k_2, \cdots, k_n} \prod_{m=1}^{n} (x_m)^{k_m}, \tag{2}$$

with

$$\sum_{m=1}^{n} k_m = N. \tag{3}$$

An expansion of the form (2) with condition (3) is called a *multinomial expansion*, and the coefficients $\binom{N}{k_1, k_2, \ldots, k_n}$ are called *multinomial coefficients*. In the special case, $n = 2$, we obtain from (2) and (3)

$$P(N, 2) = \sum_{k=0}^{N} \binom{N}{k, N-k} x_1^k x_2^{N-k} = \sum_{k=0}^{N} \binom{N}{k, N-k} x_1^{N-k} x_2^k,$$

where we write $\binom{N}{k, N-k}$ for the usual $\binom{N}{k}$, the binomial coefficient, since the symmetric form is more convenient in case $n > 2$.

The theorem stating that an expansion of the form (2) with condition (3) exists, and that

$$\binom{N}{k_1, k_2, \cdots, k_n} = N! \Big/ \prod_{m=1}^{n} k_m! \tag{4}$$

is called the *multinomial expansion theorem* (in German: Der polynomische Lehrsatz) and is found throughout the mathematical literature, mainly in treatises of combinatorial analysis, algebra, statistics and number theory.

According to [1] it was first mentioned in a letter (1695) from Leibnitz to Johann Bernoulli, who then proved it. Later the same theorem was proved in a different form by Johann's brother Jacob Bernoulli (about 1697). In [1] the theorem is proved by means of combinatorial analysis. In references we give a few examples of the occurrence of the theorem throughout the more recent literature: the theorem is proved in two different ways in [2]; it is also stated and generalized in [3]. For further examples see [4], [5], [6] and [7]. Finally, in [8], several summation formulae involving multinomial coefficients are proved. We shall refer later to some of the results obtained there. In this paper we shall prove five basic formulae concerning multinomial coefficients.

2. First summation formula. We mention first that the multinomial expansion theorem can be proved by induction. The proof is rather long. It leads to a summation formula that we shall prove by a different method.

Multiplying both sides of (2) by $P(1, n) = (x_1+x_2+\cdots+x_n)$, we obtain

$$P(1, n)P(N, n) = P(N+1, n) = \sum_{k_1=0}^{N+1}\sum_{k_2=0}^{N+1}\cdots\sum_{k_n=0}^{N+1}\binom{N+1}{k_1, k_2, \cdots, k_n}\prod_{m=1}^{n}(x_m)^{k_m},$$

but,

$$P(1, n)P(N, n) = \sum_{s=0}^{n}\sum_{k_1=0}^{N}\sum_{k_2=0}^{N}\cdots\sum_{k_n=0}^{N}\binom{N}{k_1, k_2, \cdots, k_n}\prod_{m=1}^{n}x_s(x_m)^{k_m}.$$

By equating the coefficients of the term

$$(x_1)^{k_1}(x_2)^{k_2}\cdots(x_{s-1})^{k_{s-1}}(x_s)^{k_s+1}(x_{s+1})^{k_{s+1}}\cdots(x_n)^{k_n},$$

with (3) being satisfied, we have

$$\sum_{j=1}^{n}\binom{N}{k_1, k_2, \cdots, k_{j-1}, k_j-1, k_{j+1}, \cdots, k_{s-1}, k_s+1, k_{s+1}, \cdots, k_n} = \binom{N+1}{k_1, k_2, \cdots, k_{s-1}, k_s+1, k_{s+1}, \cdots, k_n}, \tag{5}$$

or, taking $h_1+h_2+\cdots+h_n = N+1$,

$$\sum_{j=1}^{n}\binom{N}{h_1, h_2, \cdots, h_{j-1}, h_j-1, h_{j+1}, \cdots, h_n} = \binom{N+1}{h_1, h_2, \cdots, h_n}. \tag{6}$$

In case $n=2$, we obtain

$$\binom{N}{k-1, N-k-1} + \binom{N}{k, N-k} = \binom{N+1}{k, N-k+1}$$

or, with the classical notation

$$\binom{N}{k-1} + \binom{N}{k} = \binom{N+1}{k}.$$

3. Second summation formula. In [9] page 73, the following formula is proved and called "Cauchy's summation formula":

$$\sum_{j=0}^{k}\binom{p}{j}\binom{q}{k-j} = \binom{p+q}{k}.$$

This formula is also known as the "Vandermonde convolution formula." According to [1] it was first proved by Euler.

We shall find a similar relation for the multinomial coefficients. Considering the identity

$$(x_1+x_2+\cdots+x_{n-1}+1)^p(x_1+x_2+\cdots+x_{n-1}+1)^q = (x_1+x_2+\cdots+x_{n-1}+1)^{p+q}, \tag{7}$$

and the two relations

$$j_1 + j_2 + \cdots + j_n = p$$
$$k_1 + k_2 + \cdots + k_n = q + p,$$

thus $(k_1 - j_1) + (k_2 - j_2) + \cdots + (k_n - j_n) = q$, we see that the general term of the expansion of the right-hand member of (7) can be written

$$\binom{p+q}{k_1, k_2, \cdots, k_n} (x_1)^{k_1}(x_2)^{k_2} \cdots (x_{n-1})^{k_{n-1}}.$$

The corresponding terms on the left of (7) are obtained by taking

$$\binom{p}{j_1, j_2, \cdots, j_n}\binom{q}{k_1 - j_1, k_2 - j_2, \cdots, k_n - j_n} \prod_{m=1}^{n-1} (x_m)^{j_m + k_m - j_m},$$

and summing for all admissible values of the j's. Thus by equating the coefficients of the corresponding terms we obtain

$$\text{(8)} \quad \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \cdots \sum_{j_{n-1}=0}^{k_{n-1}} \binom{p}{j_1, j_2, \cdots, j_n}\binom{q}{k_1 - j_1, k_2 - j_2, \cdots, k_n - j_n} = \binom{p+q}{k_1, k_2, \cdots, k_n},$$

which is the equivalent of Cauchy's formula for the multinomial coefficients. This formula was also found by L. Carlitz and appears in [8].

4. Third summation formula. If in (1) we take $x_1 = x_3 = x_5 = \cdots = 1$, and $x_2 = x_4 = x_6 = \cdots = -1$, then $P(N, 2p) = 0$, and, $P(N, 2p+1) = 1$. Let then, for $P(N, n)$, p be the largest integer such that $2p \leqq n$, then

$$\text{(9)} \quad \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} \cdots \sum_{k_n=0}^{N} (-1)^{(k_2+k_4+\cdots+k_{2p})} \binom{N}{k_1, k_2, \cdots, k_n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

where the positive integers k_j, $j = 1, 2, \cdots, n$, satisfy (3).

5. Fourth summation formula. In [10] the numbers A_n^m and B_n^m, n and m being positive integers or zero, were defined to be quasi-orthogonal if

$$\sum_{s=m}^{n} A_n^s B_s^m = \delta_n^m,$$

where δ_n^m is the Kronecker delta. In particular the numbers $A_n^s = \binom{n}{s}$ and $B_s^m = (-1)^{s-m}\binom{s}{m}$ are quasi-orthogonal, for, observing (as the referee suggested) that

$$\binom{n}{s}\binom{s}{m} = \binom{n}{m}\binom{n-m}{s-m}$$

we have, with $s-m=t$,

$$\sum_{s=m}^{n}\binom{n}{s}(-1)^{s-m}\binom{s}{m}=\binom{n}{m}\sum_{s=m}^{n}(-1)^{s-m}\binom{n-m}{s-m}$$

$$=\binom{n}{m}\sum_{t=0}^{n-m}(-1)^{t}\binom{n-m}{t}$$

$$=\binom{n}{m}(1-1)^{n-m}=\delta_n^m.$$

We shall prove a similar relation for the multinomial coefficients. Let $s_0, s_1, \cdots, s_n$ be positive integers such that $s_0\leqq s_1\leqq s_2\leqq \cdots \leqq s_{n-1}\leqq s_n$. We can write

$$\sum_{m_1=s_0}^{s_1}(-1)^{m_1-s_0}\binom{s_1}{m_1}\binom{m_1}{s_0}=\delta_{s_0}^{s_1}$$

$$\sum_{m_2=s_1}^{s_2}(-1)^{m_2-s_1}\binom{s_2}{m_2}\binom{m_2}{s_1}=\delta_{s_1}^{s_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{m_n=s_{n-1}}^{s_n}(-1)^{m_n-s_{n-1}}\binom{s_n}{m_n}\binom{m_n}{s_{n-1}}=\delta_{s_{n-1}}^{s_n}.$$

By multiplying these n relations together we obtain

$$\sum_{m_1=s_0}^{s_1}\sum_{m_2=s_1}^{s_2}\cdots\sum_{m_n=s_{n-1}}^{s_n}(-1)^{m_1+m_2+\cdots+m_n-(s_0+s_1+\cdots+s_{n-1})}\cdot\prod_{k=1}^{n}\binom{s_k}{m_k}\binom{m_k}{s_{k-1}}$$

$$=\prod_{p=1}^{n}\delta_{s_{p-1}}^{s_p},$$

but

$$\prod_{k=1}^{n}\binom{s_k}{m_k}\binom{m_k}{s_{k-1}}=\binom{s_n}{s_n-m_n,\, m_n-s_{n-1},\, \cdots,\, s_1-m_1,\, m_1-s_0,\, s_0},$$

so that

$$\sum_{m_1=s_0}^{s_1}\sum_{m_2=s_1}^{s_2}\cdots\sum_{m_n=s_{n-1}}^{s_n}(-1)^{m_1+m_2+\cdots+m_n-(s_0+s_1+\cdots+s_{n-1})}$$

$$\cdot\binom{s_n}{s_n-m_n,\, m_n-s_{n-1},\, \cdots,\, s_1-m_1,\, m_1-s_0,\, s_0}=\prod_{p=1}^{n}\delta_{s_{p-1}}^{s_p}.$$

On the other hand, if q is a positive integer such that $1<q<n$, we can write

$$\binom{s_n}{s_n - m_n, m_n - s_{n-1}, \cdots, s_{q+1} - m_{q+1}, m_{q+1} - s_q, s_q}$$
$$\cdot\binom{s_q}{s_q - m_q, m_q - s_{q-1}, \cdots, m_1 - s_0, s_0}$$
$$= \binom{s_n}{s_n - m_n, m_n - s_{n-1}, \cdots, s_1 - m_1, m_1 - s_0, s_0},$$

so that

$$\begin{aligned}\sum_{m_1=s_0}^{s_1} \sum_{m_2=s_1}^{s_2} \cdots \sum_{m_n=s_{n-1}}^{s_n} & (-1)^{m_1+m_2+\cdots+m_n-(s_0+s_1+\cdots+s_{n-1})} \\ & \cdot\binom{s_n}{s_n - m_n, m_n - s_{n-1}, \cdots, s_{q+1} - m_{q+1}, m_{q+1} - s_q, s_q} \\ & \cdot\binom{s_q}{s_q - m_q, m_q - s_{q-1}, \cdots, m_1 - s_0, s_0} = \prod_{p=1}^{n} \delta_{s_{p-1}}^{s_p}.\end{aligned} \tag{10}$$

This relation expresses the generalized quasi-orthogonality for multinomial coefficients. Similarly, since

$$\sum_{s=m}^{n}\binom{n}{s}\binom{s}{m} = \binom{n}{m}\sum_{s=m}^{n}\binom{n-m}{s-m} = \binom{n}{m}\sum_{t=0}^{n-m}\binom{n-m}{t} = \binom{n}{m}\cdot 2^{n-m},$$

we obtain, performing the same summations, but without the power of (-1)

$$\begin{aligned}&\sum_{m_1=s_0}^{s_1} \sum_{m_2=s_1}^{s_2} \cdots \sum_{m_n=s_{n-1}}^{s_n} \binom{s_n}{s_n-m_n,\ m_n-s_{n-1}, \cdots, s_1-m_1,\ m_1-s_0,\ s_0} \\ &= \sum_{m_1=s_0}^{s_1} \sum_{m_2=s_1}^{s_2} \cdots \sum_{m_n=s_{n-1}}^{s_n} \binom{s_n}{s_n-m_n,\ m_n-s_{n-1}, \cdots, s_{q+1}-m_{q+1},\ m_{q+1}-s_q, s_q} \\ &\cdot\binom{s_q}{s_q-m_q,\ m_q-s_{q-1}, \cdots, s_1-m_1,\ m_1-s_0,\ s_0} \\ &= \prod_{k=1}^{n}\binom{s_k}{s_{k-1}}\cdot 2^{s_k-s_{k-1}} = \binom{s_n}{s_n-s_{n-1},\ s_{n-1}-s_{n-2}, \cdots, s_1-s_0,\ s_0}\cdot 2^{s_n-s_0}.\end{aligned} \tag{11}$$

This relation can be considered as the generalization of

$$\sum_{s=m}^{n}\binom{n}{s}\binom{s}{m} = \binom{n}{m}\cdot 2^{n-m}.$$

Finally, we point out corrections to [10], page 368; line 5, *read*:

$$x^n = (y-1)^n = \sum_{m=0}^{n} B_n^m y^m = \sum_{m=0}^{n} (-1)^{n-m}\binom{n}{m} y^m,$$

and line 6, *read*:

$$\text{so that } B_n^m = (-1)^{n-m}\binom{n}{m} \quad \text{and} \quad B_n^m = -B_{n-1}^m + B_{n-1}^{m-1}.$$

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ON ESTIMATES IN NUMBER THEORY

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In many problems in number theory, one is interested in estimating the number of solutions in integers of certain equations. It often happens that one does not need the most refined estimate for the number of solutions, but an estimate somewhat finer than a trivial estimate is needed. It is the purpose of this paper to present a very general method for dealing with such problems. The main result will be a consequence of a recently discovered theorem in the geometry of numbers [2, p. 270].

THEOREM. *If $f(x)$ is continuous and twice differentiable with continuous second derivative in $[a, b]$, $M=\max_{a\leq x\leq b} f(x)$, $m=\min_{a\leq x\leq b} f(x)$, and $f''(x)=0$ has d solutions in (a, b), then*

$$N < c(d+1)\{(M-m)(b-a)\}^{1/3},$$

where N is the number of solutions in integers of $y=f(x)$ and $c>0$ is an absolute constant.

Proof. It is sufficient to prove the theorem when $d=0$. For assume it has been proved in this case. Consider the intervals (a, x_1), (x_1, x_2), $\cdots$, (x_d, b) where x_j is the jth zero of $f''(x)$ in (a, b) (there are at most $d+1$ such intervals). Let N_i

denote the number of solutions in integers of $y=f(x)$ in (x_{i-1}, x_i) with $a=x_0$ and $b=x_{d+1}$. Then

$$\begin{aligned} N &\leqq \sum_{j=1}^{d+1} N_j \\ &< c \sum_{j=1}^{d+1} \{(M-m)(x_j - x_{j-1})\}^{1/3} \\ &\leqq c \sum_{j=1}^{d+1} \{(M-m)(b-a)\}^{1/3} \\ &= c(d+1)\cdot\{(M-m)(b-a)\}^{1/3}. \end{aligned}$$

If $d=0$, then the curve $y=f(x)$ forms an arc which is strictly convex since now $f''(x)$ must be of constant sign [**5**, pp. 172–3]. Also the region, C_1, bounded by $y=f(x)$ and the straight line L through P_1: $(a, f(a))$ and P_2: $(b, f(b))$ lies entirely within the rectangle R given by $m \leqq y \leqq M$, $a \leqq x \leqq b$.

Hence if A_1 is the area of C_1 and A_0 is the area of R, then $A_1 < A_0$.

Let us now pass a circle of radius λ through the two points P_1 and P_2 with center on the perpendicular bisector of $\overline{P_1P_2}$ and on the same side of L as the graph of $y=f(x)$. Let α denote the arc of this circle which lies on the opposite side of L from the graph of $y=f(x)$ and has as end points P_1 and P_2. Provided we take λ sufficiently large, we have that α and the graph of $y=f(x)$ bound a strictly convex body, C_2, with area as near that of C_1 as we please. Also we may take λ large enough so that if A_2 is the area of C_2, then $A_2 < A_0$.

By the main theorem in [**2**, p. 270],

$$A_2 > \mathcal{K}(2) N^{(2+1)/(2-1)} = \mathcal{K}(2) N^3,$$

where $\mathcal{K}(2) > 0$ is an absolute constant. Hence there exists an absolute constant $c > 0$, such that

$$\begin{aligned} N &< c A_2^{1/3} \\ &< c A_0^{1/3} \\ &= c\{(M-m)(b-a)\}^{1/3}. \end{aligned}$$

This concludes the proof of the theorem.

We now show how this theorem may be applied.

If $d(n)$ denotes the number of divisors of n, then trivially $d(n) = O(n)$. In most texts in elementary number theory, improved results are obtained only after carefully studying properties of multiplicative functions [**3**, p. 260]. By our main theorem we may quickly get a nontrivial result.

THEOREM. $d(n) = O(n^{2/3})$.

Proof. Let $f(x) = n/x$ with $a=1$, $b=n$, then $M=n$, $m=1$, $N=d(n)$, and $f''(x) = 2n/x^3 > 0$ in $(1, n)$. Hence

$$d(n) = N < c\{(n-1)(n-1)\}^{1/3}$$
$$< cn^{2/3}.$$

By a somewhat more subtle use of the main theorem, we show how a more refined result may be obtained.

THEOREM. $d(n) = O(n^{1/3} \log n)$.

Proof. Consider the r intervals $(1, n^{1/r})$, $(n^{1/r}, n^{2/r})$, $\cdots$, $(n^{(r-1)/r}, n)$. Let $f(x) = n/x$ as before. Let N_j be the number of lattice points on the curve in the jth interval $(n^{(j-1)/r}, n^{j/r})$. Then in the jth interval $M - m \leqq M = n^{1-(j-1)/r}$, and $n^{j/r} - n^{(j-1)/r} < n^{j/r}$. Hence $(M-m)(n^{j/r} - n^{(j-1)/r}) < n^{1+(1/r)}$ in the jth interval. Thus

$$d(n) = N \leqq \sum_{j=1}^{r} N_j$$
$$< c \cdot r \cdot (n^{1+(1/r)})^{1/3}.$$

Taking $r = [\log n] + 1$, we obtain the desired result.

Of course we come nowhere near the truth that $d(n) = O(n^{\epsilon})$ for any $\epsilon > 0$ [**3**, p. 260], but we obtained our results without any use of the arithmetic properties of $d(n)$.

As one sees, very seldom will best possible results be obtained. It is clear, however, that this method is useful when something stronger than a nontrivial estimate is required, and a minimal amount of information about the arithmetic properties of the equation under consideration is available.

Finally, we note that it is not possible to improve our main theorem substantially. Taking $f(x) = x^2$, $a = 0$, $b = n$, we find that the theorem implies

$$N < c(n^2 \cdot n)^{1/3} = c \cdot n.$$

Actually we see by inspection that $N = n + 1$, so that the exponent $\frac{1}{3}$ cannot be replaced by any smaller number.

Professor I. J. Schoenberg has pointed out to me the close connection between the main result in this paper and the result in [**4**]. However, Jarnik's result is weaker; for example, his result will imply $d(n) = O(n^{2/3})$, but will not imply $d(n) = O(n^{1/3} \log n)$. Actually the result of Jarnik implies the result in [**1**] in the two-dimensional case.

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A COMBINATORY DETECTION PROBLEM

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Counterfeit coins weigh 9 grams and genuine coins weigh 10 grams. Given a scale that weighs all real numbers exactly, what is the minimum number of weighings required to extract all the counterfeits from a sample of n coins? (For $n=5$ see Problem E 1399, this MONTHLY 67 (1960) 82 and 697 (Solution).)

Since weighing any set of coins tells us how many counterfeits are in the set, and gives no further information, this problem is equivalent to the following:

Detection Problem. Let there be given a set E consisting of n objects, and let X be an unknown (possibly empty) subset of E. We wish to determine the set X by experiments of the following type: we may choose any subset $Y \subset E$ and are told how many elements lie in the intersection $X \cap Y$. What is the minimum number of such experiments needed to determine X?

The precise determination of the minimal number of experiments seems very difficult, even for small n, and we propose in this paper only to obtain upper and lower bounds which are of the same order of magnitude (namely $n/\log n$; see below).

1. A lower bound. Let $f(n)$ denote the minimal number of weighings (we will use the "weighing" terminology in discussing the problem) needed for a sample of size n. Notice that schemes for finding the counterfeits fall into two types, which we will call *ramified* and *unramified*. A *ramified* scheme is one in which the choice of coins to be weighed depends at least once upon the results of the preceding weighings. An *unramified* scheme is one in which the sets of coins to be weighed are specified in advance. Denoting a set by its "characteristic function," a binary sequence of length n, the unramified problem can be formulated as follows. We will say that an $m \times n$ matrix A, all of whose entries are 0 or 1, is a *detecting matrix*, if the vectors $A\mathbf{x}_i$, $i=1, \cdots, 2^n$ are all distinct, where $\mathbf{x}_i$ ranges over all binary n-tuples (imagined as column matrices). Then, for given n, we wish to find the least m for which an $m \times n$ detecting matrix exists. It is clear that we must have $(n+1)^m \geqq 2^n$, since each component of $A\mathbf{x}_i$ is an integer in the range $[0, n]$. Hence, so far as unramified experiments are concerned,

$$f(n) \geqq \frac{n \log 2}{\log (n+1)} \cdot \tag{1}$$

But this holds also for ramified schemes. To see this define the number $r=r(X)$ for a subset X (relative to the fixed scheme in question), the number of weighings in which the experiment terminates, when X is the set of counterfeits. With each subset X we have then an ordered set of "coordinates" $a_1, \cdots, a_r$, where $a_i (0 \leqq a_i \leqq n)$ is the number of counterfeits in the ith weighed sample. Clearly no other subset can have $a_1, \cdots, a_r$ as its first r coordinates. Since any collection of such r-tuples, with r varying in the range $1 \leqq r \leqq k$, contains $\leqq (n+1)^k$

elements, (1) is established also for ramified experiments. Notice that if $r(X) < k$ for at least one X, the collection contains strictly less than $(n+1)^k$ elements, as a number of otherwise usable coordinate sequences are "killed off." This raises the question whether the minimal $f(n)$ is always attained for an unramified experiment.

2. Upper bounds. Our main result is

THEOREM. *Suppose that for two positive integers k, s (where $s<k$) there exists a $k \times k$ matrix B whose entries are zeros and ones, with the following two properties:*

(i) *rank $B=k$, with respect to the rational field*

(ii) *rank $B=k-s$, with respect to the field J_2 of integers mod 2.*

Then, whenever it is possible to determine n coins in m weighings by an unramified scheme, it is possible to determine $kn+sm$ coins in km weighings by an unramified scheme.

COROLLARY 1. $f[3^{m-1}(m+3)] \leqq 3^m$, $m=0, 1, 2, \cdots$.

COROLLARY 2. $f[5^{m-1}(2m+5)] \leqq 5^m$, $m=0, 1, 2, \cdots$.

COROLLARY 3. $\liminf_{N\to\infty} (f(N) \log N)/N \leqq 3 \log 3$.

COROLLARY 4. $f(N) = O(N/\log N)$.

Before proving the theorem, let us establish the corollaries. Once we have a pair k, s satisfying the conditions of the theorem, we have then the inequality $f(b_m) \leqq k^m$, where b_m is the solution of the difference equation,

$$b_{m+1} = kb_m + sk^m, \; m \geqq 0; \qquad b_0 = 1. \tag{2}$$

This follows from the theorem, starting from $f(1)=1$ and proceeding iteratively. Setting $b_m = k^{m-1}c_m$, we have

$$c_{m+1} = c_m + s, \qquad c_0 = k. \tag{3}$$

Hence $c_m = k+ms$ and

$$f[k^{m-1}(sm + k)] \leqq k^m, \qquad m = 0, 1, \cdots. \tag{4}$$

Thus to prove Corollaries 1 and 2 we have only to exhibit matrices with $k=3$, $s=1$ and $k=5$, $s=2$ respectively. These matrices are

$$B_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

In the case of B_2 note that (mod 2) the third row is the sum of the first and third, while the fifth row is the sum of the second and third. Thus the rank over J_2 is 3, and the rational rank is 5 since the determinant is -4. To get Corollary

3, let $N \to \infty$ through the sequence of values $N = 3^{m-1}(m+3)$ and note that

$$\lim_{m\to\infty} \frac{3^m \log N}{N} = 3 \log 3.$$

Corollary 4 follows from Corollary 3 and the fact that between k and $4k$ there lies an integer of the form $3^{m-1}(m+3)$. Numerical bounds for lim sup $(f(m) \log m)/m$ can be obtained, but we do not insist on this point as in any case we cannot close the gap with the lower bound (1) by the methods of the present paper. Bernt Lindström, of the University of Stockholm, has kindly communicated results to the authors; from his work it is now known that

$$\lim_{m\to\infty} \frac{f(m) \log m}{m} = \log 4.$$

It remains to prove the theorem. For this it is convenient to use the concept of Kronecker product of two rectangular matrices (sometimes also called cross, direct, exterior, Grassman, outer or tensor products). We recall that for rectangular matrices P, Q the Kronecker product $P\times Q$ is the partitioned matrix $\|p_{ij}Q\|$, where $P = \|p_{ij}\|$, that is, each element p_{ij} of P is replaced by the *matrix* $p_{ij}Q$. If P is $m\times n$ and Q is $r\times s$ then $P\times Q$ is an $mr\times ns$ matrix. We require the following two facts:

$$(P \times Q)(R \times S) = PR \times QS \tag{5}$$

$$\text{Rank } (P \times Q) = (\text{Rank } P)(\text{Rank } Q). \tag{6}$$

In (5) it is assumed of course that the dimensions of the matrices are such that the indicated matrix multiplications may be carried out. For proofs of these relations and further references, the reader may consult [1] pp. 81–83, where the term "right direct product" is employed. Actually, we will employ (6) only in the trivial case when Q is a unit matrix. For the proof of the theorem, suppose that A is an $m\times n$ detecting matrix and B is the matrix whose existence is postulated by the theorem.

Let D be a $k\times k$ matrix over J_2 whose rank (over J_2) is s, and such that $DB = O$ (mod 2). Such a matrix exists because B has nullity s over J_2. Let now $J = F\times I_m$, where F is a $k\times s$ matrix over J_2 having rank s and such that moreover DF has rank s (all ranks are over J_2). It is evident that such a matrix F may be found; we can take it, for example to consist of s unit column vectors of length k, the positions of the units corresponding to s linearly independent columns of d. Then the partitioned matrix

$$C = (B \times A)\|J$$

is a $km\times(kn+sm)$ matrix of zeros and ones. The proof will be complete if we show that it is a detecting matrix, i.e. that the equation

$$Cx = \boldsymbol{a}, \tag{7}$$

where $\mathbf{x}$ and $\boldsymbol{a}$ denote column matrices with $kn+sm$ and km entries respectively, has at most one solution $\mathbf{x}$ in which all x_i are 0 or 1. It is convenient also to introduce the notation $\mathbf{x}'$, $\mathbf{x}''$ to denote the vectors whose coordinates are, respectively, the first kn and the last sm components of $\mathbf{x}$. Multiplying (7) by $D\times I_m$ and noting that, by (5), $(D\times I_m)(B\times A)=(DB)\times(I_mA)\equiv O \pmod 2$, we have

$$(D \times I_m)J\mathbf{x}'' \equiv \boldsymbol{e} \pmod 2, \tag{8}$$

where $\boldsymbol{e}=(D\times I_m)\boldsymbol{a}$.

Since the rank over J_2 of $(D\times I_m)J=DF\times I_m$ is sm, (8) can be uniquely solved for $\mathbf{x}''$ (mod 2), which is the same as $\mathbf{x}''$ since all components are 0 or 1. Hence to complete the solution of (7) we have only to solve $(B\times A)\mathbf{x}'=\boldsymbol{a}-J\mathbf{x}''$. This is possible since we get, on multiplying through by $B^{-1}\times I_m$, the equation $(I_k\times A)\mathbf{x}'=\boldsymbol{b}$, where $\boldsymbol{b}=(B^{-1}\times I_m)(\boldsymbol{a}-J\mathbf{x}'')$. But this latter system of equations may be uniquely solved for $\mathbf{x}'$, since it is a union of k systems, each of the type $A\mathbf{y}=\mathbf{c}$, and A is known to be a detecting matrix. The theorem is proved.

Remarks. (1) It will be perceived that the method of forming the "compound experiment," which we have for convenience treated algebraically, has a simple intuitive content, and applies equally well if the n-coin scheme from which we start is ramified. (2) The constant 3 log 3 in Corollary 3 seems to be the best obtainable by the use of the theorem, i.e. already the simplest nontrivial application ($k=3$) seems to be the best for large n. For relatively small n, however, Corollary 2 and its analogues for larger k still give good results. (3) J need not be taken as a Kronecker product, we need only that $(D\times I_m)J$ have rank sm over J_2.

3. An example. Let us illustrate the technique of Theorem 1 in a simple case. Starting with $n=1$, $k=1$ and using the matrix B_1 we may take for J any non-null column (say 1 0 0) and obtain the new detecting matrix.

$$A_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

for determining 4 coins in 3 weighings. Here the detection scheme is very simple:

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and if a_i ($i=1, 2, 3$) denote the results of the experiments (i.e. the number of counterfeits in the respective samples) we see that $x_4\equiv \sum_1^3 a_i \pmod 2$ which determines x_4. The remaining x_i are obtained from

$$B_1\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 - x_4 \\ a_2 \\ a_3 \end{bmatrix}.$$

Using the "Kronecker" construction a second time gives the new detecting matrix

$$A_{15} = \left[\begin{array}{cccc|cccc|cccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

where for J we have chosen the matrix which comprises the last three columns —obviously other natural choices are possible. This matrix gives the scheme for determining 15 coins in 9 weighings: on the first weighing weigh coins 1, 3, 4, 9, 11, 12, 13 and so on. The actual detection scheme is again readily described.

In conclusion we present a sample table based on our theorem.

n	1	4	7	15	26	45	54
upper bound for $f(n)$	1	3	5	9	15	25	27

Reference

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MATHEMATICAL NOTES

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FINITELY GENERATED ABELIAN GROUPS AS UNIONS OF PROPER SUBGROUPS

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1. Introduction. In a previous paper [1] S. Haber and the author derived some basic results on the representability of groups as unions of proper subgroups. The present paper discusses decompositions into unions in somewhat greater detail for the case of finitely generated abelian groups or, equivalently, groups which are finite direct products of (finite or infinite) cyclic groups.

2. Union decompositions of minimal cardinality. The main result of the present paper is the following

THEOREM. *Let G be a finitely generated abelian group which is not cyclic. Then:*

1) *If ∞ occurs more than once in the type of G, G is the union of three proper subgroups and no fewer.*

2) *If ∞ occurs exactly once in the type of G, G is the union of $p+1$ proper subgroups and no fewer, where p is the smallest prime having a power in the type of G.*

3) *If ∞ does not occur in the type of G, G is the union of $p+1$ proper subgroups and no fewer, where p is the smallest prime which occurs two or more times in the type of G.*

Prefatory to proving the Theorem, the following may be observed:

PROPOSITION. *Let G be any group, H a normal subgroup of G, and let G/H be the union of the proper subgroups H_i/H; then G is the union of the proper subgroups H_i.*

COROLLARY. *If $m(G)$ denotes the smallest number of proper subgroups of which G is the union, then $m(G) \leqq m(G/H)$ for all G and all normal subgroups H of G. (It is understood that if G is not a finite union of proper subgroups, $m(G) = \infty$.)*

COROLLARY. *$m(G) \leqq m(H)$ for all G and all supplemented subgroups H of G.*

Proof of the Theorem. 1) Since [**1**] no G can be the union of two proper subgroups, the "and no fewer" of (1) is assured. By the last Corollary, it suffices to show that $Z \times Z$ (where Z is the group of integers) is the union of three proper subgroups, since $Z \times Z$ is a supplemented subgroup of G in case (1). But $Z \times Z$ is in fact the union of the three proper subgroups consisting of those pairs of integers (x, y) such that (a) x and y have the same parity, (b) x is even, (c) y is even.

2) Let $G = Z \times H$, where H is finite, and let G be the finite union $\bigcup G_j$, where the G_j are proper subgroups of G and their number cannot be reduced. Since the subgroups of a finitely generated abelian group satisfy the maximum condition (see, e.g., [**2**] Theorem 10.3), it may be assumed that the G_j are maximal subgroups of G, so that G_j has prime index p_j. If q is a prime which has no power in the type of H, then clearly G has only one subgroup of index q, namely $qZ \times H$. If this subgroup is one of the G_j, by a lemma of [**1**] it must contain the intersection of the remaining G_j's, which is impossible. It follows that the indices of the G_j must all be primes which have powers in the type of H. Since the G_j have finite indices in G, so has their intersection; furthermore, the finite group $G/\bigcap G_j$ is the union of its proper subgroups $G_j/\bigcap G_j$, which have the same indices in it as the G_j do in G. But by an argument of [**1**] the number of $G_j/\bigcap G_j$ must exceed the smallest of their indices; the "and no fewer" is thus proved.

Conversely, let p^k be in the type of G; it suffices as before to show that the supplemented subgroup $Z \times Z/(p^k)$ of G is the union of $p+1$ proper subgroups.

To this end, let the relation R on the set of pairs (x, y) of $Z\times Z/(p^k)$ such that $p\nmid y$ be defined by $(x, y)R(u, v)$ if and only if $p\mid xv-yu$. Clearly R is reflexive and symmetric. Let y^{-1} denote any integer such that $y^{-1}y\equiv 1 \pmod{p}$. If $p\mid xv-yu$ and $p\mid uz-vw$, then $p\mid zy^{-1}(xv-yu)$ or equivalently $p\mid zy^{-1}xv-zu$, whence $p\mid \text{this}+uz-vw=zy^{-1}xv-vw$, whence $p\mid yv^{-1}(zy^{-1}xv-vw)$ and $\mid xz-yw$, so that R is also transitive. The pairs $(0, y_0), \cdots, (p-1, y_0)$, where $p\nmid y_0$, are clearly all R-inequivalent. On the other hand, for any (u, v) such that $p\nmid v$ there exists an x_0, $0\leqq x_0<p$, such that $(u, v)R(x_0, y_0)$, namely, $x_0=$ the remainder mod p of $v^{-1}y_0u$. The number of equivalence classes defined by R is therefore just p. The subgroup generated by each of these equivalence classes is evidently proper, since by the linearity of $\left|\begin{smallmatrix} x & y \\ u & v\end{smallmatrix}\right|$ in each of its rows, and the fact that addition mod a power of p preserves the divisibility of the determinant by p, any pair of elements of this subgroup must have determinant divisible by p. Finally, any pair (x, y) for which $p\nmid y$ is in some R-equivalence class, hence in at least one of these subgroups. $Z\times Z/(p^k)$ is thus the union of these proper subgroups together with the proper subgroup consisting of those pairs (x, y) for which $p\mid y$, a total of (at most) $p+1$ proper subgroups.

3) Let G be finite and equal to $\cup G_j$, where G_j has index p_j as in (2). If p_j has only one power $p_j^{k_j}$ in the type of G, readily G has only one subgroup of index p_j, namely $p_jZ/(p_j^{k_j})\times H$, where $G=Z/(p_j^{k_j})\times H$. If this subgroup is one of the G_j, the others must all contain $Z/(p_j^{k_j})$, so that this subgroup cannot contain the intersection of all the others. Hence the indices of the G_j must all be primes which have at least two powers in the type of G, and the number of G_j's must exceed the smallest of these indices, which proves the "and no fewer." Conversely, let p^h and p^k be in the type of G; it suffices as usual to show that $Z/(p^h)\times Z/(p^k)$ is the union of $p+1$ proper subgroups, and this may be proved exactly as in (2).

3. Decompositions into unions of cyclic subgroups. Since every cyclic subgroup of a finitely generated abelian group is contained in a maximal cyclic subgroup, it follows that a noncyclic finitely generated abelian group is the union of its maximal cyclic subgroups, which are of course all proper. Clearly this union is irredundant, since a maximal proper cyclic subgroup cannot be contained in a union of other proper cyclic subgroups. Since [1] a group is union-indecomposable if and only if it is cyclic, this union decomposition is the minimal way of expressing the given group as a union of proper subgroups which are not themselves further union-decomposable. The minimal-cardinality union decompositions considered in Section 2 are of course not in general of this type. It is thus of interest to determine the cardinality of this maximal cyclic union decomposition—in other words, to enumerate the maximal cyclic subgroups of a given finitely generated abelian group. This is accomplished by the following

THEOREM. *Let G be a finitely generated abelian group which is noncyclic. Then*
1) *If G is infinite, it has infinitely many maximal cyclic subgroups.*

2) *If G is finite, and $p_j^{k_{j1}}, \cdots, p_j^{k_{j,m_j}}$ are the powers of the prime p_j which occur in the type of G, then the number of maximal cyclic subgroups of G is*

$$\sum_{p_j^{k_j} \mid o(G)} (1/\phi(\Pi p_j^{k_j}))\Pi_j(p_j^{\alpha} - p_j^{\beta} - p_j^{\gamma} + p_j^{\delta})],$$

where $\alpha = \sum_h \min(k_{jh}, k_j)$, $\beta = \sum_h \min(k_{jh}, k_j - 1)$, $\gamma = \sum_h \min(k_{jh} - 1, k_j)$, $\delta = \sum_h \min(k_{jh} - 1, k_j - 1)$.

Proof. 1) If ∞ occurs more than once in the type of G, and p, q are relatively prime, then any element of G of the form $(\cdots, p, q, \cdots)$, where the p and q belong to infinite cyclic direct factors of G, generates a maximal cyclic subgroup, since it clearly cannot be a proper multiple of another element. If ∞ occurs only once in the type of G, let p^k be some prime power in the type of G (such must exist since G is noncyclic). Then any element of G of the form $(\cdots, u, p^m, \cdots)$, where the u belongs to a direct factor of G which is cyclic of order p^k, $p \nmid u$, and the p^m belongs to the infinite cyclic direct factor of G, generates a maximal cyclic subgroup. (It is of course clear that these maximal cyclic subgroups are all different.)

2) If G is finite, let H_j be the direct product of the cyclic groups in the type of G whose orders are powers of p_j. Let $x \in G$, and let x_{ij} $(1 \leqq i \leqq m_j)$ be the summands of x belonging to these cyclic groups. It is easily seen that x generates a maximal cyclic subgroup of G of order $\Pi p_j^{k_j}$ if and only if $\sum_i x_{ij}$ generates a maximal cyclic subgroup of H_j of order $p_j^{k_j}$. This in turn is readily equivalent to the assertions that some x_{ij} is non-divisible by p_j, no x_{ij} has order $> p_j^{k_j}$, and some x_{ij} has order exactly $p_j^{k_j}$. Now the number of m_j-tuples (x_{ij}) in which every term has order $\leqq p_j^{k_j}$ is $\Pi_h p_j^{\alpha}$, where $\alpha = \min(k_{jh}, k_j)$; the number in which also some term has order exactly $p_j^{k_j}$ is obtained by subtracting from this the number of (x_{ij}) in which every term has order $\leqq p_j^{k_j - 1}$, or $\Pi_h p_j^{\beta}$, where $\beta = \min(k_{jh}, k_j - 1)$. To impose the further restriction that some term be nondivisible by p_j, one need only subtract from this result the number of (x_{ij}) in which every term has order $\leqq p_j^{k_j}$, some term has order exactly $p_j^{k_j}$, and every term is divisible by p_j; this number is

$$\Pi_h p_j^{\gamma} - \Pi p_j^{\delta}, \qquad (\gamma = \min(k_{jh} - 1, k_j) \text{ and } \delta = \min(k_{jh} - 1, k_j - 1)).$$

The number of x's which generate maximal cyclic subgroups of G of order $\Pi p_j^{k_j}$ is obtained by multiplying together these numbers of m_j-tuples (product over j). Since a cyclic subgroup of order $\Pi p_j^{k_j}$ has $\phi(\Pi p_j^{k_j})$ generators, and no two such subgroups can have a common generator, the number of such subgroups is obtained by dividing the number of generators just found by $\phi(\Pi p_j^{k_j})$. The total number of maximal cyclic subgroups of G is then obtained by summing these numbers of subgroups over all possible orders.

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THE MINIMAL REGULAR GRAPH CONTAINING A GIVEN GRAPH

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Let G be an ordinary graph of order n which is not regular and whose maximum degree is $v>0$. Let H denote any regular graph of degree v which contains a subgraph isomorphic to G. We seek the minimal order possible for H. Let x_i denote the degree of the ith vertex in G, so $v-x_i$ is the "deficiency" of that vertex; let $\sigma=\Sigma(v-x_i)$ be the sum of the deficiencies and d be the maximum deficiency

THEOREM. *The necessary and sufficient condition that $m+n$ be the minimal order possible for H is that m be the least positive integer such that:* (1) $m\geqq\sigma/v$; (2) $m^2-(v+1)m+\sigma\geqq 0$, (3) $m\geqq d$ *and* (4) $(m+n)v$ *is an even integer. The maximum value of m is n, and for each $n>3$ there exists a graph G such that $m=n$.*

Proof. Necessity. It is known that finite graphs H exist, so there is a minimal solution, say a graph H of order $m+n$, and $(m+n)v$ is clearly an even integer.

Let G' be the subgraph of H isomorphic to G and let A be the subgraph induced on the vertices of H not in G'. Then in H there are σ joins between the subgraphs G' and A. Since each of the m vertices of A receives at most v of these joins, $mv\geqq\sigma$, and clearly $m\geqq d$.

Denote by $m(A)$ the number of joins in A. The sum of the degrees of the vertices of A, as points of A, must be $mv-\sigma$, hence

(i) $m(A)=\frac{1}{2}(mv-\sigma)$.

Then from $m(m-1)/2\geqq m(A)$, it follows that

(ii) $m^2-(v+1)m+\sigma\geqq 0$,

so all four conditions are necessary.

To establish the sufficiency, let m be the least positive integer satisfying conditions (1)–(4). Define a graph H by beginning with G and m extra independent points $a_1, a_2, \cdots, a_m$. Let $p_1, p_2, \cdots, p_k$ denote the points of G with positive deficiencies $d_1, \cdots, d_k$. Let the completion of G be done in the following way. First, p_1 is completed by joins to the points $a_1, a_2, \cdots, a_{d_1}$ in succession. Then p_2 is completed by joins to successive points a_i, starting with a_{d_1+1}, which is taken cyclically to be a_1 if $d_1=m$. These completions are possible because $m\geqq d$. The degrees attained by points of A in this construction cannot differ from one another at any stage by more than one. So this is also true when the points of G are all complete.

Now let $\sigma/m=h+r/m$, where h and r are nonnegative integers and where $r<m$, and $h<v$ if $r>0$. Then when the vertices of G have been completed the

set $\mathfrak{A}$ of vertices a_i, $i=1, \cdots, m$, consists of r points of degree $h+1$ and $m-r$ points of degree h. Since there are as yet no joins between points in $\mathfrak{A}$, any point of the greatest remaining deficiency $v-h$ can be completed if $v-h \leqq m-1$. But condition (2) can be written in the form

(iii) $v-\sigma/m \leqq m-1$,

from which it follows that

(iv) $v-h \leqq m-1+r/m$.

Because $0 \leqq r/m < 1$, while $v-h$ and $m-1$ are integers, (iv) implies that

(v) $v-h \leqq m-1$.

Thus there are in $\mathfrak{A}$ sufficient points so that each point individually can be completed.

Finally, the collective completion of all the points in $\mathfrak{A}$ will be possible if the sum of the deficiencies is an even integer, that is, if

(vi) $r(v-h-1)+(m-r)(v-h)=mv-\sigma$

is even. But

(vii) $mv-\sigma=mv-[nv-2m(G)]=(m+n)v-2[nv-m(G)]$.

By assumption $(m+n)v$ is even, hence $mv-\sigma$ is even and the completion of all points in $\mathfrak{A}$ is possible.

Since $\sigma < nv$, the condition $m \geqq \sigma/v$ cannot force $m > n$. Similarly $m^2-(v+1)m+\sigma \geqq 0$ always holds for $m=v+1$, and $v+1=n$. Condition (3) cannot force m to exceed $n-1$. The maximum possible value $m=n$, satisfying conditions (1) and (2) cannot be increased by condition (4), since $(m+n)v=(n+n)v$ is necessarily even. Thus in all cases $m \leqq n$.

If $n>3$, let G be the graph obtained from a complete graph of order n by deleting one join. Then $v=n-1$ and $\sigma=2$, and the condition

(viii) $m^2-nm+2 \geqq 0$ implies that $m \geqq n$.

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TOTAL BOUNDEDNESS, GÂL'S THEOREM, AND COMPLETION FOR REGULAR TOPOLOGICAL SPACES

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In this note a theorem by Gâl [3] on compactness is proven by means of "indexed systems of neighborhoods" [2] and the theorem is then used to show that Weil's theory of uniformities is "minimal" if one wishes to deal only with spaces which have completions without exception. Roughly speaking, the result is that regular spaces which have regular completions without exception are already completely regular. In order to obtain from Gâl's theorem this result on completion, portions of a theory of totally bounded spaces will be presented. Surprisingly many of the theorems on total boundedness for uniform spaces hold for spaces in general.

In [2] it was shown how the topology of any topological space may be defined by an indexed system of neighborhoods. Here it will be more convenient to speak in terms of relations rather than of functions which assign neighborhoods

to points. A family $N_I = \{N_\alpha\}_{\alpha \in I}$ of dyadic relations over a set X which satisfies the postulates given in [2] for an indexed system of neighborhoods will be called a *structure* for X. Either (X, N_I) or $(X, \mathfrak{I})$, where $\mathfrak{I}$ is the topology defined by N_I, will be referred to as a *space*, the context indicating which is meant. A structure N_I is called *symmetric* if for each $\alpha \in I$ there is a $\beta \in I$ such that $N_\beta^{-1} \subseteq N_\alpha$, and *transitive* if for each $\alpha \in I$ there is a $\beta \in I$ and a $\gamma \in I$ such that $N_\gamma N_\beta \subseteq N_\alpha$. (Composition is denoted by juxtaposition and, for powers, by superscripts.) Thus a uniform structure is a reflexive, symmetric, and transitive family of relations and, in this sense, is a generalized equivalence relation. N_I is *locally transitive* if it satisfies the "local triangle inequality": for each $x \in X$, each $\alpha \in I$, $N_\gamma N_\beta(x) \subseteq N_\alpha(x)$, where β and γ may depend on x as well as on α. It was shown in [2] that a topological space is regular if and only if it admits a symmetric locally transitive structure. Such structures themselves will be called *regular*.

We shall use the definitions of *semiregular net*, *Cauchy filter*, and *completeness* given in [2]. In addition, we shall need the idea of "total boundedness." A structure N_I for X is *finite* if for each $\alpha \in I$ there is a finite subset $F_\alpha \subseteq X$ such that $X \subseteq N_\alpha(F_\alpha)$. The space (X, N_I), when N_I is finite, is *totally bounded.* It happens that every topological space admits a finite structure:

THEOREM 1. *There is a finite structure defining the topology of any given space. If the space is R_0 (or R_2, or completely regular), then that structure may be chosen also to be symmetric (and locally transitive, and transitive).*

Proof. For spaces in general, the construction used in the proof of Theorem 1 of [2] is easily seen to yield finite structures. Given an R_0-space $(X, \mathfrak{I})$, let I be the family of all finite open coverings. Then for each $\alpha \in I$, define $N_\alpha = \bigcup_{G \in \alpha} G \times G$. Clearly N_I is reflexive and symmetric. Moreover,

$$\begin{aligned} N_\alpha \cap N_\beta &= \left[\bigcup_{G \in \alpha} G \times G\right] \cap \left[\bigcup_{H \in \beta} H \times H\right] = \bigcup_{G,H} ([G \times G] \cap [H \times H]) \\ &= \bigcup_{G \cap H} ([G \cap H] \times [G \cap H]) = N_\gamma, \end{aligned}$$

where γ is the finite open covering whose members are of the form $G \cap H$ with $G \in \alpha$ and $H \in \beta$. Thus N_I is closed under intersection. Now given $G \in \mathfrak{I}$ and $x \in G$, let $\alpha = [G, X - \{x\}^-]$. Then $\alpha \in I$ and $N_\alpha(x) = G$. So N_I defines $\mathfrak{I}$. Finally, (X, N_I) is totally bounded: Given $\alpha \in I$, let F_α be a selection of points, one from each member of α. Then $X = N_\alpha(F_\alpha)$ and F_α is finite. Now if $(X, \mathfrak{I})$ happens to be regular, an argument similar to the one used in the proof of Theorem 4 of [2] will show that N_I is locally transitive. Proofs of existence of finite uniform structures for completely regular spaces are well known.

As in the theory of uniformities, total boundedness may be characterized in terms of the existence of semiregular nets:

THEOREM 2. *A space is totally bounded if and only if each net (filter) has a semiregular subnet (Cauchy superfilter).*

Proof. Using universal subnets (or ultra-filters), this may be proven as in the theory of uniform spaces. (E.g., see p. 199 of [4].)

As usual, a topological space is said to be *compact* if each of its open coverings has a finite subcovering. Totally bounded spaces are "precompact" in the following sense:

THEOREM 3. *A space has a compact topology if and only if it is totally bounded and complete.*

Proof. Compact spaces (X, N_I) are totally bounded since each covering of the form $\{N_\alpha(x)\}_{x\in X}$ has a finite subcovering. (If N_I is not an open neighborhood structure, choose an equivalent structure which is.) Since *every* filter has a limit point, compact spaces are also complete. Now, conversely, suppose (X, N_I) is totally bounded and complete. By Theorem 2, every net in X has a semiregular subnet which, by completeness, converges. It is well known that the existence of convergent subnets without exception is equivalent to compactness.

For compact spaces, not only are all uniformities equivalent (a standard result) but indeed all regular structures are equivalent:

THEOREM 4. *All regular structures defining a compact topology are equivalent; all are uniform.*

Proof. First we show that if N_I is a regular structure defining a compact topology for X, then $\overline{N}_I=\{\overline{N}_\alpha\}_{\alpha\in I}$ (closure taken in the product topology) generates an equivalent structure. We need two auxiliary results:

LEMMA A. *Let $(X, \mathfrak{I})$ be defined by a symmetric structure N_I. For each*

$$R\subseteq X\times X,\ \overline{R}=\bigcap_{\alpha\in I} N_\alpha R N_\alpha.$$

LEMMA B. *If N_I is a regular structure, then $\bigcap_{\beta\in I} N_\beta^3=\bigcap_{\alpha\in I} N_\alpha$.*

The first may be proven just as it is in the theory of uniformities; the second follows from the fact that, for each x and each α, there is a β such that $N_\beta^3(x)\subseteq N_\alpha(x)$.

Now, given $\alpha\in I$, we have

$$\bigcap_{\beta\in I} \overline{N}_\beta = \bigcap_{\beta\in I}\bigcap_{\gamma\in I} N_\gamma N_\beta N_\gamma \subseteq \bigcap_{\gamma\in I} N_\gamma^3 = \bigcap_{\beta\in I} N_\beta \subseteq N_\alpha.$$

That is, $(X\times X)-N_\alpha\subseteq\bigcap_{\beta\in I}[(X\times X)-\overline{N}_\beta]$, so that N_α together with $\{(X\times X)-\overline{N}_\beta\}_{\beta\in I}$ form an open covering of the compact set $X\times X$. The existence of a finite subcovering gives us $\bigcap_{i=1}^n \overline{N}_{\beta_i}\subseteq N_\alpha$, for some choice of $\beta_1, \cdots, \beta_n$. Thus the structure generated by $\overline{N}_I$ (by taking finite intersections) is as fine as the

structure N_I. But then it is clear that the two are in fact equivalent, as was to be shown.

To finish the proof of Theorem 4, one may now proceed as in the proof of the corresponding theorem for uniformities, showing that all regular structures are equivalent to the structure consisting of the family of all neighborhoods of Δ in $X\times X$ and that this family is in fact a uniformity. The crucial ingredient in the standard proof (e.g., see p. 197 of [4]) is the existence of a structure defining $\mathfrak{J}$ all of whose members are closed, and this is provided by $\overline{N}_I$.

THEOREM 5. (Gál) *Let $\overline{X}$ have a regular topology and let X be dense in $\overline{X}$. If every net in X has a cluster point in $\overline{X}$, then $\overline{X}$ is compact.*

Proof. Let N_I be a regular structure for $\overline{X}$. We shall show that each net $x_D=\{x_n\}_{n\in D}$ in $\overline{X}$ has a cluster point. For each $(\alpha, n)\in I\times D$, choose $x_{\alpha,n} \in N_\alpha(x_n)\cap X$. If $I\times D$ is directed by letting $(\alpha, n)\geqq(\beta, m)$ mean that $n\geqq m$ and $N_\alpha\subseteq N_\beta$, then $x_{I\times D}$ is a net in X and so has a cluster point p in $\overline{X}$. Now given $\alpha\in I$, let $\beta\in I$ be such that $N_\beta^2(p)\subseteq N_\alpha(p)$. Given $n\in D$, let $m\geqq n$ and $\gamma\geqq\beta$ be such that $x_{\gamma,m}\in N_\beta(p)$. Then since also $x_{\gamma,m}\in N_\gamma(x_m)\subseteq N_\beta(x_m)$, we have $x_m\in N_\beta^2(p)$ $\subseteq N_\alpha(p)$. Thus p is a cluster point for x_D. We conclude (using the cluster-points-for-nets characterization of compactness) that $\overline{X}$ must be compact, q.e.d.

Remark. Presupposing Theorem 4 of [2], the use of structures simplifies and illuminates the proof of Gál's theorem considerably. (Cf. [3].)

Now we are ready to consider the question of completions for regular spaces. By a *completion* of a space (X, N_I) we mean a complete space which contains an isomorphic replica of (X, N_I) as a dense subspace. Our result is a negative one: it says that there can be no general procedure for constructing regular completions of spaces which do not admit uniform structures.

THEOREM 6. *Let $(X, \mathfrak{J})$ be a regular topological space. If for each regular structure N_I defining $\mathfrak{J}$ there is a regular completion of (X, N_I), then $(X, \mathfrak{J})$ is completely regular.*

Proof. Given a regular space $(X, \mathfrak{J})$, there is, by Theorem 1, a finite regular structure N_I' defining $\mathfrak{J}$. Let $(\overline{X}, N_I)$ be a completion of (X, N_I') whose topology is regular. Since (X, N_I') is totally bounded, each of its nets has a semiregular subnet (Theorem 2) and each such subnet converges to a point in $\overline{X}$. That is, each net in X has a cluster point in $\overline{X}$. Then by Theorem 5, $(\overline{X}, N_I')$ has a compact topology. It is well known that subspaces of compact regular spaces are completely regular. Since we have shown that $(X, \mathfrak{J})$ is such a space, our proof is done.

Remark. Cohen [1] has found a way of completing spaces whose structures seem to be intermediate between those which are regular and those which are uniform. He has not shown, however, nor has the present author, that the topologies of these spaces are not necessarily completely regular. Counterexamples abound, of course, to show that spaces which admit regular structures need not admit uniformities, so Theorem 6 is not without some significance. It should be

noted, however, that our theory of completeness depends heavily on the particular generalization of "Cauchy net (filter)" that was chosen. A different choice might yield a more elegant theory.

This material is taken from a chapter of *Structures and Topology*, a dissertation submitted to the University of Oklahoma in partial fulfillment of requirements for the Ph.D. degree.

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ON THE SIEVE OF ERATOSTHENES

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Introduction. Let $\alpha_{n_1}, \cdots, \alpha_{n_m}$ be any m integers. The Sieve of Eratosthenes yields the expression (1) in Theorem 2 for the number of elements less than α which are not multiples of any of the α_{n_i} for $i=1, \cdots, m$. This expression can be characterized by a few of its properties. We shall give this characterization and generalize the results to the normed Euclidean space described below.

Let E denote a Euclidean ring with a finite number of units $\lambda_1, \lambda_2, \cdots, \lambda_k$, and let $N(x)$ denote a norm on E. We shall assume that E and $N(x)$ have the following properties:

a) If $N(x)=1$, then x is a unit element.

b) Given $\alpha\in E$, there exist at most a finite number of elements x of E such that $N(x)\leqq N(\alpha)$.

Let S denote a sequence $\{\alpha_1, \cdots\}$ of elements of E such that $N(\alpha_i)\geqq 2$ for $i=1, 2, \cdots$ and such that S contains all primes of E. Let D be the class of all pairs (α, S_α), where $\alpha\in E$, $S_\alpha\subset S$ and such that $N(x)\leqq N(\alpha)$ for every $x\in S_\alpha$. For each α we include all possible S_α having the above property. Let T_α be the set of nonzero elements of E of norm $\leqq N(\alpha)$. For any set S let $|S|$ denote the number of elements of S.

DEFINITION. $A(\alpha, S_\alpha)=$ *(the number of elements of T_α which are not multiples of any of the elements of S_α)$/k$.*

In the following let ϕ denote the null set, $S_\alpha=\{\alpha_{i_1}, \cdots, \alpha_{i_{m+1}}\}$, and $S_\alpha'=\{\alpha_{i_1}, \cdots, \alpha_{i_m}\}$.

THEOREM 1. *A is characterized by the following*:

I. $A(\alpha, \phi)=|T_\alpha|/k$;

II. $1\leqq A(\alpha, S_\alpha)\leqq A(\alpha, S_\alpha')$;

III. $A(\alpha, S_\alpha)=A(\alpha, S'_\alpha)$ *iff* $\alpha_{i_{m+1}}$ *is a multiple of* α_{i_j} *for some* $j=1, \cdots, m$, *i.e.*, $\alpha_{i_{m+1}}=\xi\alpha_{i_j}$ *where* $\xi\in E$;

IV. A *is integral-valued.*

Proof. Let us denote the A of our definition by $\overline{A}$. Firstly $A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}) \geqq \overline{A}(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\})$. We note that if β is not a multiple of any of the α_i then $\lambda_t\beta$ is also not a multiple of any of the α_{i_j} for $t=1, \cdots, k$ and for $j=1, \cdots, m$. Let $\beta_1, \cdots, \beta_{i_q}$ be those elements of T_α which are not multiples of any of the α_{i_j} for $j=1, \cdots, m$, and which satisfy the condition that if β appears among the β_i then $\lambda_t\beta$ does not for $\lambda_t\neq 1$. Furthermore, assume that $N(\beta_1)\geqq N(\beta_2)\geqq \cdots \geqq N(\beta_t)$. Now β_j cannot be a multiple of β_r for $j>r$, since $\beta_j=a\beta_r$ implies that $N(\beta_j)=N(a)N(\beta_r)\geqq N(\beta_r)$, and from our hypothesis it would follow that $N(a)=1$ and that a is a unit, which cannot be the case. Thus we get

$$
\begin{aligned}
&A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}) \\
&\qquad > A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}, \beta_1\}) > \cdots > A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}, \beta_1, \cdots, \beta_q\}) \geqq 1
\end{aligned}
$$

or

$$A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}) \geqq q = \overline{A}(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}).$$

On the other hand $A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\})\leqq \overline{A}(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\})$. For let $\bar{\alpha}_1, \cdots, \bar{\alpha}_r$ be those elements of T_α which are multiples of some element of S_α. Again from each set $\lambda_1\bar{\alpha}, \lambda_2\bar{\alpha}, \cdots, \lambda_k\bar{\alpha}$ we allow only one element to appear among the $\bar{\alpha}_i$ for $i=1, \cdots, r$. Furthermore, it is assumed that $N(\bar{\alpha}_1)\geqq N(\bar{\alpha}_2) \geqq \cdots \geqq N(\bar{\alpha}_r)$. Thus we get

$$
\begin{aligned}
&A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}) \\
&= A(\alpha, \{\bar{\alpha}_1, \cdots, \bar{\alpha}_r\}) < A(\alpha, \{\bar{\alpha}_1, \cdots, \bar{\alpha}_{r-1}\}) < \cdots < A(\alpha, \{\bar{\alpha}_1\}) < |T_\alpha|/k.
\end{aligned}
$$

Hence $A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\})\leqq |T_\alpha|/(k-r)=\overline{A}(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\})$ and our theorem is proved.

Noting that the above proof is valid for $E=$ the set of positive integers, with $N(x)$ the usual norm, we get

THEOREM 2. *If*

$$
\begin{aligned}
A(\alpha, S_\alpha) &= \alpha - \sum_{i=1}^{m}\left[\frac{\alpha}{\alpha_{n_i}}\right] + \sum_{1\leq i<j\leq m}\left[\frac{\alpha}{\langle\alpha_{n_i}, \alpha_{n_j}\rangle}\right] \\
&\quad - \sum_{1\leq i<j<k\leq m}\left[\frac{\alpha}{\langle\alpha_{n_i}, \alpha_{n_j}, \alpha_{n_k}\rangle}\right] + \cdots + (-1)^m\left[\frac{\alpha}{\langle\alpha_{n_1}, \cdots, \alpha_{n_m}\rangle}\right],
\end{aligned}
\tag{1}
$$

where $\langle a_1, a_2, \cdots, a_n\rangle$ *denotes the least common multiple of* $a_1, a_2, \cdots$ *and* a_n, *and* $[x]$ *denotes the integral part of* x, *then* A *is characterized by the following*:

I′. $A(\alpha, \phi)\leqq\alpha$;

II′. $1\leqq A(\alpha, S_\alpha)\leqq A(\alpha, S'_\alpha)$;

III$'$. $A(\alpha, S_\alpha) = A(\alpha, S'_\alpha)$ *iff* $\alpha_{i_{m+1}}$ *is a multiple of* α_{i_j} *for some* $j = 1, \cdots, m$;
IV$'$. A *is integral-valued.*

Here ϕ, S_α and S'_α are used as they were in Theorem 1.

Remark. Let I$'$ be replaced by the following:

I$''$. Given α, there exists an S_α such that $A(\alpha, S_\alpha) = 1$.

In this case it is still possible to prove that $A(\alpha, S_\alpha)$ has many of the properties of the functions defined by (1). For example: if $m = \pi(\alpha)$, the number of primes not exceeding α, then $A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}) = 1$ iff $\{\alpha_{i_1}, \cdots, \alpha_{i_m}\} = \{P_1, \cdots, P_m\}$, where $P_1, \cdots, P_m$ are the primes not exceeding α. We have $1 \leqq A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}, P_1\}) \leqq A(\alpha, \{\alpha_{i_1}, \cdots, \alpha_{i_m}\}) = 1$. Since this is only possible if P_1 is a multiple of α_{i_j} for some $j = 1, \cdots, m$, we get $P_1 = \alpha_{i_j}$. In a similar manner we find that the other α's are each one of these primes.

The second part of our assertion follows from the first and I$''$.

THE NUMBER OF REPRESENTATIONS OF A NUMBER AS A SUM OF TWO SQUARES

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Let $d_1(n)$ $(n > 0)$ denote the number of those divisors of n that are of the form $4k+1$, and $d_3(n)$ the number of those of the form $4k+3$. Then it is well known that the total number of representations $r_2(n)$ of n as a sum of two squares of integers (positive, negative, or zero) is given by

$$r_2(n) = 4\{d_1(n) - d_3(n)\}.$$

In what follows, we write $\rho(n)$ for $d_1(n) - d_3(n)$. We denote by $d(n)$ the number of divisors of n as usual.

The object of this note is to obtain an apparently new expression for $r_2(n)$ in terms of the exponents in the prime-power decomposition of n.

THEOREM 1. *$\rho(n)$ is a multiplicative function.*

Proof. Let $n = m_1 m_2$, where $(m_1, m_2) = 1$. Then

$$d_1(n) = d_1(m_1)d_1(m_2) + d_3(m_1)d_3(m_2);$$

and

$$d_3(n) = d_3(m_1)d_1(m_2) + d_1(m_1)d_3(m_2).$$

Subtracting, we have $\rho(n) = \rho(m_1)\rho(m_2)$.

THEOREM 2. *If p is an odd prime, then*

$$\rho(p^\alpha) = \left[\frac{\alpha+2}{2}\right] + (-1)^{[p/2]}\left[\frac{\alpha+1}{2}\right], \qquad \alpha \geqq 0.$$

Proof. If $p \equiv 1 \pmod 4$, then p^α has no divisor of the d_3-type and

$$\rho(p^\alpha) = d(p^\alpha) = \alpha + 1.$$

If $p \equiv 3 \pmod 4$, then the divisors of p^α which are of the d_1-type are

$$1, p^2, p^4, \cdots, p^{2[\alpha/2]},$$

and those which are of the d_3-type are

$$p, p^3, p^5, \cdots, p^{2[(\alpha-1)/2]+1}.$$

Hence $d_1(p^\alpha) = [\alpha/2]+1 = [(\alpha+2)/2]$, and $d_3(p^\alpha) = [(\alpha-1)/2]+1 = [(\alpha+1)/2]$, so that

$$\rho(p^\alpha) = \left[\frac{\alpha+2}{2}\right] - \left[\frac{\alpha+1}{2}\right] = 1 \text{ or } 0$$

according as α is even or odd. The result stated in the theorem covers both the cases.

Since $\rho(2^\beta) = 1$ when $\beta \geqq 0$ because 1 is the only odd divisor of 2^β and it is of the d_1-type, we have from Theorems 1 and 2 the following

THEOREM 3. *If $n = 2^\beta \Pi p^\alpha$, $\beta \geqq 0$, be the canonical factorisation of n, then*

$$\rho(n) = \Pi\left\{\left[\frac{\alpha+2}{2}\right] + (-1)^{[p/2]}\left[\frac{\alpha+1}{2}\right]\right\}.$$

If we write $n = 2^\beta MN$, where M has no prime divisor of the form $4k+1$, and N has no prime divisor of the form $4k+3$, the theorem reduces to $\rho(n) = d(N)$ or 0 according as M is or is not a square.

Editorial Note: A paper entitled "Functions which represent all integers," by E. N. Gilbert, appeared in this MONTHLY, 70 (1963) 736–738 (August–September). In a letter dated October 1, 1963, Gilbert calls attention to the fact that a similar theorem and other examples appear in a paper by J. Lambek and L. Moser, "Inverse and complementary sequences of natural numbers," this MONTHLY, 61 (1954) 454–458. This reference became available too late for acknowledgement when Gilbert's paper appeared.

CLASSROOM NOTES

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THE RATIONAL CANONICAL FORM OF A FUNCTION OF A MATRIX

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1. Introduction. Once one has been introduced to the canonical form of a matrix A, it is natural to ask, "What are the canonical forms of $A^2, A^3, \cdots$ and in general of $f(A)$, where $f(x)$ is any polynomial?" This problem has in fact a long and interesting history, the most comprehensive treatment being due to M. C. R. Butler [2], while the first complete solution using only rational operations was given by N. H. McCoy [3]. I feel, however, that the solution given

here is the simplest one to date and that it is suitable for teaching to students. Although I only deal with the primary rational canonical form (p.r.c.f.) ([1], p. 332) the corresponding results for other canonical forms can be readily deduced. See for example the techniques in [5].

2. Statement of Theorem. The elements of all matrices and the coefficients of all polynomials are assumed to belong to an arbitrary field. Let $f(x)$ be our given polynomial,

$$f(x) = \sum_{i=0}^{m} \alpha_i x^{m-i}, \qquad \alpha_0 \neq 0,$$

let A be our given square matrix, and let $\phi(x)$ be the characteristic polynomial of A,

$$\phi(x) = \sum_{i=0}^{n} \beta_i x^{n-i}, \qquad \beta_0 = 1.$$

We need

LEMMA 1. *The characteristic polynomial $\psi(x)$ of $f(A)$ is $(-1)^{m(n+1)}R(x)$, where $R(x)$ is the resultant whose two principal rows are $\alpha_0, \alpha_1, \cdots, \alpha_{m-1}, \alpha_m - x$, and $\beta_0, \beta_1, \cdots, \beta_n$.*

Now if H is a nonsingular matrix of order n then $H^{-1}f(A)H = f(H^{-1}AH)$. Hence we may assume that A is in p.r.c.f. This form is a direct sum of companion matrices C_i, and so $f(A)$ will be the direct sum of the matrices $f(C_i)$. Hence we may further assume that A is a single companion matrix, and then $\phi(x)$ must be a power $p^a(x)$ of a monic irreducible polynomial $p(x)$. Having made this assumption we introduce the following notation.

If the derivative $\psi'(x)$ of the characteristic polynomial $\psi(x)$ is not 0, then we write $q(x)$ for the quotient of $\psi(x)$ and the highest common factor $(\psi(x), \psi'(x))$. If $\psi'(x) = 0$, then ([6], p. 119) we must be in a field of nonzero characteristic P, and there is a largest integer e such that $\psi(x) = \omega(x^{P^e})$, for some polynomial $\omega(x)$. In this case we put $q_0(x) = \omega(x)/(\omega(x), \omega'(x))$, and by testing whether the coefficients of $q_0(x)$ have P^s-th roots in the field, we find the largest integer s such that $q_0(x^{P^e}) = q^{P^s}(x)$ for some polynomial $q(x)$. Next we let t be the largest integer for which $p^t(x)$ divides $q(f(x))$. Then writing c and d for the respective degrees of $p(x)$ and $q(x)$, we define numbers b, h, k by the relations

$$\begin{aligned} &(b-1)t < a \leqq bt, \\ (1) \qquad &hd = \{a - (b-1)t\}c, \\ &kd = \{bt - a\}c. \end{aligned}$$

Then the main result is

THEOREM 1. *The p.r.c.f. of $f(A)$ is the direct sum of h companion matrices of $q^b(x)$ and k companion matrices of $q^{b-1}(x)$.*

3. The Proofs. The proof of Lemma 1 is easy. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the latent roots of A then ([4], p. 113) the latent roots of $f(A)$ are $f(\lambda_1), \cdots, f(\lambda_n)$. The resultant $R(x)$ vanishes for $x=\xi$ if and only if the polynomials $f(x)-\xi$ and $\phi(x)$ have a common root ([6], p. 84). Hence $R(x)=\pm\psi(x)$ and the lemma follows.

As a first step towards proving Theorem 1 we observe that $p(x)$ is the polynomial of lowest degree satisfied by any λ_i, and the roots of $\psi(x)$ are $f(\lambda_i)$, $1 \leqq i \leqq n$. Suppose that $q_1(x)$, $q_2(x)$ are factors of positive degree of $\psi(x)$, with highest common factor $h(x)$. Then $p(x)$ divides both $q_1(f(x))$ and $q_2(f(x))$, and hence $h(f(x))$. Assuming that $q_1(x)$, $q_2(x)$ are irreducible we see that $\psi(x)$ is a power of $q(x)$, where $q(x)$ is the monic polynomial of lowest degree such that $p(x)$ divides $q(f(x))$. Clearly $q(x)$ is irreducible, has a degree d which divides the degree c of $p(x)$, and may be found by the process described in section 2. Since $q(x)$ is irreducible, the p.r.c.f. of $f(A)$ must be a direct sum of companion matrices of polynomials $q^{e_\nu}(x)$, $1 \leqq \nu \leqq l$, with $e_\nu \geqq 1$. Therefore it only remains to determine l and $e_1, e_2, \cdots, e_l$.

The order of A is n, and for $i=0, 1, 2, \cdots$, we let U_i and V_i be the set of all $n\times 1$ column matrices annihilated by $p^i(A)$ and $q^i(f(A))$ respectively. Clearly the U_i's and V_i's are vector spaces with $U_i \subseteq U_{i+1}$ and $V_i \subseteq V_{i+1}$ for $i=0, 1, 2, \cdots$. Moreover

$$\text{(2)} \qquad \text{dimension } U_i = ic \qquad \text{for } i = 1, 2, \cdots, a,$$

while

$$\text{(3)} \qquad \text{dimension } V_j = \sum_{\substack{1\leq\nu\leq l \\ e_\nu > j}} jd + \sum_{\substack{1\leq\nu\leq l \\ e_\nu\leq j}} e_\nu d \qquad \text{for } j = 1, 2, \cdots.$$

If $p^i(x)$ divides $q^j(f(x))$ then $U_i \subseteq V_j$, for

$$u \in U_i \Rightarrow p^i(A)u = 0 \Rightarrow q^j(f(A))u = 0 \Rightarrow u \in V_j.$$

Next suppose that i is the largest integer for which $p^i(x)$ divides $q^j(f(x))$. Then there is a polynomial $g(x)$, prime to $p(x)$, such that $q^j(f(x))=g(x)p^i(x)$, moreover $g(A)$ is nonsingular. Hence

$$u \in V_j \Rightarrow q^j(f(A))u = 0 \Rightarrow g(A)p^i(A)u = 0 \Rightarrow p^i(A)u = 0 \Rightarrow u \in U_i,$$

and so $V_j \subseteq U_i$.

Now since $p(x)$ is irreducible, by definition of t, for $j=1, 2, \cdots$, the largest integer i such that $p^i(x)$ divides $q^j(f(x))$ is $i = jt$. Altogether therefore, if $a=(b-1)t+r$, $0 \leqq r < t$, then we have

$$\text{(4)} \qquad U_{jt} = V_j \qquad \text{for } j = 0, 1, \cdots, (b-1),$$

and

$$\text{(5)} \qquad U_a = V_j \qquad \text{for } j = b, b+1, \cdots.$$

Suppose now that $b>1$. Then putting $j=1$ in (4) shows that $U_t = V_1$, and since $e_\nu \geqq 1$, using (2) and (3) to compare dimensions gives us

$$tc = ld. \tag{6}$$

If we repeat the operation with $j=b-1$ we obtain

$$\sum_{\substack{1\leq\nu\leq l\\ e_\nu>(b-1)}} (b-1)d + \sum_{\substack{1\leq\nu\leq l\\ e_\nu\leq(b-1)}} e_\nu d = (b-1)\{tc\} = (b-1)\{ld\},$$

and hence $e_\nu \geqq b-1$ for $\nu=1, 2, \cdots, l$.

Since $U_a = V_b$ by (5), and $bt>a$, in a similar way we find that $e_\nu \leqq b$ for $\nu=1, 2, \cdots, l$. If h of the e_ν's are b, and k of them are $b-1$, then (6) becomes $tc=(h+k)d$, (5), (2) and (3) show that $ac=hbd+k(b-1)d$, and trivially

$$(h+k)bd \geqq ac > (k+h)(b-1)d.$$

Relations (1) now follow easily for the case $b>1$. If $b=1$ then (5) shows that $e_\nu=1$ for all ν, and again relations (1) hold because, even though k may not be 0, the companion matrix of $q^0(x)$ is empty.

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THE DERIVED SET OPERATOR

FRANK REESE HARVEY, Carnegie Institute of Technology

A topology for a set X can be characterized by specifying the open sets, the closed sets, a closure operator, an interior operator, or a derived set operator. The set of all accumulation points of a set A in a topological space is sometimes called the *derived set* of A. A list of derived set operator axioms does not appear to be in print. The following is one possible list.

A *derived set operator* on a set X is a function α, which maps the subsets A of the set X into subsets A^α of X, and which satisfies the following four statements.

I. $\varnothing^\alpha=\varnothing$
II. $(A\cup B)^\alpha = A^\alpha \cup B^\alpha$ for each A and B
III. $x\in A^\alpha$ if and only if $x\in(A\sim\{x\})^\alpha$ for each A.
IV. $(A\cup A^\alpha)^\alpha \subset A\cup A^\alpha$ for each A.

It is easily verified that the derived sets of a topological space satisfy I through IV.

THEOREM. *Let α be a derived set operator on X, let $\mathfrak{F}$ be the family of all subsets A of X for which $A^\alpha \subseteq A$, and let $\mathfrak{I}$ be the family of all complements of members of $\mathfrak{F}$. Then $\mathfrak{I}$ is a topology for X, and A^α is the set of all $\mathfrak{I}$-accumulation points of A for each subset A of X.*

Proof. Define the operator c by $A^c = A \cup A^\alpha$ for each subset A of X; c is a Kuratowski closure operator. (a) $\varnothing^c = \varnothing$ because $\varnothing^\alpha = \varnothing$. (b) $(A \cup B)^c = A^c \cup B^c$ because $(A \cup B)^\alpha = A^\alpha \cup B^\alpha$. (c) $A \subset A^c$ because $A^c = A \cup A^\alpha$. (d) $A^{cc} = A^c$. $(A \cup A^\alpha)^\alpha \subset A \cup A^\alpha$ implies $A^{cc} = (A \cup A^\alpha) \cup (A \cup A^\alpha)^\alpha = A \cup A^\alpha = A^c$. Therefore c characterizes a topology $\mathfrak{I}'$ for X. Let $\overline{A}$ denote the $\mathfrak{I}'$-closure of A. Then it is well known that $\overline{A} = A^c$. It is also true that $\overline{A} = A \cup A'$, where A' is the set of all $\mathfrak{I}'$-accumulation points of A. Therefore $A \cup A' = A \cup A^\alpha$ for each subset A of X. In particular $(A \sim \{x\}) \cup (A \sim \{x\})' = (A \sim \{x\}) \cup (A \sim \{x\})^\alpha$ for $x \in X$. It follows that $x \in (A \sim \{x\})'$ if and only if $x \in (A \sim \{x\})^\alpha$. Axiom III says $x \in (A \sim \{x\})^\alpha$ if and only if $x \in A^\alpha$. Since $\mathfrak{I}'$ is a topology for X we know $x \in A'$ if and only if $x \in (A \sim \{x\})'$. Therefore $x \in A'$ if and only if $x \in A^\alpha$, or $A' = A^\alpha$ for each subset A of X. Let $\mathfrak{F}'$ be the family of all subsets B of X such that $B' \subset B$. Then $\mathfrak{I}'$ is the family of all complements of members of $\mathfrak{F}'$. Since $A^\alpha = A'$ for each subset A of X, $\mathfrak{I}' = \mathfrak{I}$.

This problem was suggested in a topology course taught by Dr. Victor J. Mizel.

Reference

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ON SECOND ORDER DIFFERENTIAL EQUATIONS

HARRY HOCHSTADT, Polytechnic Institute of Brooklyn

A standard method of solving second order differential equations of the type

$$\frac{d^2x}{dt^2} + f(x) = 0, \qquad x(0) = c_1, \ \frac{dx(0)}{dt} = c_2 \tag{1}$$

is to multiply the equation by dx/dt. Then it can be rewritten in the form

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 + f(x)\frac{dx}{dt} = 0.$$

This last equation can be integrated to obtain

$$\left(\frac{dx}{dt}\right)^2 + F(x) = c_2^2 + F(c_1), \qquad x(0) = c_1, \tag{2}$$

where $dF(x)/dx = 2f(x)$. In this fashion the second order equation (1) has been reduced to a first order equation (2). Presumably once (2) has been solved (1) has also been solved.

But it can happen that (1) is of a type which has unique solutions, whereas (2) does not. For example, the problem

$$\frac{d^2x}{dt^2} + x = 0, \qquad x(0) = 0, \ \frac{dx}{dt}(0) = 1 \tag{3}$$

can be reduced to

$$\left(\frac{dx}{dt}\right)^2 + x^2 = 1, \qquad x(0) = 0. \tag{4}$$

One can easily check that each of the following is an acceptable solution of (4).

$$\begin{aligned}
x_1 &= \sin t & 0 &\leqq t < \infty \\
x_2 &= \sin t & 0 &\leqq t < \pi/2 \\
&= 1 & \pi/2 &\leqq t < \infty \\
x_3 &= \sin t & 0 &\leqq t < \pi/2 \\
&= 1 & \pi/2 &\leqq t < T \\
&= \cos(t - T) & T &\leqq t < \infty.
\end{aligned}$$

Only x_1, however, also satisfies (3). The reason for the multiplicity of solutions is related to the fact that whenever $x^2 = 1$ neither value of dx/dt from (4) satisfies a Lipschitz condition. The question therefore arises, as to what additional conditions must be imposed on (2) in order to be sure that the selected solution also satisfies (1). We must select a solution of (2) which is at least doubly differentiable everywhere. For (3) and (4) x_1 evidently is distinguished from the others by this requirement. Actually all solutions of (3) must be analytic.

In general, it may not even be enough to impose on the solutions of (2) the requirement of double differentiability or even, when applicable, analyticity. For example, the problem

$$\frac{d^2x}{dt^2} + x = 0, \qquad x(0) = 1, \ \frac{dx}{dt}(0) = 0 \tag{5}$$

can be reduced to

$$\left(\frac{dx}{dt}\right)^2 + x^2 = 1, \qquad x(0) = 1. \tag{6}$$

Aside from an infinity of nonanalytic solutions, (6) has the following analytic solutions:

$$x_1 = \cos t, \qquad x_2 = 1.$$

But only x_1 satisfies (5).

These problems have much physical interest since in (2), $(dx/dt)^2$ can be considered as a kinetic energy term and $F(x)$ a potential energy term so that (2) is a mathematical statement of the law of conservation of energy. (1) is the corresponding force equation.

ON CONDITIONS FOR DIFFERENTIABILITY

P. V. KRISHNAIAH AND K. V. RAJESWARA RAO, Andhra University, Waltair, India

A function $f(x, y)$ is said to be differentiable at (a, b) if there exist numbers p, q and functions $g(x, y)$ and $h(x, y)$ such that (i) $g(x, y)$ and $h(x, y)$ tend to zero as $(x, y) \to (a, b)$ and (ii) $f(x, y) \to f(a, b) = (x - a)\{p + g(x, y)\} + (y-b)\{q+h(x, y)\}$.

Under the above conditions it is easy to see that $f_x(a, b)$ and $f_y(a, b)$ exist and equal p and q respectively. Thus, in order to test a function f for differentiability at (a, b), one has to search for functions g and h which tend to zero as $(x, y)\to(a, b)$ and which satisfy (ii) above with $p=f_x(a, b)$ and $q=f_y(a, b)$.

One of the purposes of this note is to construct such functions in general (see Theorem 1). Differentiability at (a, b) of a function $f\ (x,y)$ is clearly equivalent to requiring that the function $\phi(x, y)$, defined by $\phi(x, y)=f(x+a, y+b) -f(a, b)-xf_x(a, b)-yf_y(a, b)$, which, together with its first partial derivatives, vanishes at $(0, 0)$, is differentiable at $(0, 0)$.

THEOREM 1. *Suppose* $\phi(0, 0)=\phi_x(0, 0)=\phi_y(0, 0)=0$. *A necessary and sufficient condition for* ϕ *to be differentiable at* $(0, 0)$ *is that* $x\phi(x, y)/(x^2+y^2)$ *and* $y\phi(x, y)/(x^2+y^2)$ *tend to* 0 *as* $(x, y)\to(0, 0)$.

Proof. Necessity. By definition, there exist functions $g(x, y)$ and $h(x, y)$ which tend to 0 as $(x, y)\to(0, 0)$, such that $\phi(x, y)=xg(x, y)+yh(x, y)$ for $(x, y)\neq(0, 0)$. Hence,

$$\begin{aligned}\left|\frac{x\phi(x, y)}{x^2+y^2}\right| &= \left|\frac{x^2g(x, y)}{x^2+y^2}+\frac{xyh(x, y)}{x^2+y^2}\right| \\ &\leqq \frac{x^2}{x^2+y^2}\,|g(x,y)| + \frac{|xy|}{x^2+y^2}\,|h(x, y)| \\ &\leqq |g(x,y)| + |h(x,y)| \qquad \text{for all } (x, y) \neq (0, 0).\end{aligned}$$

Thus $x\phi(x, y)/(x^2+y^2)\to0$ as $(x, y)\to(0, 0)$. Similarly $y\phi(x, y)/(x^2+y^2)\to0$ as $(x, y)\to(0, 0)$.

Sufficiency. Write $g(x, y)=x\phi(x, y)/(x^2+y^2)$ and $h(x,y)=y\phi(x, y)/(x^2+y^2)$.

Our next purpose is to discuss the converse of the following well-known theorem: *If* $f(x, y)$ *is differentiable at* $(0, 0)$, $g(t)$ *and* $h(t)$ *are derivable at* 0 *and* $g(0) =h(0)=0$, *then* $f(g(t), h(t))$ *is derivable at* 0 *and its derivative equals* $g'(0)f_x(0, 0) +h'(0)f_y(0, 0)$.

The following question arises naturally: If, for every pair $(g(t), h(t))$ of functions, derivable at 0, such that $g(0)=h(0)=0, f(g(t), h(t)$ is derivable at 0, then does it follow that f is differentiable at $(0, 0)$?

Consideration of the function $x^{1/3}y^{2/3}$ shows that the answer is in the negative.

Now the following question may be asked: If, for every pair $(g(t), h(t))$ of functions, derivable at 0, such that $g(0)=h(0)=0, f(g(t), h(t))$ is derivable at 0

(this implies the existence of $f_x(0, 0)$ and $f_y(0, 0)$) and its derivative equals $g'(0)f_x(0, 0)+h'(0)f_y(0, 0)$, then does it follow that f is differentiable at $(0, 0)$?

This can be reduced to the case where $f_x(0, 0)=f_y(0, 0)=0$ and we prove the following

THEOREM 2. *Suppose* $f_x(0, 0)=f_y(0, 0)=0$. *A necessary and sufficient condition for* f *to be differentiable at* $(0, 0)$ *is that, for every pair* $(g(t), h(t))$ *of functions, derivable at* 0, *such that* $g(0)=h(0)=0$, *the derivative of* $f(g(t), h(t))$ *exists and vanishes at* 0.

Proof. Necessity. A special case of the well-known theorem above.

Sufficiency. We shall suppose, as we may without loss of generality, that $f(0, 0)=0$. In virtue of Theorem 1, it is enough to prove that $xf(x, y)/(x^2+y^2)$ and $yf(x, y)/(x^2+y^2)$ tend to 0 as $(x, y)\to(0, 0)$. If $g(t)/t$ and $h(t)/t$ tend to limits as $t\to 0$ then, by hypothesis,

$$\frac{f(g(t), h(t))}{t} \to 0$$

as $t\to 0$. If, further, $\lim_{t\to 0} g(t)/t\neq 0$, then each of

$$\left|\frac{g(t)f(g(t), h(t))}{(g(t))^2+(h(t))^2}\right| \quad \text{and} \quad \left|\frac{h(t)f(g(t), h(t))}{(g(t))^2+(h(t))^2}\right|$$

is less than or equal to $|f(g(t), h(t))/g(t)|$ in a deleted neighbourhood of 0 and hence tends to 0 as $t\to 0$. Similarly the same conclusion holds when $\lim_{t\to 0} h(t)/t \neq 0$.

Thus we have proved that $xf(x, y)/(x^2+y^2)$ and $yf(x, y)/(x^2+y^2)$ tend to 0 as (x, y) approaches $(0, 0)$ along any curve that has a tangent at $(0, 0)$. The required result now follows from this in virtue of the following

LEMMA. *If the single-valued function* ϕ *is continuous at* P *along every convex curve through* P *which is (at least once) differentiable at* P, *then* ϕ *is continuous at* P.

This has been proved by A. Rosenthal in *Mathematische Zeitschrift*, 63 (1955) 31–38.

The two questions leading to Theorem 2 were put to us by our teacher, the late Dr. V. Ramaswami.

The generalization of Theorems 1 and 2 above to n variables is obvious.

ON IRREDUCIBLE POLYNOMIALS IN GALOIS FIELDS

DENNIS TRAVIS, Columbia College

Dickson, in his classic work on Linear Groups, gives the following expression for $B(n, m)$, the number of monic irreducible polynomials of degree n in $GF(p^m)$.

Let $n=q_1^{s_1}q_2^{s_2}\cdots q_r^{s_r}$, where the q_i are distinct primes and let $Q(i_1, \cdots, i_k)=mn/q_{i_1}\cdot q_{i_2}\cdots q_{i_k}$. Then

$$(1) \quad n \cdot B(n, m) = p^{mn} - \sum_{i_1} p^{Q(i_1)} + \sum_{i_1 i_2} p^{Q(i_1, i_2)} - \cdots + (-1)^r p^Q(i_1, \cdots, i_r).$$

Both this statement and its proof, however, are cumbersome. In this paper I present a proposition of wider generality, from which Dickson's expression can readily be derived. In addition, this proposition easily yields results for various special cases, and also gives rise to some interesting number theoretical properties.

THEOREM. $\sum_{h|n} h \cdot B(h, m) = p^{mn}$.

Proof. Consider $GF(p^{mn})$. It contains a subfield A with p^m elements. Every algebraic extension of a Galois field is normal and separable, and a Galois field is completely determined up to isomorphism by the number of its elements. It follows from the above remarks that if $f(x)$ is an irreducible polynomial of degree k with coefficients in A, then a root of $f(x) = 0$, adjoined to A, generates a field of p^{mk} elements in which $f(x)$ splits completely into distinct linear factors; and this field is isomorphic to a subfield of $GF(p^{mn})$ if and only if k divides n. From this we can conclude that an irreducible polynomial with coefficients in A splits completely into distinct linear factors in $GF(p^{mn})$ if and only if k divides n.

Now consider the following equivalence relation defined on $GF(p^{mn})$. We say $a \equiv b$ if and only if a and b are conjugates over $A[x]$. Then $GF(p^{mn})$ is partitioned into a finite number of disjoint equivalence classes by this relation. From the preceding paragraph, it follows that the equivalence classes correspond to monic irreducible polynomials that have degrees which are divisors of n, and each equivalence class contains all the zeros of its respective polynomial. Since $B(h, m)$ is the number of monic irreducible polynomials of degree h in $A[x]$, it is clear that there are $B(h, m)$ equivalence classes in $GF(p^{mn})$, each having exactly h elements. Since we need only consider divisors of n, then by summing up the number of terms in the equivalence classes we arrive at $\sum_{h|n} h \cdot B(h, m) = p^{mn}$. Q.E.D.

COROLLARY. *If $n = r^q$ and r is prime, then $B(n, m) = (p^{mn} - p^{m(n/r)})/n$.*

Proof. The proof is clear since the only divisors of $n = r^q$ are n itself and the divisors of $r^{q-1} = n/r$.

As a final remark I should like to point out that Dickson's original expression can be derived from the statement just proven by application of the Möbius inversion formula.

Acknowledgement. I should like to thank the referee for some helpful suggestions concerning the presentation of this theorem. The fact that Dickson's expression could be derived from the theorem proven in this paper was first pointed out by Professor Leonard Carlitz, Duke University. I would also like to thank Professor Norman Oler, Columbia University, for some useful pointers in the preparation of the manuscript.

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ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

COLLABORATING EDITOR: C. W. DODGE, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1641. *Proposed by W. C. Waterhouse, Harvard University*

Prove that for any integer $k>1$ and any positive integer n, n^k is the sum of n consecutive odd integers.

E 1642. *Proposed by Ralph Greenberg, University of Pennsylvania*

Prove that a polynomial $P(x)$ with rational coefficients which assumes rational values for rational x and irrational values for irrational x must be linear.

E 1643. *Proposed by J. E. MacDonald, Jr., International Business Machines Corporation, Poughkeepsie, New York*

Given that $\sum_{i=1}^{n} b_i=b$ with each b_i a nonnegative integer, prove that $\sum_{j=1}^{n-1} b_jb_{j+1}\leqq b^2/4$.

E 1644. *Proposed by T. R. Curry, College on Long Island, State University of New York*

Prove that the sum of the sines of the angles of a triangle never exceeds $3\sqrt{3}/2$, with equality when and only when the triangle is equilateral.

E 1645. *Proposed by E. R. Barnes, Morgan State College*

Let r be a nonnegative integer and let $A=[a_{ij}]$ be an $n\times n$ matrix where $a_{ij}=(i+j+r-2)!/(i-1)!(j+r-1)!$. Show that $|A|=1$.

E 1646. *Proposed by L. J. Green, Case Institute of Technology*

Exhibit a finite topological space and a subset A such that A, A' (the set of all limit points of A), A'', $\cdots$, $A^{(k)}$ are all different, none of them closed.

E 1647. *Proposed by P. R. Chernoff, Harvard University*

Sum the infinite series $\sum_{n=0}^{\infty}1/(kn)!$, where k is any positive integer.

E 1648. *Proposed by David Singmaster, University of California at Berkeley*

(1) What is the order of the smallest nontrivial commutative ring with identity which is not a field? Find two such rings with this minimal order. Are there more?

(2) How many rings of order four are there?

E 1649. *Proposed by R. A. Jacobson, South Dakota State College*

Let "$*$" be a binary operation on a set S. Do the following axioms define a group?

A1. There exists an element $e \in S$ such that $x * e = x$ for all $x \in S$, $x \neq e$.

A2. For all $x, y, z \in S$, $z \neq e$, $x * (y * z) = (x * y) * z$.

A3. $x * w = y$ has a unique solution $w \in S$, for each ordered pair $x, y \in S$, $x \neq y$.

E 1650. *Proposed by J. L. Brenner, Stanford Research Institute*

Let α, β be any two real numbers and ϵ any positive quantity. Prove that there are integers a, b, m, q such that the inequalities

$$|q\alpha - a - b\sqrt{m}| < \epsilon q, \qquad |q\beta - a + b\sqrt{m}| < \epsilon q$$

hold simultaneously, and m is not a perfect square.

SOLUTIONS

Product of the Edges of a Trirectangular Tetrahedron

E 1563 [1963, 210]. *Proposed by D. L. Silverman, National Security Agency, Fort Meade, Maryland*

If a tetrahedron with a solid right angle has integral edges, prove that the product of the edges is a multiple of 5,702,400.

Solution by Leonard Carlitz, Duke University. Consider the system

$$b^2 + c^2 = r^2 \qquad a^2 + c^2 = s^2, \qquad a^2 + b^2 = t^2.$$

If we assume b even and c odd, then the second equation requires a even, so that t is also even. Indeed it follows from the first and second equations that a and b are divisible by 4, so that t is also. Then the third equation becomes

$$(a/4)^2 + (b/4)^2 = (t/4)^2,$$

so that either $a/4$ or $b/4$ is divisible by 4. Thus, in all, $abcrst$ is divisible by at least 4^4.

Next, if b is divisible by 3 but c is not, it follows, from the second equation, that a is divisible by 3, and hence, by the third equation, t is also divisible by 3. Thus we get

$$(a/3)^2 + (b/3)^2 = (t/3)^2,$$

so that either $a/3$ or $b/3$ is divisible by 3. Hence $abcrst$ is divisible by at least 3^4.

If $5 \mid b$ but $5 \nmid c$, the second equation implies $5 \mid as$. Similarly if $5 \mid c$ but $5 \nmid bc$. Since $5 \mid bcr$, it follows that $abcrst$ is divisible by 5^2.

Finally we show divisibility by 11. Assume $11 \nmid abcrst$, and treat the given system as a congruence (mod 11). The quadratic residues (mod 11) are 1, 3, 4, 5, 9. There is no loss in generality in assuming that $b^2 \equiv 1$ (mod 11). Hence $c^2 \equiv 3$ or 4 and also $b^2 \equiv 3$ or 4. But then the first congruence, $b^2 + c^2 \equiv r^2$, is impossible.

Thus *abcrst* is divisible by $4^4 \cdot 3^4 \cdot 5^2 \cdot 11 = 5{,}702{,}400$.

Also solved by D. I. A. Cohen, Philip Franklin, D. C. B. Marsh, and the proposer.

Cohen and Marsh called attention to Problem 271 [1918, 304], which already shows that the product of the edges is divisible by $4^4 \cdot 3^4 \cdot 5^2$. The proposer remarked that the only known triples (a, b, c) each two members of which are the legs of a Pythagorean triple are (85, 132, 720) and (44, 117, 240). The products obtained by multiplying the three members of each triple by the the three hypotenuses involved are both divisible by $4^4 \cdot 3^4 \cdot 5^4 \cdot 11 \cdot 61$.

Another Triangle Inequality

E 1564 [1963, 210]. *Proposed by Leon Bankoff, Los Angeles, California*

Show that the sum of the distances from the incenter to the vertices of an acute triangle is not less than twice the sum of the distances from the orthocenter to the sides.

Solution by the proposer. Let I, H, r denote the incenter, the orthocenter, and the inradius of triangle ABC, and let D, E, F denote the feet of the altitudes on sides BC, CA, AB. Then

$$\sum AI + 3r \geqq \sum AD = \sum AH + \sum HD \geqq \sum AI + \sum HD$$

(see the solution to E 1397 [1960, 695]). Hence $3r \geqq \sum HD$. But, by the Erdös-Mordell Theorem, $\sum AI \geqq 6r$. It follows that $\sum AI \geqq 2\sum HD$.

Also solved by A. N. Aheart, Leonard Carlitz, José Gallego-Diaz, Ned Harrell, and F. Leuenberger.

A Power of 2 with a String of n Zeros

E 1565 [1963, 210]. *Proposed by R. L. Graham, University of California, and F. D. Parker, University of Alaska*

Show that for any positive integer n there exists a power of 2 with a string of more than n successive zeros.

I. *Solution by E. J. Burr, University of New England, Armidale, N.S.W.' Australia.* Let $\epsilon = \log_{10}(1+10^{-n-1})$. Since $\log_{10} 2$ is irrational, there exist integers $a > 0$ and $b > n$ for which

$$0 < s \log_{10} 2 - b < \epsilon,$$

whence

$$10^b < 2^a < 10^b(1 + 10^{-n-1}).$$

Therefore 2^a, in the scale of ten, begins with a one and $n+1$ or more successive zeros.

II. *Solution by J. A. Schatz, Sandia Corporation, Albuquerque, N. M.* For any positive integer t, there are positive integers k and r such that $2^k = 1 + r \cdot 5^t$, since 2 is a nondivisor of zero in the ring of integers modulo 5^t. Taking $t = 2n$ and multiplying by 2^{2n}, we obtain $2^{k+2n} = r \cdot 10^{2n} + 2^{2n}$, which has a string of at least n consecutive zeros since $2^{2n} < 10^n$.

Also solved by Jack Abad, Leonard Carlitz, D. I. A. Cohen, W. A. Edelstein, Stephen Fisk, James Foster, Ralph Greenberg, G. A. Heuer, Erwin Just and Norman Schaumberger (jointly), Harry Lass and Jack Lorell (jointly), D. C. B. Marsh, Leo Moser, F. D. Parker, Guy Torchinelli, R. A. Wonderly, and Leonard Zacks.

Leo Moser and Nathaniel Macon have shown, more generally, that every finite sequence of digits appears as the first digits of some power of 2; see their paper, "On the distribution of first digits of powers," *Scripta Mathematica*, XVI (1950) 290–2.

A Sheep in Wolf's Clothing

E 1566 [1963, 210]. *Proposed by Michael Skalsky, Southern Illinois University*

Show that

$$\sum_{k=0}^{n} (m+k)_m = (m+n+1)_{m+1}/(m+1),$$

where $(x)_m = x(x-1) \cdots (x-m+1)$.

I. *Solution by D. C. Shreve, North Carolina State College.* We have

$$\begin{aligned}\sum_{k=0}^{n} (m+k)_m &= \sum_{k=0}^{n} [(m+k+1)_{m+1} - (m+k)_{m+1}]/(m+1) \\ &= (m+n+1)_{m+1}/(m+1).\end{aligned}$$

II. *Solution by Jiang Luh, University of Michigan.* We will prove the desired result by induction on n. Clearly the identity holds for $n=0$. Now assume that

$$\sum_{k=0}^{n-1} (m+k)_m = (m+n)_{m+1}/(m+1).$$

Then

$$\begin{aligned}\sum_{k=0}^{n} (m+k)_m &= \sum_{k=0}^{n-1} (m+k)_m + (m+n)_m \\ &= (m+n)_{m+1}/(m+1) + (m+n)_m \\ &= (m+n)_m(n+m+1)/(m+1) \\ &= (n+m+1)_m/(m+1).\end{aligned}$$

III. *Solution by E. R. Barnes, Morgan State College.* The desired relation is nothing but the well-known identity

$$\sum_{k=0}^{n} \binom{m+k}{m} = \binom{m+n+1}{m+1}$$

with each side multiplied by $m!$.

Also solved by Nadhla Abdul-Halim, A. N. Aheart, P. G. Albert, K. F. Bailie, B. W. Banks, T. L. Beatty, M. T. L. Bizley, W. H. Bonney, G. T. Boswell, Maurice Brisebois, R. E. Brown, Stephen Carley, F. V. Cavoto, D. I. A. Cohen, M. J. Cohen, Robert Cohen, J. F. Dillon, P. F. Duvall, Jr., Ragnar Dybvik, L. M. Elewitz, C. G. Fain, J. A. Faucher, Stephen Fisk, Michael Fried, Michael Friedman, Joseph Gayda, Maurice Glaymann, Michael Goldberg, Richard Goodman, Ralph Greenberg, Arthur Greenspan, Cornelius Groenewoud, J. R. Hanna, E. R. Hansen, J. C. Hickman, C. R. Hutchinson, R. A. Jacobson, K. R. Jones, Erwin Just and Norman Schaumberger (jointly), M. S. Klamkin, David Klappholz, Lawrence Kratz, Joel Kugelmass, A. E. Livingston, W. R. McEwen, Robert Maas, Thomas Maddock, Andrzej Makowski, C. F. Marion, D. C. B. Marsh, Helen M. Marston, M. J. Maybury, S. G. Mohanty, Stephen Montague, J. W. Moon, P. N. Muller, M. G. Murdeshwar, M. J. Pascual, Stanton Philipp, H. J. Ricardo, T. K. Roney, Perry Scheinok, E. M. Scheuer, G. R. Schubert, Donna J. Seaman, R. R. Seeber, D. L. Silverman, Arnold Singer, F. C. Smith, O. E. Stanaitis, L. A. Steen, P. D. Thomas, Dmitri Thoro, Guy Torchinelli, Dennis Travis, Simon Vatriquant, J. E. Vinson, Julius Vogel, W. C. Waterhouse, R. E. Wilder, Clement Winston, Leonard Yap, Leonard Zacks, L. A. Zalcman, Walter Zayachkowski, A. R. Zingher, and the proposer.

Bizley pointed out that the result is the special case of the Jeffrey identity

$$\sum_{j=0}^{N}\binom{j}{t}\binom{N-j}{M-t}=\binom{N+1}{M+1},\ \text{(regardless of the value of } t\text{)},$$

obtained by setting $N=m+n$, $M=m=t$, and then multiplying through by $m!$.

Re Integral Solutions of $x+y+z=n$

E 1567 [1963, 210]. *Proposed by Hugh Noland, Chico State College*

If n is a positive integer, determine the number of integral solutions of the equation $x+y+z=n$ satisfying the conditions $x\geqq y\geqq z\geqq 0$.

I. *Solution by Guy Torchinelli, State University of New York, Buffalo.* Let w be the number of integral solutions of the equation $x+y+z=n$ satisfying $x\geqq y\geqq z\geqq 0$ and let w' be the number of integral solutions satisfying $x\geqq 0$, $y\geqq 0$, $z\geqq 0$. Now w' is equal to the number of terms of the multinomial expansion of $(a+b+c)^n$, because of the one-to-one correspondence $(x, y, z)\leftrightarrow a^xb^yc^z$. Thus

$$w' = (n+1)(n+2)/2.$$

Let p' be the number of these w' solutions for which $x=y=z$; let q' be the number of these solutions for which exactly two of x, y, z are equal; let r' be the number of these solutions for which no two of x, y, z are equal. Then

$$w = p' + q'/3 + r'/6.$$

If $n\equiv 0 \pmod 6$, $w=1+n/2+n^2/12$;
if $n\equiv 2$ or $4 \pmod 6$, $w=0+(n+2)/2+(n^2-4)/12$;
if $n\equiv 1$ or $5 \pmod 6$, $w=0+(n+1)/2+(n^2-1)/12$;
if $n\equiv 3 \pmod 6$, $w=1+(n-1)/2+(n^2+3)/12$.
In each case we have $w=[(n^2+6n+12)/12]$.

II. *Solution by P. T. Bateman, University of Illinois.* Putting

$$u = x-y, \qquad v = y-z, \qquad w = z,$$

or

$$x = u + v + w, \qquad y = v + w, \qquad z = w,$$

we see that the number of integral solutions of

$$x + y + z = n, \qquad x \geqq y \geqq z \geqq 0$$

is equal to the number of integral solutions of

$$u + 2v + 3w = n, \qquad u \geqq 0, \qquad v \geqq 0, \qquad w \geqq 0.$$

But, by Problem 25 of Part I of G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, the latter number is equal to the integer nearest to $(n+3)^2/12$.

Also solved by H. D. Abramson, Merrill Barnebey, E. R. Barnes, W. H. Bonney, Robert Brooks, R. E. Brown, D. I. A. Cohen, Frank Dapkus, P. F. Duvall, Jr., C. G. Fain, Stephen Fisk, Michael Fried, C. M. Frye, Ralph Greenberg, W. J. Hansen, R. A. Jacobson, Robert Maas, D. C. B. Marsh, S. G. Mohanty and P. J. Schillo (jointly), M. G. Murdeshwar, J. B. Muskat, Stanton Philipp, H. J. Ricardo, S. J. Ryan, Perry Scheinok, L. A. Steen, D. J. Vitek, J. D. Watson, and the proposer.

A Consequence of the Prime Number Theorem

E 1568 [1963, 210]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Evaluate $T = \sum_{n=1}^{\infty} (-1)^s(1/n)$, where s is the number of prime factors of n.

I. *Solution by S. Chowla and S. L. Segal, University of Colorado.* If s denotes the *total* number of prime factors of n, then $(-1)^s = \lambda(n)$ (Liouville's function) and the fact that $T=0$ is well known to be equivalent to the prime number theorem (without error term).

If s denotes the number of *distinct* prime factors of n, then for $\mathrm{Re}(w) > 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^s n^{-w} &= \prod_p (1 - p^{-w} - p^{-2w} - \cdots) = \prod_p \left(1 - \frac{1}{p^w - 1}\right) \\ &= \frac{1}{\zeta(w)} \prod_p \left(1 - \frac{1}{(p^w - 1)^2}\right).\end{aligned} \tag{1}$$

The last product converges and represents an analytic function with an absolutely convergent Dirichlet series representation for $\mathrm{Re}(w) > 1/2$, while the Dirichlet series for $1/\zeta(w)$ converges for $\mathrm{Re}(w) \geqq 1$ and absolutely for $\mathrm{Re}(w) > 1$ (this is again equivalent to the prime number theorem). Hence by a well-known theorem of Landau (*Handbuch*, p. 755) (1) also holds on the line $\mathrm{Re}(w) = 1$ and hence again $T = 0$.

II. *Solution by D. I. A. Cohen, Brooklyn, New York.* In Hardy and Wright, *Theory of Numbers*, 4th ed., p. 255, we find the following more general result:

$$\zeta(2s)/\zeta(s) = \sum_{n=1}^{\infty} \lambda(n)/n^s,$$

where $\lambda(n)=(-1)^{s'}$, s' being the total number of prime factors of n. When $s=1$ this sum is T, and since $\zeta(2)$ is bounded and $\zeta(1)$ diverges, $T=0$.

III. *Remarks by P. T. Bateman, University of Illinois.* If multiple prime factors are counted according to their multiplicities, Euler gave an heuristic argument that $T=0$ ("Variae observationes circa series infinitas," *Commentationes Academiae Scientiarum Imperialis Petropolitanae*, 9 (1737) 160–188). It is easy to show rigorously that if we assume the convergence of the series, then its sum must be zero (cf. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Section 166). The fact that the series converges is known (op. cit., Section 167), but is equivalent to the prime number theorem.

If s denotes the number of *distinct* prime factors of n, the same assertions are true and can be easily derived from those for the previous case.

Also solved by W. E. Briggs, Michael Fried, Joel Kugelmass, Helen M. Marston, J. W. Moon, Stanton Philipp, T. K. Roney, D. Suryanarayana, and the proposer.

A Hyperbolic Paraboloid

E 1569 [1963, 210]. *Proposed by R. H. Moorman, Tennessee Polytechnic Institute*

Find the locus of points equidistant from two skew lines.

Solution by W. R. McEwen, University of Minnesota. Let $2a$ be the length of the common perpendicular to the two skew lines. Choose a set of axes with the origin at the midpoint of the common perpendicular, the x-axis coinciding with it, and the z-axis oriented so that the equations of the lines are

$$\frac{x-a}{0}=\frac{y}{\cos B}=\frac{z}{\sin B}, \qquad \frac{x+a}{0}=\frac{y}{-\cos B}=\frac{z}{\sin B}.$$

The equation of the required locus is then

$$\begin{vmatrix} y & z \\ \cos B & \sin B \end{vmatrix}^2+\begin{vmatrix} z & x-a \\ \sin B & 0 \end{vmatrix}^2+\begin{vmatrix} x-a & y \\ 0 & \cos B \end{vmatrix}^2=$$
$$\begin{vmatrix} y & z \\ -\cos B & \sin B \end{vmatrix}^2+\begin{vmatrix} z & x+a \\ \sin B & 0 \end{vmatrix}^2+\begin{vmatrix} x+a & y \\ 0 & -\cos B \end{vmatrix}^2,$$

which reduces to $yz \sin B \cos B=ax$, a hyperbolic paraboloid.

Also solved by W. H. Bonney, J. M. Cardosa, D. I. A. Cohen, Frank Dapkus, José Gallego-Diaz, Michael Goldberg, Kit Hanes, R. T. Hood, D. C. B. Marsh, Amos Nannini, P. D. Thomas, Simon Vatriquant, and Roscoe Woods.

Cardosa pointed out that this problem is solved in Cherubino and Checcucci, *Esercizi di geometria analitica* (Pisa, Vallerini Ed.), 1953, p. 285. The more general problem of finding the locus of points whose distances from two skew lines are in a constant ratio is proposed as Exercise 7 in Castelnuova, *Lecciones de geometria analitica*, 1943, p. 175. Dapkus found this more general problem (without solution) in W. F. Osgood and W. C. Graustein, *Plane and Solid Analytic Geometry*, 1930, p. 582. Gallego-Diaz observed that the given problem is equivalent to each of the following two problems:

1. Find the locus of the centers of all spheres tangent to two given skew lines.
2. Find the locus of the axes of revolution of all hyperboloids of revolution containing two given skew lines.

A Cubic with Three Real Roots

E 1570 [1963, 211]. *Proposed by J. S. Brock, David Taylor Model Basin, Washington, D. C.*

Prove that the roots of the cubic equation

$$x^3 - (a + b + c)x^2 + (ab + bc + ca - d^2 - e^2 - f^2)x + (ae^2 + bf^2 + cd^2 - abc - 2def) = 0$$

are all real when a, b, c, d, e, f are real numbers.

Solution by R. G. Albert, Smithsonian Institute, Cambridge, Mass. The given equation E is the characteristic equation of the matrix

$$\begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}.$$

Hence the roots of E, being the eigenvalues of a real symmetric matrix, are real.

Also solved by A. N. Aheart, Marjorie R. Bicknell, R. T. Borochoff and Thomas Maddock (jointly), H. E. Bray, Leonard Carlitz, D. I. A. Cohen, José Gallego-Diaz, W. R. McEwen, S. P. H. Mandel, D. C. B. Marsh, Perry Scheinok, Arnold Singer, O. E. Stanaitis, H. W. Vayo, Roscoe Woods, and the proposer.

For an elementary direct proof see Snyder and Sisam, *Analytic Geometry of Space*, 1914, Sections 68–69, or Hall and Knight, *Higher Algebra*, 1955 edition, pp. 486–7.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College
COLLABORATING EDITORS: L. CARLITZ, Duke University, H. S. M. COXETER, University of Toronto; and A. WILANSKY, Lehigh University

PROBLEMS FOR SOLUTION

5151. *Proposed by L. Ehrenpreis, New York University, and D. J. Newman, Yeshiva University*

If ξ is an irrational number and x any number, it is well known that integers m and n can be found such that $m+n\xi$ is arbitrarily close to x. Show, further, that the m, n can be chosen to be relatively prime.

5152. *Proposed by J. D. Sondow, Princeton University*

Prove that the decomposition space whose points are the interior points of the solid torus in E_3 and the longitudinal circles of the boundary is topologically

the 3-sphere. Intuitively this amounts to identifying all meridians with one of them.

5153. *Proposed by Seymour Kass, Illinois Institute of Technology*

Let G be a group of order p^2q, where p and q are distinct primes. Prove that if G has a composition series with sequence of indices (p, p, q), then G is abelian.

5154. *Proposed by Allen S. Davis, University of Oklahoma*

Show that, with respect to a suitable metric, a square stochastic matrix M with positive entries defines a contraction map of the space X of probability vectors. (Hence, as is well known, $xM=x$ has a unique solution in X.)

5155. *Proposed by Joseph Hammer, University of Sydney, Australia*

Given a convex domain D, its area $A(D)$ and its half perimeter, $S(D)$. Prove that if $A(D)>rS(D)$, where r is any positive integer, then D contains r lattice points. (See Bender, *Area-perimeter relations for two-dimensional lattices*, this MONTHLY, 69 (1962) 742–744, where the case $r=1$ is proved.)

5156. *Proposed by Don Kirkham, Iowa State University*

For $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$, prove that

$$\theta \cot\theta + \tfrac{1}{3}\cot^3\theta(\tan\theta - \theta) - \tfrac{1}{5}\cot^5\theta(-\tfrac{1}{3}\tan^3\theta + \tan\theta - \theta) + \tfrac{1}{7}\cot^7\theta(\tfrac{1}{5}\tan^5\theta - \tfrac{1}{3}\tan^3\theta + \tan\theta - \theta) - \cdots = \tfrac{1}{8}(\pi^2 - 4\theta^2).$$

5157. *Proposed by R. L. Graham, Bell Telephone Laboratories*

Suppose that $S=(s_1, s_2, \cdots)$ is a monotone sequence of positive real numbers which has the property that every sufficiently large rational number is of the form $\sum_{k=1}^{\infty}\epsilon_k s_k$, where ϵ_k is 0 or 1 and all but a finite number of the ϵ_k are 0. Prove (or disprove) that all positive rationals are of the form $\sum_{k=1}^{\infty}\epsilon_k s_k$.

5158. *Proposed by S. D. Chatterji, University of New South Wales, Australia*

Given a finite commutative group $A=\{a_1, \cdots, a_n\}$. What is the product $a_1a_2\cdots a_n$ equal to?

5159. *Proposed by S. D. Chatterji, University of New South Wales, Australia*

Show that the monoid M (semigroup with identity) of objects (k, l), k and l nonnegative integers, with the composition law

$$(k, l)\circ(k', l') = (k+k', l+l'+2ll')$$

is isomorphic to the positive integers considered as a monoid under multiplication. If one orders M by the rules $(k, l)<(k', l')$ if $k<k'$, or if $k=k'$ and $l<l'$, then M is also order isomorphic to the positive integers (with natural ordering).

5160. *Proposed by Alvin Hausner, The City College of New York*

Let A and B be $n\times n$ (real) orthogonal matrices. Prove that the rank of $A+B$ is $n-2k$, $0\leqq k\leqq \frac{1}{2}n$, if A, B are both proper-orthogonal or both improper-

orthogonal. Further, prove that the rank of $A+B$ is $n-(2k+1)$, $0 \leqq k \leqq \frac{1}{2}(n-1)$, if A is proper and B improper.

SOLUTIONS

Plane section of a tetrahedron

5006 [1962, 63; 1963, 338]. *Proposed by E. Ehrhart, Lycée Kleber, Strasbourg, France*

Is the following proposition valid: Every plane section of a tetrahedron is smaller (in area) than the largest face?

II. *Solution by H. G. Eggleston, Bedford College, University of London, and University of Washington, Seattle.* The following observations provide an affirmative answer. By a compactness argument, there is a plane p whose section with the given tetrahedron T has the largest area of all such plane sections. The plane p cannot cut T in a quadrilateral; for, if it did, no vertex of this quadrilateral $ABCD$ could be a vertex of T (otherwise the section would be a triangle), also the area of $ABCD$ is the product of the length of one diagonal, say AC, by one half the length of the projection of the other diagonal BD onto a plane perpendicular to AC. If we consider the effect of a sufficiently small rotation of p about the line AC, we see that both B and D vary and their projections vary on two lines (the projections of the edges of T on which they lie) in such a way that the projected segment passes through a fixed point (the projection of the line AC). But then the sense of rotation of p about AC can be chosen so as to increase this projected length. This contradicts the extremal property of p.

Thus p meets T in a triangle, and a similar argument (rotating p about one of the edges of this triangle) shows that every vertex of this triangle must be a vertex of T.

Editorial Note. The proposer points out a difficulty with the treatment of quadrilateral sections as given in the earlier solution [1963, 338]. In particular it is, in general, false that "any linear parameter μ determining the place of D on m is a linear function of a linear parameter λ determining the place of C on l." The solution given above avoids this difficulty.

Left field of quotients

5059 [1962, 1012]. *Proposed by E. R. Gentile, Universidad del Sur, Argentina*

Let R be an integral ring (possessing unity element and without divisors of zero). R is said to have a left field of quotients if any two nonzero left ideals have a nonzero intersection. Let A be a left R-module. An element a of A is called a (left) R-torsion element if $\mu a=0$ for some $\mu \neq 0$ in R. Let tA be the set of all torsion elements of A. Prove that R has a left field of quotients if and only if for every left R-module A, tA is a submodule.

I. *Solution by G. J. Janusz, University of Wisconsin.* Let R have a left field of quotients. For any nonzero elements u, $v \in R$, there exist u', $v' \in R$ such that $v'u=u'v\neq 0$. Now if A is any left R-module with x, $y \in tA$ then $ux=vy=0$ for

some u, $v\in R$, $uv\neq 0$. Then $u'v(x-y)=0$ and so $x-y\in tA$. Also for any $r\in R$, $r\neq 0$, and $r'u=u'r\neq 0$ we have $0=r'ux=u'rx$ which implies $rx\in tA$. Thus tA is a submodule of A.

For the converse, let L be any nonzero left ideal of R. Define $L^*=\{u\in R: Ru\cap L\neq 0\}\cup\{0\}$. We easily see that L^*/L is exactly the set of torsion elements in the R-module $A=R/L$. Hence L^*/L is a submodule and thus L^* is a left ideal. Since $1\in R$, we have $1\in L^*$ and so $L^*=R$. Thus for any nonzero left ideal H, we have $H=RH$ and $H\cap L=RH\cap L\neq 0$, since $H\subseteqq L^*$.

II. *Solution by Barbara L. Osofsky, Douglass College.* A slight generalization of this problem, with the word "left" replaced by "right," is given as Theorem 1.4 of Lawrence Levy, *Torsion-free and Divisible Modules over Nonintegral Domains*, Canad. J. Math., 15 (1963) 132–151.

Also solved by Marshall Barton, David Carlson, Harley Flanders, J. P. Jans, Kwangil Koh, Ronald McHaffey, Veselin Perić, J. D. Reid, Richard Stroud, E. W. Swokowski, Dov Tamari, Dennis Travis, Seth Warner, Richard Wiegandt, and the proposer.

Infinitely differentiable functions which are nowhere analytic

5061 [1963, 1013]. *Proposed by Eric D. Nix, New York, N. Y.*

Exhibit (or disprove the existence of) a function which is C^∞ on an interval but nowhere analytic.

Editorial Note. I. N. Baker proves that the following function $F(x)$ is infinitely differentiable but not analytic anywhere on the real axis: $F(x)=\sum_{n=1}^{\infty} 2^{-n}f_n(x)$, where $f(x)=\exp\ (-1/x^2)$, $x\neq 0, f(0)=0$, and $f_n(x)=f(x-p_n)$, with p_n the rational numbers in a sequence.

Many examples are in the literature. The following references are cited by readers: Morgenstern, Math. Nachr., 12 (1954) 74; H. Cartan, Actualités Sci. Ind., no. 867 (1940) 20–22; A. Pringsheim, Math. Ann., 42 (1893) 153–184; S. Mandelbrojt, *Analytic Functions and Classes of Infinitely Differentiable Functions*, The Rice Institute Pamphlet, v. XXIX, pp. 2–3; Zahorski, Fundamenta Math., 34 (1947) 183–245; Goursat, Mathematical Analysis, VII, part I, p. 213.

Bicompact topological space

5062 [1962, 1013]. *Proposed by Jonathan Sondow, University of Wisconsin*

Prove that a topological space X is bicompact if and only if every cover of X has an irreducible subcover.

Solution by J. A. Zilber, Ohio State University. By cover we understand open cover; otherwise the case X a closed interval is easily seen to be a counterexample. In one direction the assertion is obvious. In the other direction, suppose X is not bicompact, so that it has a cover such that every subcover is infinite; then we must exhibit a cover such that every subcover is reducible. If every subcover is reducible, we are done. If not, let C be an irreducible subcover; thus (1) C is infinite and (2) for every $V\in C$ there is an $x\in V$ which is covered by no other member of C. Let $V_1, V_2, \cdots$ be a sequence of distinct members of C, and let W_0 be the union of the remaining members, and put $W_n=V_1\cup\cdots\cup V_n$, $n\geqq 1$. Then $D=\{W_0, W_1, \cdots\}$ is a cover with the desired property. For clearly

any subcover consists of an infinite subsequence of D, and it remains a cover if we drop from it any one W_n, $n \geqq 1$.

Also solved by B. H. McCandless, Barbara L. Osofsky, W. J. Pervin, W. C. Waterhouse, Richard Wiegandt, and the proposer.

Two special summations

5063 [1962, 1013]. *Proposed by R. E. Shafer, University of California, Livermore*

Show that

$$(1) \quad \sum_{n=1}^{\infty} \frac{1}{\sinh^2 n\pi} = \frac{1}{6} - \frac{1}{2\pi}, \qquad (2) \quad \sum_{n=1}^{\infty} \frac{1}{\cosh^2 (2n-1)\pi/2} = \frac{1}{2\pi}.$$

I. *Solution by A. E. Livingston, University of Alberta.* Apply the Residue Theorem to $\int_C \cot z\, dz/\sinh^2 z$ for (1) and to $\int_\Gamma \tan z\, dz/\cosh^2 z$ for (2) where C and Γ are rectangles in the complex plane with vertices at $\pm(m+\frac{1}{2})(1\pm i)\pi$ and $\pm m(1\pm i)\pi$, respectively ($m=1, 2, \cdots$), and take the limit as $m\to\infty$. We offer only a sketch of the solution.

Recalling that $\tan z$ and $\cot z$ have simple poles, it is immediate that

$$\text{Res}\left\{\frac{\cot z}{\sinh^2 z}; n\pi\right\} = 1/\sinh^2 n\pi \qquad \text{for } n \neq 0 \text{ and}$$

$$\text{Res}\left\{\frac{\tan z}{\cosh^2 z}; \frac{2n-1}{2}\pi\right\} = -1/\cosh^2 \frac{2n-1}{2}\pi.$$

The residues at the remaining singularities are most easily obtained from the Laurent series

$$\frac{\cot z}{\sinh^2 z} = \frac{1}{z^3} - \frac{2}{3z} + \cdots$$

$$= \frac{\cot in\pi}{(z-in\pi)^2} - \frac{\csc^2 in\pi}{z-in\pi} + \cdots \qquad n \neq 0,$$

$$\frac{\tan z}{\cosh^2 z} = -\frac{\tan a}{(z-a)^2} - \frac{\sec^2 a}{z-a} + \cdots, \qquad a = \frac{2n-1}{2}\pi i.$$

Since $\csc it = -i \operatorname{csch} t$ and $\sec it = 1/\cosh t$, we have

$$\int_C = 2\pi i\left(-\frac{2}{3} + 4\sum_{n=1}^{m} \frac{1}{\sinh^2 n\pi}\right),$$

$$\int_\Gamma = 2\pi i\left(4\sum_{n=1}^{m} \frac{1}{\cosh^2 (2n-1)\pi/2} - \frac{2}{\cosh^2 (2m-1)\pi/2}\right).$$

It is easily seen that each of the integrals along the vertical sides of the

appropriate rectangles is $O(me^{-2m})$. Along the combined horizontal sides the integrals become

$$2i\int_{-(m+\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi}\frac{\operatorname{Im}\{\cot[x+(m+\frac{1}{2})\pi i]\}}{\cosh^2 x}dx,$$

$$-2i\int_{-m\pi}^{m\pi}\frac{\operatorname{Im}\{\tan(x+n\pi i)\}}{\cosh^2 x}dx,$$

and, hence, tend to $2i$ I and $-2i$ I, respectively, as $m\to\infty$, where

$$I\equiv\int_{-\infty}^{\infty}\operatorname{sech}^2 x\,dx = 2.$$

The conclusions now follow upon passing to the limit in the expressions for $\int_C$ and $\int_\Gamma$ above.

II. *Solution by J. Raleigh, Temple University.* Following Halphen, *Traité des Fonctions Elliptiques*, Paris, 1886, p. 281, we consider the lemniscatic function $\wp(u;\ 2\omega,\ 2\omega')$ with $\omega'=i\omega$, $\eta\omega=\pi/4$, $q=e^{-\pi}$, $e_1=1$, $e_0=0$, $e_3=-1$, $g_2=4$, $g_3=0$. Then from formulas (33) and (35), Halphen p. 404, our two results follow at once.

The expansion in partial fractions of $\pi^2/\sin^2\pi z=\sum_{n=-\infty}^{\infty}(z+n)^{-2}$ shows that the problem in question is intimately related to the modular form $\sum'^{\infty}_{m,n=-\infty}(mz+n)^{-2}$. Attempts to deal directly with such double series would imply reproducing or simulating proofs to be found in the literature on elliptic or modular functions. The corresponding formulae for different values of g_2 and g_3 would supply a whole class of identities similar to (1) and (2).

Also solved by P. Baldwin, P. T. Bateman, J. Boersma, L. Carlitz, P. J. de Doelder, M. L· Glasser, J. Koekoek, H. Schmidt and K. Kiyek, and the proposer.

Regular Semigroup

5064 [1962, 1013]. *Proposed by S. Lajos, L. Eötvös University, Hungary*

A semigroup S is called regular if to each element $a\in S$ corresponds at least one semi-inverse element $x\in S$ such that $axa=a$. Prove that S is regular if and only if $(a)_R(a)_L=(a)_R\cap(a)_L$, for all $a\in S$, where $(a)_L$ denotes the principal left ideal of S generated by a.

Solution by Edgar R. Guillot (undergraduate), *Louisiana State University in New Orleans.* Assume that S is regular. Let $a\in S$, and a' be such that $aa'a=a$. Then $(a)_R=aS\cup a=aS$ since $a=aa'a=a(a'a)\in aS$; similarly, $(a)_L=Sa$. Let $z\in(a)_R(a)_L$; then $z=axya$ for some $x,y\in S$. Since $z=a(xya)\in(a)_R$ and $z=(axy)a\in(a)_L$, it follows that $(a)_R(a)_L\subseteq(a)_R\cap(a)_L$. Now let $z\in(a)_R\cap(a)_L$; then there exist x, $y\in S$ such that $z=ax=ya$. Then $z=ya=yaa'a=axa'a\in(a)_R(a)_L$; so $(a)_R\cap(a)_L=(a)_R(a)_L$.

Conversely, suppose that $(a)_R\cap(a)_L=(a)_R(a)_L$ for all $a\in S$. Obviously $a\in(a)_R\cap(a)_L$; therefore $a\in(a)_R(a)_L$, so that $a=aa$, whence $aaa=a$ and a is its

own semi-inverse. Let $b \neq a$ and $b \in (a)_R \cap (a)_L$; therefore $b \in (a)_R(a)_L$ so that $b=(as_j)(s_i a)$, with $s_i, s_j \in S$, whence $s_j s_i$ is the semi-inverse of b. Thus S is regular.

Also solved by Robert Bowen, Dennis R. Brown, Bruce Erickson, E. C. Ingraham, M. F. Janowitz and E. A. Schreiner, A. E. Livingston, Jiang Luh, D. B. McAlister, R. A. Mahaffey, Barbara L. Osofsky, Veselin Perić, J. D. Reid, Richard Stroud, D. Topping, Oscar Travis, Chester E. Tsai, A. J. Umen, Bro. T. C. Wesselkamper, Richard Wiegandt, and the proposer.

Editorial Note. In his paper, *A remark on regular semigroups* [Proc. Japan Acad., 37 (1961) 29–30], the proposer shows that the following conditions concerning a semigroup S are equivalent: (1) S is regular, (2) $R \cap L = RL$ for every right ideal R and every left ideal L of S, (3) $(a)_R(b)_L = (a)_R \cap (b)_L$ for every $a, b \in S$, (4) $(a)_R \cap (a)_L = (a)_R(a)_L$ for all $a \in S$.

Characterizing properties of sin x

5065 [1963, 96]. *Proposed by Robert Spira, University of California, Berkeley*

For distinct integers k and l let

$$\int_0^\pi f(kx)f(lx)dx = 0, \qquad \int_0^\pi [f(kx)]^2 dx = \tfrac{1}{2}\pi.$$

What additional restrictions on f are needed to insure that $f(x) = \sin x$?

I. *Solution by A. E. Livingston, University of Alberta.* If $f(x) = \sin x$, then

(A) $f(x) \in C[0, \pi]$,
(B) $\{f(nx)\}_{n=1}^{\infty}$ is a complete orthogonal system in $L^2[0, \pi]$, and
(C) $2\pi^{-1}\int_0^\pi f(nx) \sin x dx = \delta_{n1}$ $(n=1, 2, \cdots)$.

(Conclusions (A) and (C) are obvious. For (B), see G. H. Hardy and W. W. Rogosinski, *Fourier Series*, Cambridge University Press, (1950) 20–21.)

On the other hand, $\sin x \in L^2[0, \pi]$, so that (B) and the Riesz-Fischer theorem guarantee that $\sin x$ has a Fourier series $\sum_{n=1}^{\infty} c_n f(nx)$, with $c_n = 2\pi^{-1}\int_0^\pi f(nx) \sin x dx$, which converges to $\sin x$ in the L^2-sense. But then there is a sequence of positive integers $\nu_1 < \nu_2 < \nu_3 < \cdots$ such that

$$(*) \qquad \sin x = \lim_{m\to\infty} \sum_{n=1}^{\nu_m} c_n f(nx)$$

almost everywhere in $[0, \pi]$. If, now, (C) is true, then (*) tells us that $\sin x = f(x)$ almost everywhere in $[0, \pi]$. Finally, (A) gives $\sin x = f(x)$ everywhere in $[0, \pi]$.

Thus, (A), (B), and (C) are necessary and sufficient conditions that $f(x) = \sin x$, given that $2\pi^{-1}\int_0^\pi f(kx)f(lx)dx = \delta_{kl}$.

II. *Note by Matts Essén, University of Uppsala, Sweden.* The problem is not new. It has been treated by Bourgin and Mendel (Trans. Amer. Math. Soc., 57 (1945) 332–363) and by Bourgin (Proc. Nat. Acad. Sci., USA, 32 (1946) 1–5, and Trans. Amer. Math. Soc., 60 (1946) 478–518). It was later rediscovered by Essén (Arkiv för Matematik, 3, nr. 47 (1958) 505–510) who treated the corresponding problem for almost periodic functions.

Semigroup with Left Inverses and Identities

5066 [1963, 96]. *Proposed by P. J. Sally, Jr., Boston College*

In an article in this MONTHLY (Oct. 1961, p. 795) the authors have posed the following question. If G is a semigroup with local left identities and inverses, that is, writing the semigroup operation multiplicatively,

1) for each a in G, there is an element $e=e(a)$ such that $ea=a$.
2) for each e in G such that $ea=a$, there is an element $a'=a'(e)$ in G such that $a'a=e$,

is G then a group? This question is answered negatively with a simple example. A theorem is stated to the effect that a commutative semigroup with properties 1) and 2) is a group. The restriction to commutativity is not essential and precludes the application of these conditions to the noncommutative case. Show that the necessary and sufficient condition for a semigroup G satisfying 1) and 2) to be a group is that every left identity is in $C(G)$, the center of G with respect to the semigroup structure.

Solution by E. A. Schreiner, Wayne State University, Detroit, Mich.

The necessity of the condition is clear, since the only local left identity in a group is the identity, which is central.

Now let a, b be arbitrary elements of G. Then there are elements e and a' with $ea=a$, $a'a=e$, and $e\in C(G)$. Since $e(eb)=a'aeb=a'eab=a'ab=eb$, there is an $(eb)'$ with $(eb)'eb=e$. Then $a(eb)'eb=ae=a$. Thus for arbitrary a, $b\in G$ there is an $x\in G$ such that $xb=a$.

Let i be a left identity corresponding to b. Then $a=xb=xib=ixb=ia=ai$, so that i is the identity for G. Further, for any $g\in G$ there is an x with $xg=i$, so G is a group and the condition is sufficient.

Also solved by Raymond Achilles, Edward S. K. Ansah, K. F. Bailie, J. L. Chrislock, L. V. Di Bello, John M. Franks, A. W. Fuller, Michael Gemignani, J. F. Leetch, Jiang Luh, Veselin Peric, Dennis Travis, Chester Tsai, W. C. Waterhouse, and the proposer.

Quadratic residues

5067 [1963, 97]. *Proposed by L. Carlitz, Duke University*

I. Given $a^2=4p^2b^2+c^2$, $(a, 2p)=1$, $a>0$, where p is an odd prime. Show that a is a quadratic residue, mod p.

II. Given $a^2=p^2b^2+4c^2$, $(a, 2p)=1$, $a>0$, where p is an odd prime. Show that $2a$ is a quadratic residue, mod p.

Solution by Ralph Greenberg, University of Pennsylvania. We must assume $(a, b)=1$. Then, if $a^2=4p^2b^2+c^2$, it is well known that $a=u^2+v^2$, $pb=uv$, $c=u^2-v^2$ for some integers u, v with $(u, v)=1$. Then p divides u or v but not both, and $a\equiv x^2 \pmod p$ is solvable. Similarly for II, $a=u^2+v^2$, $pb=u^2-v^2$ imply $2a+2pb=4u^2$, so that $2a$ is a quadratic residue, mod p.

The proposed statement is untrue for a written in either of the above forms if (a, b) is a quadratic nonresidue, mod p.

Also solved by W. J. Blundon, Robert Breusch, Daniel I. A. Cohen, Lawrence Corwin, Harley Flanders, Roy H. Ogawa, Harlan Stevens, M. Sugunamma, and the proposer.

presented, an automobile manufacturing problem is solved using linear programming, etc.

In Chapter III there is a discussion of flow diagrams, for, as the authors rightly point out, an accurate flow diagram permits "easy description of complicated computational processes such as the Monte Carlo simulation calculations of Chapter IV . . . " and construction of a good flow diagram is an extremely important part of computer programming.

Almost all of the topics treated are well motivated, and the inexperienced reader is constantly reminded of the relevance of each topic to business and industrial problems. In this reviewer's opinion, however, the treatment of compound statements at the outset and the use of elements of the vocabulary of symbolic logic at various points throughout the text (in the discussion of probability and of polyhedral convex sets in Chapter VII, for example) could well have been omitted—given the objective of the book. For the discussion of compound statements contributes little to understanding those topics of finite mathematics (discussed in later chapters) of real relevance from the manager's point of view, and in fact is somewhat redundant in view of the well motivated treatment of some of the same ideas from the set theoretic point of view in Chapter II. Also, this terminology is rare in the management science literature.

In sum, this text does a good job of treating a wide variety of topics in finite mathematics on a level that is elementary enough to be understood by the unsophisticated reader, but at the same time just deep enough to give such readers a feel for the increasingly important role mathematics is playing in the analysis of business and industrial problems. It is a fine addition to a growing number of texts on this level.

G. M. KAUFMAN, School of Industrial Management, M.I.T.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo) Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. H. Bing, President of the MAA, represented the Association at the Centennial Celebration of the National Academy of Sciences during October 21–24, 1963.

Professor D. E. Christie, Bowdoin College, represented the Association at the special academic convocation commemorating the 100th Anniversary of Bates College on November 20, 1963.

Dean R. D. Larsson, Mohawk Valley Community College, represented the Association at the inauguration of W. L. Whitson as President of Clarkson College of Technology on September 20, 1963.

Professor J. C. Polley, Wabash College, represented the Association at the inauguration of W. E. Kerstetter as President of DePauw University on October 12, 1963.

Professor R. R. Stoll, Oberlin College, represented the Association at the inauguration of R. I. White as President of Kent State University on October 25, 1963.

American University: Associate Professors Grace S. Quinn and S. H. Schot have been promoted to Professors; Professor S. H. Schot has been awarded a Post Doctoral Fellowship by NSF and is on sabbatical leave at the Max Planck Institute, Göttingen, Germany.

Eastern Michigan University: Mrs. Nelly Ullman, Polytechnic Institute of Brooklyn, has been appointed Assistant Professor; Assistant Professor H. G. Falahee has been promoted to Associate Professor.

Franklin and Marshall College: Assistant Professor W. H. Leser has been promoted to Associate Professor; Associate Professor J. R. Holzinger has been promoted to Professor of Mathematics and Astronomy; Associate Professor Clifford Marburger retired September 1, 1963 with the title of Professor Emeritus.

Georgetown University: Visiting Assistant Professor Witold Bogdanowicz and Assistant Professor J. E. LeBel have been promoted to Associate Professors.

Kansas State University: Dr. Robert Bechtel, Purdue University, has been appointed Assistant Professor; Associate Professor L. J. Dixon, Arkansas State College, has been appointed Associate Professor; Professor W. L. Stamey has been promoted to Associate Dean of the College of Arts and Sciences.

Mary Washington College: Misses Lois J. Reid and Mary Montgomery and Mr. L. J. Jones have been promoted to Assistant Professors; Assistant Professor Anna M. Harris has been promoted to Associate Professor.

Michigan State University: Dr. Michael Edelstein, Israel Institute of Technology, and Associate Professor Diran Sarafyan, U. S. Army Research Center, University of Wisconsin, have been appointed Visiting Professors; Assistant Professor R. H. Wasserman has been promoted to Associate Professor; Associate Professor W. E. Deskins has been promoted to Professor.

University of New Mexico: Associate Professor M. F. Janowitz, Wayne State University, and Assistant Professor Burt Morse, St. John's University, have been appointed Assistant Professors; Professor Bernard Epstein, Yeshiva University, has been appointed Professor; Professor M. S. Hendrickson has been appointed Acting Dean of Arts and Sciences; Professor J. R. Blum has been appointed Acting Chairman of the Mathematics Department.

North Carolina State College: Dr. J. W. Querry, Air Force Office of Scientific Research, Washington, D. C., has been appointed Associate Professor; Professor Hans Sagan, University of Idaho, has been appointed Professor; Mrs. Ruth B. Honeycutt and Miss Thelma J. Caraway have been promoted to Assistant Professors; Assistant Professor J. B. Wilson has been promoted to Associate Professor.

Stephen F. Austin State College: Assistant Professor F. D. Alexander has been promoted to Associate Professor; Professor W. I. Layton has been named Coordinator of the Data Processing Center.

Syracuse University: Dr. R. T. J. Mahoney, Washington University, has been appointed Assistant Professor; Assistant Professor F. J. Kosier, University of Wisconsin, has been appointed Visiting Assistant Professor; Assistant Professor G. T. Cargo has been promoted to Associate Professor; Associate Professors R. B. Davis and Erik Hemmingsen have been promoted to Professors.

University of Toledo: Dr. S. E. Spielberg, University of Minnesota, has been appointed Assistant Professor; Assistant Professor J. L. Bailey, Case Institute of Technology, has been appointed Associate Professor; Professor R. F. Jackson, University of Delaware, has been appointed Professor; Assistant Professor Edward Ebert has been appointed Assistant Chairman of the Mathematics Department; Assistant Professor Camilla Hayden retired June 1963.

Dean A. A. Albert, University of Chicago, has been elected to a corresponding membership in the National Academy of Sciences of Buenos Aires, Argentina.

Associate Professor Ferrel Atkins, Eastern Illinois University, has been promoted to Professor.

Assistant Professor Adi Ben-Israel, Carnegie Institute of Technology, has been appointed Senior Lecturer at the Technion-Israel Institute of Technology, Haifa, Israel.

Assistant Professor May Blackstock, Shorter College, has been promoted to Associate Professor.

Associate Professor W. E. Briggs, University of Colorado, has been appointed Acting Dean of the College of Arts and Sciences for the academic year 1963–64.

Mrs. John Burr, Hood College, has been promoted to Assistant Professor.

Associate Professor H. E. Campbell, Michigan State University, has been appointed Professor and Head of the Mathematics Department of the University of Idaho.

Rev. Robert Case, OSB, Yeshiva University, has been appointed Assistant Professor at St. Anselm's College.

Assistant Professor R. F. DeMar, Miami University, has returned from a year's leave of absence at the National Bureau of Standards and has been promoted to Associate Professor.

Dr. S. I. Drobnies, General Dynamics Corporation, Fort Worth, Texas, has been appointed Assistant Professor at San Diego State College.

Associate Professor W. H. Fleming, Brown University, has been promoted to Professor.

Dr. Abraham Franck, Tron-chemics Research, has been appointed Staff Scientist at Fabri-Tek's Research and Development Center, Hopkins, Minnesota.

Assistant Professor R. P. Goblirsch, University of Rochester, has been appointed Assistant Professor at the University of Colorado.

Professor Arnold Grudin, Denison University, has been appointed Chairman of the Mathematics Department.

Dr. E. C. Johnsen, National Bureau of Standards, Washington, D. C., has been appointed Lecturer at the University of California, Santa Barbara.

Associate Professor J. B. Kelly, Michigan State University, has been appointed Associate Professor at Arizona State University.

Professor Joseph Lehner, Michigan State University, has been appointed Professor at the University of Maryland.

Professor Howard Levi, Columbia University, has been appointed Professor at Hunter College. (This is a correction of an item which appeared in the June–July issue.)

Associate Professor M. E. Mahowald, Syracuse University, has been appointed Professor at Northwestern University.

Associate Professor D. G. Mead, Pratt Institute of Technology, has been appointed Associate Professor at the University of Santa Clara.

Professor W. G. Miller, Clemson College, has been appointed Professor at Trinity University.

Dr. D. M. Paine, University of Wisconsin, has been appointed Assistant Professor at Wells College.

Professor Sallie E. Pence, University of Kentucky, retired July 1, 1963 with the title of Professor Emeritus.

Mr. Edward Rayher, Central High School, Paterson, New Jersey, has been appointed Assistant Professor at Hartwick College.

Assistant Professor R. J. Roth, University of Kentucky, has been appointed Professor at the University of South Florida.

Lt. Col. W. G. Saunders, Virginia Military Institute, has been promoted to Colonel and Professor of Mathematics.

Mr. C. P. Seguin, Beloit College, has been promoted to Assistant Professor.

Mr. H. K. Stumpff, University of New Mexico, has been appointed Assistant Professor at Central Missouri State College.

Dr. D. H. Trahan, University of Pittsburgh, will be a Lecturer in Mathematics under a Fulbright award at University College, Galway, Ireland during the academic year 1963–64.

Dr. D. R. Traylor, Auburn University, has been appointed Assistant Professor at the University of Houston.

Assistant Professor R. G. Van Meter, Geneva College, has been appointed Associate Professor at Allegheny College.

Assistant Professor Eugene Wermer, Radford College, has been appointed Assistant Professor at Norwich University.

Sister Mariola Dobbin, Professor Emerita of Rosary College, died on August 12, 1963. She was a charter member of the Association.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the five undergraduates ranking highest on the score of the General Mathematics Examination of the May 1963 Preliminary Actuarial Examinations are as follows: The first prize of $200 was awarded to Ralph E. Walde of the University of Minnesota. Additional prizes of $100 each went to Beverly D. Causey of Princeton University; James H. Foster of the University of Notre Dame; Michael J. Razar of Harvard University; and Mark G. Tanenbaum of Massachusetts Institute of Technology. Information concerning the Preliminary Actuarial Examinations can be obtained from the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The twenty-second annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Brooklyn College on April 27, 1963. Dr. Francis P. Kilcoyne, Dean of Administration, Brooklyn College, welcomed the gathering. Professor Abraham Schwartz, Collegiate Vice-Chairman of the Section, presided at the morning session and Mr. Lester Schlumpf, High School Vice-Chairman, presided at the afternoon session. One hundred eighteen persons, including fifty-nine members of the Association, attended the meeting.

Professor Jules P. Russell, Chairman of the Section, presided at the business meeting. Professor Charles T. Salkind announced the names of the local winners in the mathematics contest sponsored by the Mathematical Association of America and the Society of Actuaries. Several of these winners were present and were introduced to the gathering.

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EDITED BY RAOUL HAILPERN, SUNY at Buffalo

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